Algebra Colloquium **10**:4 (2003) 461-478

Algebra Colloquium

On Local Properties of Unary Algebras

S. Bogdanović*, M. Ćirić†, B. Imreh‡, T. Petković§,†, M. Steinby§

**University of Niš, Faculty of Economics*
P.O. Box 91, 18000 Niš, Yugoslavia
E-mail: sbogdan@pmf.pmf.ni.ac.yu

†Department of Mathematics, Faculty of Sciences and Mathematics
University of Niš, Ćirila i Metodija 2, P.O. Box 224, 18000 Niš, Yugoslavia
E-mail: ciricm@bankerinter.net, tanjapet@pmf.pmf.ni.ac.yu

‡Department of Informatics, University of Szeged, Árpád tér 2
H–6720 Szeged, Hungary
E-mail: imreh@inf.u-szeged.hu

§Turku Centre for Computer Science, and Department of Mathematics,
University of Turku, FIN-20014 Turku, Finland
E-mail: tatpet@utu.fi, steinby@utu.fi

Received 7 August 2001

Communicated by M. Ito

Abstract. If every finitely generated subalgebra of an algebra $\mathcal{A}$ has a property P, then $\mathcal{A}$ is said to have P as a *local property*. In this paper we study classes $L(\mathbf{K})$ and $M(\mathbf{K})$, where $\mathbf{K}$ is a class of unary algebras, $L(\mathbf{K})$ consists of all algebras such that every finitely generated subalgebra is in $\mathbf{K}$, and $M(\mathbf{K})$ is the class of all algebras with every monogenic subalgebra in $\mathbf{K}$. In particular, the cases where $\mathbf{K}$ is a variety, a generalized variety or a pseudovariety, are considered. It is also shown that the monogenic closure of a variety or a pseudovariety equals the regularization of the class. Finally, we note that some of our central concepts are derived from the theory of finite automata and hence many of the results can be interpreted in that theory.

2000 Mathematics Subject Classification: 08A60, 03C05, 08B99, 68Q70

Keywords: unary algebras, automata, varieties, pseudovarieties, regular identities

1 Introduction

We say that an algebra has a property *locally* if all of its finitely generated subalgebras have this property. The idea is formalized by means of an operator L, introduced in [19], that assigns to each class $\mathbf{K}$ of algebras (of a given type) the class $L(\mathbf{K})$ of all algebras $\mathcal{A}$ such that every finitely generated subalgebra of $\mathcal{A}$ is in $\mathbf{K}$. We study also an operator M that assigns to $\mathbf{K}$ the class $M(\mathbf{K})$ of all algebras $\mathcal{A}$ such that $\mathbf{K}$ contains every monogenic subalgebra of $\mathcal{A}$. Especially as we consider unary algebras only, the operator M has some special properties. The classes $L(\mathbf{K})$ and $M(\mathbf{K})$ are called the *local* and the *monogenic closure* of $\mathbf{K}$, respectively.

Many of the notions discussed in this paper are derived from the theory of finite automata, but we shall consider more generally classes of X-algebras $\mathcal{A} = (A, X)$, where neither the set A of elements nor the set X of unary operation symbols is assumed to be finite. Only in connection with pseudovarieties finiteness is required.

In the Preliminaries most of the general notation is fixed, and several algebraic notions are recalled. In Section 3 the operators L and M are defined and some of their basic properties are recorded.

In Section 4 we study algebras belonging to the monogenic closure of the class **Conn** of so-called *connected* X-algebras. This class will appear also later in the general theory.

In Section 5 we note that M applied to varieties of X-algebras is a closure operator. It is also shown that the monogenic closure of a variety $\mathbf{V}$ of X-algebras equals the least regular variety containing $\mathbf{V}$, the *regularization* $R(\mathbf{V})$ of $\mathbf{V}$; a variety of unary algebras is *regular* if it can be defined by identities in which the variable is the same on both sides. We show that the monogenic closure (and hence, the regularization) of a variety $\mathbf{V}$ is both the Mal'cev product and the join of $\mathbf{V}$ itself and the variety $\mathbf{D}$ of discrete X-algebras, i.e., X-algebras in which every operation is the identity mapping. Finally, we show, following Płonka [**?**, **?**], how to obtain a set of identities defining $M(\mathbf{V})$.

In Section 6 the operators L and M are applied to generalized varieties of X-algebras. In this case the operators are not closure operators, but again the regularity of a generalized variety $\mathbf{G}$ of X-algebras can be expressed in terms of discrete X-algebras. Moreover, the two classes $L(\mathbf{G})$ and $M(\mathbf{G})$ are closely related to each other. For example, $L(\mathbf{G}) = M(\mathbf{G})$ if $\mathbf{G}$ is regular, and $M(\mathbf{G}) = L(\mathbf{G}) \circ \mathbf{D} = L(\mathbf{G} \circ \mathbf{D})$ quite generally.

In Section 7 we consider pseudovarieties of X-algebras and the operators L and M are modified accordingly so that finite algebras only are included in the local and monogenic closures. After noting how the regularity of a pseudovariety can be expressed in terms of finite discrete algebras, we show that the monogenic closure of a pseudovariety and its regularization are the same class, and how this class can be constructed in two ways from the original pseudovariety and the pseudovariety of finite discrete X-algebras.

2 Preliminaries

In what follows, X is always a non-empty alphabet, but not necessarily finite, X^* denotes the set of all words over X, the empty word is denoted by e. With the catenation of words as the operation and e as the unit element, X^* is the free monoid generated by X. However, we treat X also as a set of unary operation symbols and words over X are then regarded as X-terms over a one-element set of variables $\{\varepsilon\}$, written in reverse Polish notation: the empty word represents the term ε, and a nonempty word $x_1x_2\cdots x_n$ ($n \geq 1$) the term $\varepsilon x_1x_2\cdots x_n$. An *$X$-algebra* $\mathcal{A} = (A, X)$ is a system where A is a nonempty set and each symbol $x \in X$ is realized as a unary operation $x^{\mathcal{A}} : A \to A$. For any $a \in A$ and $x \in X$, we write $ax^{\mathcal{A}}$ for $x^{\mathcal{A}}(a)$. For any word $w = x_1x_2\ldots x_n \in X^*$, $w^{\mathcal{A}} : A \to A$ is defined as the composition of the mappings $x_1^{\mathcal{A}}, x_2^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}$, that is to say, $aw^{\mathcal{A}} = ax_1^{\mathcal{A}}x_2^{\mathcal{A}}\cdots x_n^{\mathcal{A}}$ for any $a \in A$. In particular, $e^{\mathcal{A}}$ is the identity mapping 1_A of A. If $\mathcal{A}$ is known from the context, we write simply aw instead of $aw^{\mathcal{A}}$. The X-algebra $\mathcal{A}$ is *finite* if A is a finite set, and it is *trivial* if A has only one element. In what follows, we often assume, without saying so, that $\mathcal{A}$ is the X-algebra (A, X).

The basic algebraic notions are defined as for algebras in general (cf. [4], for example). Hence an X-algebra $\mathcal{B} = (B, X)$ is a *subalgebra* of the X-algebra $\mathcal{A} = (A, X)$ if $B \subseteq A$ and $bx^{\mathcal{B}} = bx^{\mathcal{A}}$ for all $b \in B$ and $x \in X$. This means that B is a *closed subset* of $\mathcal{A}$, i.e., $bx^{\mathcal{A}} \in B$ for all $b \in B$ and $x \in X$. On the other hand, each nonempty closed subset of $\mathcal{A}$ corresponds to a unique subalgebra of $\mathcal{A}$. As usual, we call also nonempty closed subsets subalgebras, and we denote their set by $\mathrm{Sub}\mathcal{A}$. For any nonempty $H \subseteq A$, the least subalgebra of $\mathcal{A} = (A, X)$ containing H, i.e., the subalgebra generated by H, is denoted by $\langle H \rangle$. It is obvious that $\langle H \rangle = \{aw \,|\, a \in H, w \in X^*\}$. If H is a one-element set, $\langle H \rangle$ is a *monogenic subalgebra* of $\mathcal{A}$, and if H is a finite set, then $\langle H \rangle$ is a *finitely generated subalgebra* of $\mathcal{A}$. For any $a_1, \ldots, a_n \in A$, we write just $\langle a_1, \ldots, a_n \rangle$ instead of $\langle \{a_1, \ldots, a_n\} \rangle$. Obviously, $\langle a_1, \ldots, a_n \rangle = \langle a_1 \rangle \cup \ldots \cup \langle a_n \rangle$.

A word $u \in X^*$ is a *directing word* of an X-algebra $\mathcal{A}$ if $au = bu$ for all $a, b \in A$, and then d_u denotes the element au. The set of all directing words of $\mathcal{A}$ is denoted by $DW(\mathcal{A})$ and $\mathcal{A}$ is called *directable* if it has a directing word. The class of all directable X-algebras is denoted by **Dir**. For any given $u \in X^*$, we call $\mathcal{A}$ *u-directable* if $u \in DW(\mathcal{A})$. The class of u-directable X-algebras is denoted by $\mathbf{Dir_u}$. A *trap* of an X-algebra $\mathcal{A}$ is an element $a_0 \in A$ such that $a_0x = a_0$ for every $x \in X$. An X-algebra is *reverse n-definite*, $n \in \mathbb{N}$, if every word of length n takes any element of $\mathcal{A}$ into a trap. Furthermore, if $\mathcal{A}$ has a unique trap, then it is *n-nilpotent*. The classes of all reverse n-definite and n-nilpotent X-algebras are denoted by $\mathbf{RDef_n}$ and $\mathbf{Nilp_n}$, respectively. Although not really needed here, the reader may consult [6], [8] or [11] for the relevant general theory of finite automata, and [3] or [12] for directable automata and further references to them.

An equivalence θ of A is a *congruence* on $\mathcal{A} = (A, X)$ if for all $a, b \in A$ and $x \in X$, $ax \,\theta\, bx$ whenever $a \,\theta\, b$. Obviously, the *diagonal relation* $\Delta_A = \{(a, a) \,|\, a \in A\}$ and the *universal relation* $\nabla_A = A \times A$ are congruences on $\mathcal{A}$. Note that if θ is a congruence on $\mathcal{A}$ and $a \,\theta\, b$, then $aw \,\theta\, bw$ for all

$w \in X^*$. For any congruence θ of $\mathcal{A}$ the *quotient algebra* $\mathcal{A}/\theta = (A/\theta, X)$ is defined on the set $A/\theta = \{a/\theta \mid a \in A\}$, where $a/\theta = \{b \in A \,|\, a\,\theta\,b\}$, so that $(a/\theta)x^{\mathcal{A}/\theta} = (ax^{\mathcal{A}})/\theta$ for all $a/\theta \in A/\theta$ and $x \in X$.

A *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $\phi : A \to B$ such that $(ax^{\mathcal{A}})\phi = (a\phi)x^{\mathcal{B}}$ for all $a \in A$ and $x \in X$, and we write then $\phi : \mathcal{A} \to \mathcal{B}$. An *epimorphism* is a surjective homomorphism and an *isomorphism* is a bijective homomorphism. If there exists an epimorphism $\phi : \mathcal{A} \to \mathcal{B}$, then $\mathcal{B}$ is a *homomorphic image* of $\mathcal{A}$, and $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic*, $\mathcal{A} \cong \mathcal{B}$ in symbols, if there is an isomorphism $\phi : \mathcal{A} \to \mathcal{B}$.

The *Rees congruence* $\varrho_{\mathcal{B}}$ on an X-algebra $\mathcal{A}$ modulo a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is defined so that $(a, b) \in \varrho_{\mathcal{B}}$ if and only if $a = b$ or $a, b \in B$. The corresponding *Rees quotient* $\mathcal{A}/\varrho_{\mathcal{B}}$ is denoted by $\mathcal{A}/\mathcal{B}$, and the X-algebra $\mathcal{A}$ is said to be an *extension* of $\mathcal{B}$ by an X-algebra $\mathcal{C}$, if $\mathcal{A}/\mathcal{B} \cong \mathcal{C}$.

We identified words over X with terms of the form εw in one variable. More generally, we define *X-terms* over any set G of *variables* as expressions of the form gw, where $g \in G$ and $w \in X^*$, and we denote by GX^* the set of all such terms. The letters s and t, possibly with subscripts, will denote terms without indicating the variable. The *term X-algebra* $\mathcal{T}(G) = (GX^*, X)$ over G is defined so that $(gu)x^{\mathcal{T}(G)} = gux$ for all $gu \in GX^*$ and $x \in X$ (see §1.6 of [8]). An *identity* of type X over G is an expression $gu = hv$, where $gu, hv \in GX^*$. An X-algebra $\mathcal{A}$ is said to *satisfy* an identity $gu = hv$, and we write $\mathcal{A} \models gu = hv$, if $\alpha(g)u^{\mathcal{A}} = \alpha(h)v^{\mathcal{A}}$ for all valuations $\alpha : G \to A$ of the variables. An X-algebra $\mathcal{A}$ *satisfies monogenically* an identity $gu = hv$, which we express by writing $\mathcal{A} \models_M gu = hv$, if $\langle a \rangle \models gu = hv$ for every $a \in A$. Since at most two variables appear in any identity, we fix a two-element set $G = \{g, h\}$ of variables. The term X-algebra $\mathcal{T}(\{g, h\})$ is denoted simply by $\mathcal{T}$, and the set of all identities of type X over G by Id.

An identity of the form $iu = iv$, where the same variable $i \in G$ appears on both sides, is called *regular*, while identities $iu = jv$, where i and j are different variables are *irregular*. For any set Σ of identities, the sets of regular and irregular members of Σ are denoted by Σ_R and Σ_N, respectively. The sets of all identities, all regular identities and all irregular identities satisfied by all the members of a class $\mathbf{K}$ of X-algebras, are denoted by $\mathrm{Id}\,(\mathbf{K})$, $\mathrm{Id}_R\,(\mathbf{K})$ and $\mathrm{Id}_N\,(\mathbf{K})$, respectively. For $\mathbf{K} = \{\mathcal{A}\}$, we write simply $\mathrm{Id}\,(\mathcal{A})$, $\mathrm{Id}_R\,(\mathcal{A})$ and $\mathrm{Id}_N\,(\mathcal{A})$.

The class $[\Sigma]$ of all X-algebras satisfying every member of a given set Σ of identities, is called the *variety* defined by Σ. It is well-known that a class of X-algebras is a variety if and only if it is closed under forming subalgebras, homomorphic images and direct products. A variety of X-algebras definable by a set of regular identities is *regular*, but otherwise it is called *irregular*. As shown by Płonka [15], every variety $\mathbf{V}$ is contained in a least regular variety $R(\mathbf{V})$, the *regularization* of $\mathbf{V}$, and this is defined by the set of identities $\mathrm{Id}_R\,(\mathbf{V})$.

A binary relation on a set I is a *quasi-order* on I if it is reflexive and transitive, and a *quasi-ordered set* is a pair $(I, \preceq)$ consisting of a set I and a quasi-order $\preceq$ on I. A quasi-order $\preceq$ on I is *directed* if for all $i, j \in I$ there exists a $k \in I$ such that $i \preceq k$ and $j \preceq k$. A non-empty subset F of a quasi-ordered set $(I, \preceq)$ is a *filter* if for all $i, j \in I$, $i \preceq j$ and $i \in F$ imply $j \in F$. A subset C of a quasi-ordered set $(I, \preceq)$ is *cofinal* if for every $i \in I$ there is a $j \in C$ such that $i \preceq j$.

A set Σ of identities presented in the form $\{s_i = t_i\}_{i \in I}$, where $(I, \preceq)$ is a directed quasi-ordered set, is called a *directed set of identities*. In this case an X-algebra $\mathcal{A}$ is said to *ultimately satisfy* Σ, in symbols $\mathcal{A} \models_u \Sigma$, if there exists a $k \in I$ such that $\mathcal{A} \models s_i = t_i$ for every $i \succeq k$. The class of all X-algebras ultimately satisfying Σ is denoted by $[\Sigma]_u$ or $[s_i = t_i \,|\, i \in I]_u$. If $\mathbf{K}$ is a class of X-algebras such that $\mathbf{K} = [\Sigma]_\mathbf{u}$ for a directed set of identities Σ, then $\mathbf{K}$ is said to be *ultimately defined* by Σ. In particular, a set of identities indexed by the natural numbers with the usual ordering is a *sequence of identities*.

A class of X-algebras is a *generalized variety* if it is closed under subalgebras, homomorphic images, finite direct products and arbitrary direct powers. As proved by Ash [1], a class of X-algebras is a generalized variety if and only if it is ultimately defined by a directed set of identities, or equivalently, if it is a directed union of varieties.

A *pseudovariety* of X-algebras is a class of finite X-algebras closed under subalgebras, homomorphic images and finite direct products. By the well-known result of Eilenberg and Schützenberger [7] (see also [6], [14], [11]), for any finite X, a class $\mathbf{K}$ of finite X-algebras is a pseudovariety if and only if it is ultimately defined by a sequence of identities, whereas Ash [1] proved that $\mathbf{K}$ is a pseudovariety if and only if it is the class of all finite X-algebras of some generalized variety.

If $\mathbf{K}$ is a class of X-algebras, $\underline{\mathbf{K}}$ denotes the class of all finite members of $\mathbf{K}$. For varieties $\mathbf{V_1}$ and $\mathbf{V_2}$, let $\mathbf{V_1} \vee \mathbf{V_2}$ be the least variety containing both of them, and if $\mathbf{V_1}$ and $\mathbf{V_2}$ are pseudovarieties, then the same notation is used for the least pseudovariety containing both of them.

An X-algebra $\mathcal{A}$ is the *direct sum* of some of its subalgebras $\mathcal{A}_\alpha$, $\alpha \in Y$, if $A = \bigcup_{\alpha \in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \varnothing$ for all $\alpha, \beta \in Y$ such that $\alpha \neq \beta$. The partition of A into the *direct summands* $\mathcal{A}_\alpha$ is a *direct sum decomposition* of $\mathcal{A}$, and the corresponding equivalence relation is a congruence on $\mathcal{A}$ called a *direct sum congruence*. The *greatest direct sum decomposition* of $\mathcal{A}$ is the decomposition corresponding to the least direct sum congruence $\sigma_\mathcal{A}$ on $\mathcal{A}$. An X-algebra $\mathcal{A}$ is *direct sum indecomposable* if the universal relation ∇_A is the only direct sum congruence on $\mathcal{A}$. More on direct sum decompositions can be found in [5].

The class of all one-element X-algebras is denoted by $\mathbf{O}$. Direct sums of trivial X-algebras are *discrete X-algebras*, and the class of all discrete X-algebras is denoted by $\mathbf{D}$. In particular, the two-element discrete X-algebra is denoted by $\mathcal{D}_2$.

The *Mal'cev product* $\mathbf{K_1} \circ \mathbf{K_2}$ of two classes $\mathbf{K_1}$ and $\mathbf{K_2}$ of X-algebras consists of all X-algebras $\mathcal{A}$ which have a congruence ϱ such that $\mathcal{A}/\varrho \in \mathbf{K}_2$ and every ϱ-class which is a subalgebra of $\mathcal{A}$ belongs to $\mathbf{K_1}$. For example, $\mathbf{K} \circ \mathbf{D}$ is the class of all direct sums of X-algebras from $\mathbf{K}$.

3 Local and Monogenic Properties of Algebras

Let $\mathbf{K}$ be a class of X-algebras. An X-algebra $\mathcal{A}$ is said to *belong locally* to $\mathbf{K}$ if every finitely generated subalgebra of $\mathcal{A}$ belongs to $\mathbf{K}$. The class of all such X-algebras is denoted by $L(\mathbf{K})$. On the other hand, $\mathcal{A}$ *belongs monogenically* to $\mathbf{K}$, if all of its monogenic subalgebras are in $\mathbf{K}$. The class of all X-algebras belonging monogenically to $\mathbf{K}$ is denoted by $M(\mathbf{K})$. We call $L(\mathbf{K})$ and $M(\mathbf{K})$, respectively, the *local closure* and the *monogenic closure* of $\mathbf{K}$.

Let us note some general properties of the operators L and M. All of them follow directly from the definitions.

Lemma 3.1 *Let $\mathbf{K}$, $\mathbf{K_1}$, $\mathbf{K_2}$ and $\mathbf{K_i}$, $i \in I$, be arbitrary classes of X-algebras and let O denote either one of the operators L and M. Then*

(a) *$O(\mathbf{K})$ is closed under subalgebras;*
(b) $\mathbf{K_1} \subseteq \mathbf{K_2} \;\Rightarrow\; \mathbf{O}(\mathbf{K_1}) \subseteq \mathbf{O}(\mathbf{K_2})$;
(c) $O\left(\bigcap_{i\in I} \mathbf{K_i}\right) = \bigcap_{i\in I} O(\mathbf{K_i})$;
(d) *If $\mathbf{K}$ is closed under subalgebras, then $\mathbf{K} \subseteq \mathbf{O}(\mathbf{K})$;*
(e) $O(\mathbf{K}) = \mathbf{O^2}(\mathbf{K})$;
(f) *If $\mathbf{K}$ is closed under homomorphic images, then $O(\mathbf{K})$ is closed under homomorphic images;*
(g) *If $\mathbf{K}$ is closed under subalgebras and (finite) direct products, then $O(\mathbf{K})$ is closed under (finite) direct products.*

4 Monogenically Connected Algebras

In this section we consider the monogenic closure of a special class of X-algebras which will play a role also in the general theory.

An X-algebra $\mathcal{A}$ is *connected* if $\langle a\rangle \cap \langle b\rangle \neq \varnothing$ for all $a, b \in A$. Let **Conn** denote the class of all connected X-algebras. If every monogenic subalgebra of $\mathcal{A}$ is connected, $\mathcal{A}$ is said to be *monogenically connected*. In this section we characterize $M(\mathbf{Conn})$ as well as the monogenic closures of two subclasses of **Conn**.

A nonempty subset D of an X-algebra $\mathcal{A}$ is called a *dual subalgebra* of $\mathcal{A}$ if for any $a \in A$ and $x \in X$, $ax \in D$ implies $a \in D$ (cf. [5]). For a nonempty $H \subseteq A$, $\langle H\rangle^d = \{a \in A \mid (\exists u \in X^*)\, au \in H\}$ is the least dual subalgebra of $\mathcal{A}$ containing H.

Theorem 4.1 *The following conditions on an X-algebra $\mathcal{A}$ are equivalent:*

(1) *$\mathcal{A}$ is monogenically connected;*
(2) $(\forall a \in A)(\forall p, q \in X^*)(\exists u, v \in X^*)\; apu = aqv$;

(3) $\mathcal{A}$ *is a direct sum of connected* X*-algebras;*
(4) $\langle B\rangle^d \in \mathrm{Sub}\mathcal{A}$ *for every* $B \in \mathrm{Sub}\mathcal{A}$.

Proof. The equivalence of (1) and (2) is evident.

(2)⇒(4). Let $B \in \mathrm{Sub}\mathcal{A}$, $a \in \langle B\rangle^d$ and $q \in X^*$. Then $ap \in B$ for some $p \in X^*$, and by (2) it follows that $apu = aqv$ for some $u, v \in X^*$. But, $ap \in B$ implies $aqv = apu \in B$, which means that $aq \in \langle B\rangle^d$. Therefore, $\langle B\rangle^d$ is a subalgebra of $\mathcal{A}$.

(4)⇒(3). As shown in [5], $\mathcal{A}$ is the direct sum of some direct sum indecomposable X-algebras $\mathcal{A}_\alpha$, $\alpha \in Y$. Let us consider any $\alpha \in Y$ and $a \in A_\alpha$. According to Theorem 3.6 of [5], A_α is the least subset of $\mathcal{A}$ containing a which is both a subalgebra and a dual subalgebra. Since $\langle\langle a\rangle\rangle^d$ also is both a subalgebra and a dual subalgebra containing a, we must have $A_\alpha = \langle\langle a\rangle\rangle^d$. Now, $b \in \langle\langle a\rangle\rangle^d$ for any $b \in A_\alpha$, so $bv \in \langle a\rangle$, and hence $bv = au$ for some $u, v \in X^*$. Hence, $\mathcal{A}_\alpha$ is connected.

(3)⇒(1). Let $\mathcal{A}$ be the direct sum of some connected X-algebras $\mathcal{A}_\alpha$, $\alpha \in Y$. For any $a \in A$ we have that $a \in A_\alpha$ for some $\alpha \in Y$, and then $\langle a\rangle \subseteq A_\alpha$. Since $\mathcal{A}_\alpha$ is connected and the class of connected X-algebras is closed under subalgebras, $\langle a\rangle$ is also connected. □

An X-algebra $\mathcal{A}$ is said to be *trap-connected* if it has a trap a_0 such that $a_0 \in \langle a\rangle$ for every $a \in A$, and it is *monogenically trap-connected* if all of its monogenic subalgebras are trap-connected.

Theorem 4.2 *The following conditions on an* X*-algebra* $\mathcal{A}$ *are equivalent:*
(1) $\mathcal{A}$ *is monogenically trap-connected;*
(2) $(\forall a \in A)(\forall p, q \in X^*)(\exists u, v \in X^*)(\forall w \in X^*)\ apu = aqvw$*;*
(3) $\mathcal{A}$ *is a direct sum of trap-connected* X*-algebras;*
(4) $\mathcal{A}$ *is a subdirect product of a discrete* X*-algebra and a trap-connected* X*-algebra;*
(5) $\mathcal{A}$ *is a subalgebra of the direct product of a discrete* X*-algebra and a trap-connected* X*-algebra.*

Proof. (1)⇒(2). Assume that $\mathcal{A}$ is monogenically trap-connected and consider any $a \in A$ and $p, q \in X^*$. Since $\langle a\rangle$ is trap-connected, there are words $u, v \in X^*$ such that $apu = aqv = a_0$, where a_0 is the trap of $\langle a\rangle$. Obviously, $aqvw = a_0 = apu$ for every $w \in X^*$, and hence (2) holds.

(2)⇒(1). According to Theorem 4.1, $\mathcal{A}$ is monogenically connected. Hence $\langle a\rangle$ is connected for any given $a \in A$, and by (2) there exist $u, v \in X^*$ such that $au = avw$, for every $w \in X^*$. This implies that au is a trap in $\langle a\rangle$ as $aux = avxx = au$ for all $x \in X$.

(1)⇒(3). By Theorem 4.1, $\mathcal{A}$ is the direct sum of some connected X-algebras $\mathcal{A}_\alpha$, $\alpha \in Y$. Moreover, every $\mathcal{A}_\alpha$ has a trap since any $\langle a\rangle$ with $a \in A_\alpha$ has a trap.

(3)⇒(4). Let $\mathcal{A}$ be the direct sum of some trap-connected X-algebras $\mathcal{A}_\alpha$, $\alpha \in Y$. For each $\alpha \in Y$, let a_α be the trap of $\mathcal{A}_\alpha$. Then $B = \{a_\alpha \mid \alpha \in$

$Y\}$ defines a subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{B}$ is trap-connected. If σ is the congruence on $\mathcal{A}$ whose classes are the subalgebras A_α, $\alpha \in Y$, then $\mathcal{A}/\sigma$ is a discrete X-algebra isomorphic to $\mathcal{B}$. Moreover, $\mathcal{A}$ is a subdirect product of $\mathcal{A}/\mathcal{B}$ and $\mathcal{A}/\sigma$ since evidently $\varrho_{\mathcal{B}} \cap \sigma = \Delta_A$. Thus, (4) holds.

(4)⇒(5). This implication is trivial.

(5)⇒(1). Let $\mathcal{A}$ be a subalgebra of the direct product of a discrete X-algebra $\mathcal{B}$ and a trap-connected X-algebra $\mathcal{C}$. For any $(b,c) \in A$, we may write $\langle (b,c)\rangle = \{b\} \times \langle c\rangle$, and hence $\langle (b,c)\rangle$ is isomorphic to $\langle c\rangle$. On the other hand, $\langle c\rangle$ is trap-connected because $\mathcal{C}$ is trap-connected, and therefore $\langle (b,c)\rangle$ is trap-connected and (1) holds. □

We conclude this section by formulating a result by Thierrin [20] in our terminology. An X-algebra $\mathcal{A}$ is *strongly connected* if $\langle a\rangle = A$ for every $a \in A$, and *monogenically strongly connected* if all of its monogenic subalgebras are strongly connected.

Theorem 4.3 *The following conditions on an X-algebra $\mathcal{A}$ are equivalent:*
(1) *$\mathcal{A}$ is monogenically strongly connected;*
(2) $(\forall a \in A)(\forall u \in X^*)(\exists v \in X^*)\ auv = a;$
(3) *$\mathcal{A}$ is a direct sum of strongly connected X-algebras.*

Various other characterizations of monogenically strongly connected X-algebras can be found in [9], [5] and [13].

5 Monogenic Closures of Varieties

In this section we shall consider the effect of the operator M on a variety of X-algebras. In particular, it is shown that M acts differently on regular and irregular varieties. Moreover, it is proved that for varieties M coincides with the regularization operator.

Theorem 5.4 *The operators M and L are closure operators on the lattice of varieties of X-algebras, and $L(\mathbf{V}) = \mathbf{V} \subseteq \mathbf{M}(\mathbf{V})$ for any variety $\mathbf{V}$ of X-algebras.*

Proof. That L and M have the three defining properties of a closure operator follows from statements (b), (d) and (e) of Lemma 3.1. Moreover, if $\mathbf{V}$ is a variety then $L(\mathbf{V})$ and $M(\mathbf{V})$ are varieties such that $\mathbf{V} \subseteq \mathbf{L}(\mathbf{V}), \mathbf{M}(\mathbf{V})$ by statements (a), (f), (g) and (d) of Lemma 3.1. The inclusion $L(\mathbf{V}) \subseteq \mathbf{V}$ follows from the obvious fact that any algebra in $L(\mathbf{V})$ satisfies the identities defining $\mathbf{V}$. □

Note that $M(\mathbf{V}) = \mathbf{V}$ does not always hold. For example, $M(\mathbf{O}) = \mathbf{D}$. The following observation is often useful.

Lemma 5.2 *An X-algebra $\mathcal{A}$ satisfies an irregular identity $gu = hv$ if and only if $u, v \in DW(\mathcal{A})$ and $d_u = d_v$.*

Proof. If $\mathcal{A} \models gu = hv$, then $\mathcal{A}$ satisfies also the identities $hu = hv$ and $gu = gv$. Therefore $\mathcal{A} \models gu = hu$ and $\mathcal{A} \models gv = hv$. Hence $u, v \in DW(\mathcal{A})$, and as $\mathcal{A} \models gu = hv$ means that $au = bv$ for all $a, b \in A$, we also have $d_u = d_v$.

Conversely, let $u, v \in DW(\mathcal{A})$ and $d_u = d_v$. Since $au = d_u$ and $bv = d_v$ for all $a, b \in A$, we may conclude that $\mathcal{A} \models gu = hv$. □

For any X-algebra $\mathcal{A}$ and any $u, v \in X^*$, the relation $\varrho^{\mathcal{A}}_{u,v}$ on A is defined so that

$$(a, b) \in \varrho^{\mathcal{A}}_{u,v} \;\Leftrightarrow\; au = bv.$$

We shall use the simpler notation $\varrho_{u,v}$ when $\mathcal{A}$ is understood from the context. Moreover, we call a relation $\rho(\subseteq A \times A)$ *positive* if $(a, ax) \in \rho$ for all $a \in A$ and $x \in X$.

Lemma 5.3 *Let $\mathcal{A}$ be an X-algebra and let $u, v \in X^*$.*

(a) $\mathcal{A} \models gu = gv$ *if and only if* $\varrho_{u,v}$ *is a reflexive relation. In that case* $\varrho_{u,v}$ *is an equivalence on* A.
(b) $\mathcal{A} \models_M gu = gv$ *if and only if* $\mathcal{A} \models gu = gv$.
(c) $\mathcal{A} \models gu = hv$ *if and only if* $\varrho_{u,v} = \nabla_A$.
(d) $\mathcal{A} \models_M gu = hv$ *if and only if* $\varrho_{u,v}$ *is reflexive and positive. In that case* $\varrho_{u,v} = \sigma_{\mathcal{A}}$, *where* $\sigma_{\mathcal{A}}$ *is the least direct sum congruence on* $\mathcal{A}$.

Proof. The first assertion in (a) is evident, so we prove only the second one.

Let $(a, b) \in \varrho_{u,v}$, i.e., $au = bv$. Since $\mathcal{A} \models gu = gv$ implies $au = av$ and $bu = bv$, we get $bu = av$ and $(b, a) \in \varrho_{u,v}$. Hence $\varrho_{u,v}$ is symmetric. Similarly, if $(a, b), (b, c) \in \varrho_{u,v}$, then $au = bv$ and $bu = cv$ together with $bu = bv$, which follows from $\mathcal{A} \models gu = gv$, yield $au = cv$, i.e., $(a, c) \in \varrho_{u,v}$. Therefore, $\varrho_{u,v}$ is also transitive and hence an equivalence relation on A.

Assertions (b) and (c) are obvious. It remains to prove (d). If $\mathcal{A} \models_M gu = hv$, then $\mathcal{A} \models gu = gv$, and by (a) the relation $\varrho_{u,v}$ is reflexive. It is evidently also positive. By (a) we have that $\varrho_{u,v}$ is an equivalence relation on A, and by Lemma 3.1 of [5], $\varrho_{u,v}$ is a direct sum congruence on $\mathcal{A}$. But, by Theorem 3.1 of [5], $\varrho_{u,v} \subseteq \sigma_{\mathcal{A}}$ and $\sigma_{\mathcal{A}}$ is the least direct sum congruence on $\mathcal{A}$, so we conclude that $\varrho_{u,v} = \sigma_{\mathcal{A}}$. Conversely, let $\varrho_{u,v}$ be reflexive and positive. To prove that $\mathcal{A} \models_M gu = hv$, consider any $a \in A$ and $b, c \in \langle a \rangle$. Again by Theorem 3.1 of [5] it follows that $(b, c) \in \sigma_{\mathcal{A}} = \varrho_{u,v}$, i.e., $bu = cv$. Therefore, $\mathcal{A} \models_M gu = hv$ and the proof is complete. □

Using the previous result we can prove the following theorem characterizing regular varieties of X-algebras.

Theorem 5.5 *The following conditions on a variety* $\mathbf{V}$ *of* X*-algebras are equivalent:*

(1) $\mathbf{V}$ *is a regular variety;*
(2) $M(\mathbf{V}) = \mathbf{V}$*;*
(3) $\mathbf{D} \subseteq \mathbf{V}$*;*
(4) $\mathcal{D}_2 \in \mathbf{V}$.

Proof. (1)⇒(2). Let $\mathbf{V}$ be a regular variety. By Lemma 5.3 (b), for any X-algebra $\mathcal{A}$, $\mathcal{A} \models_M \mathrm{Id}_R(\mathbf{V})$ if and only if $\mathcal{A} \models \mathrm{Id}_R(\mathbf{V})$. Since $\mathrm{Id}_R(\mathbf{V}) = \mathrm{Id}(\mathbf{V})$, we may conclude that $M(\mathbf{V}) = \mathbf{V}$.

(2)⇒(3). For every discrete X-algebra $\mathcal{D}$ and every variety $\mathbf{V}$ we have that $\mathcal{D} \in M(\mathbf{O}) \subseteq \mathbf{M}(\mathbf{V})$, so (2) yields $\mathcal{D} \in \mathbf{V}$. This proves (3).

The implications (3)⇒(4) and (4)⇒(1) are completely obvious. □

The previous theorem, Lemma 5.2, and the fact that $\mathbf{D}$ is a variety generated by any of its non-trivial members, yield the following characterization of irregular varieties of X-algebras.

Corollary 5.1 *The following conditions on a variety* $\mathbf{V}$ *of* X*-algebras are equivalent:*

(1) $\mathbf{V}$ *is an irregular variety;*
(2) $\mathbf{D} \not\subseteq \mathbf{V}$*;*
(3) $\mathcal{D}_2 \notin \mathbf{V}$*;*
(4) $\mathbf{D} \cap \mathbf{V} = \mathbf{O}$*;*
(5) $\mathbf{V} \subseteq \mathbf{Dir}$.

Next we present a connection between the monogenic closure operator and the regularization of varieties, as well as some other facts.

Theorem 5.6 *For any variety* $\mathbf{V}$ *of* X*-algebras,*

$$M(\mathbf{V}) = \mathbf{R}(\mathbf{V}) = \mathbf{V} \circ \mathbf{D} = \mathbf{D} \vee \mathbf{V}.$$

Proof. The equalities $R(\mathbf{V}) = \mathbf{V} \circ \mathbf{D} = \mathbf{V} \vee \mathbf{D}$ are due to Płonka [?, ?], but we give a new proof.

If $\mathbf{V}$ is a regular variety, then $R(\mathbf{V}) = \mathbf{V}$ and, by Theorem 5.5, $\mathbf{D} \vee \mathbf{V} = \mathbf{V} = \mathbf{M}(\mathbf{V})$. Evidently, $\mathbf{V} \subseteq \mathbf{V} \circ \mathbf{D}$. On the other hand, a member of $\mathbf{V} \circ \mathbf{D}$ satisfies every regular identity holding in $\mathbf{V}$, and so $\mathbf{V} \circ \mathbf{D} \subseteq \mathbf{V}$ as $\mathbf{V}$ is regular. Hence, $\mathbf{V} = \mathbf{V} \circ \mathbf{D}$.

Let $\mathbf{V}$ now be irregular. Any $\mathcal{A} \in M(\mathbf{V})$ is by Theorem 3.6 of [5] the direct sum of some direct sum indecomposable X-algebras $\mathcal{A}_\alpha$, $\alpha \in Y$. For the corresponding direct sum congruence σ it follows from Lemma 5.3 (d) that $\sigma = \varrho_{u,v}$ for each pair $(u, v) \in X^* \times X^*$ such that $gu = hv \in \mathrm{Id}_N(\mathbf{V})$. From this it follows that $\mathcal{A}_\alpha \models \mathrm{Id}_N(\mathbf{V})$ for each $\alpha \in Y$. On the other hand, by Lemma 5.3 (b), $\mathcal{A}_\alpha \models \mathrm{Id}_R(\mathbf{V})$ for every $\alpha \in Y$. Therefore, $\mathcal{A}_\alpha \models \mathrm{Id}(\mathbf{V})$, that is to say, $\mathcal{A}_\alpha \in \mathbf{V}$ for each $\alpha \in Y$, and hence $\mathcal{A} \in \mathbf{V} \circ \mathbf{D}$. This shows that $M(\mathbf{V}) \subseteq \mathbf{V} \circ \mathbf{D}$.

If $\mathcal{A} \in \mathbf{V} \circ \mathbf{D}$, then $\mathcal{A}$ is the direct sum of some X-algebras $\mathcal{A}_\alpha \in \mathbf{V}$, $\alpha \in Y$. Since every monogenic subalgebra of $\mathcal{A}$ is a subalgebra of some $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha \in \mathbf{V}$, every monogenic subalgebra of $\mathcal{A}$ is in $\mathbf{V}$, i.e., $\mathcal{A} \in M(\mathbf{V})$. Hence, $\mathbf{V} \circ \mathbf{D} \subseteq \mathbf{M}(\mathbf{V})$.

If $\mathcal{A} \in M(\mathbf{V})$, then $\mathcal{A} \models_M \mathrm{Id}(\mathbf{V})$, and therefore $\mathcal{A} \models \mathrm{Id}_R(\mathbf{V})$ by Lemma 5.3 (b). Hence $\mathcal{A} \in R(\mathbf{V})$. On the other hand, by the obvious fact that

$\mathcal{D}_2 \in M(\mathbf{V})$ and Theorem 5.5, $M(\mathbf{V})$ is a regular variety containing $\mathbf{V}$, so also $R(\mathbf{V}) \subseteq \mathbf{M}(\mathbf{V})$ holds.

Furthermore, $\mathbf{V} \subseteq \mathbf{R}(\mathbf{V})$ and $\mathbf{D} \subseteq \mathbf{R}(\mathbf{V})$ by Theorem 5.5, so $\mathbf{D} \vee \mathbf{V} \subseteq \mathbf{R}(\mathbf{V})$. Conversely, $\mathbf{D} \vee \mathbf{V}$ is a regular variety by Theorem 5.5 and it contains $\mathbf{V}$, so $R(\mathbf{V}) \subseteq \mathbf{D} \vee \mathbf{V}$. This completes the proof of the theorem. □

Now we use an idea by Płonka [16] to describe a set of identities defining the monogenic closure of an irregular variety.

Theorem 5.7 *Let* $\mathbf{V}$ *be an irregular variety of* X*-algebras defined by a set of identities* Σ*, and let* $gu = hv$ *be an arbitrary fixed irregular identity in* Σ*. Then* $M(\mathbf{V}) = [\Sigma']$ *for* $\Sigma' = \Sigma_R \cup \Gamma \cup \Lambda$*, where*

$$\Gamma = \{gu = gv\} \cup \{gxu = gv \,|\, x \in X\}$$

and

$$\Lambda = \{gu' = guu', gu' = gv', gv' = gvv' \,|\, gu' = hv' \in \Sigma_N \setminus \{gu = hv\}\}.$$

Proof. Let $\mathcal{A}$ be an arbitrary X-algebra. We have to prove that $\mathcal{A} \models_M \Sigma$ if and only if $\mathcal{A} \models \Sigma'$.

Let $\mathcal{A} \models_M \Sigma$. Then $\mathcal{A} \models \Sigma_R$ by Lemma 5.3 (b). Clearly, $\mathcal{A} \models gu = gv$ and $\mathcal{A} \models gxu = gv$ for each $x \in X$. Consider any identity $gu' = hv' \in \Sigma_N \setminus \{gu = hv\}$. It is obvious that $\mathcal{A} \models_M gu' = hv'$ implies $\mathcal{A} \models_M gu' = hu'$ and $\mathcal{A} \models_M gv' = hv'$. But this yields $\mathcal{A} \models gu' = guu'$ and $\mathcal{A} \models gv' = gvv'$. Moreover, $\mathcal{A} \models_M gu' = hv'$ implies $\mathcal{A} \models gu' = gv'$. Therefore, $\mathcal{A} \models \Sigma'$.

Conversely, if $\mathcal{A} \models \Sigma'$, then $\mathcal{A} \models \Sigma_R$ implies $\mathcal{A} \models_M \Sigma_R$, and it remains to prove that $\mathcal{A} \models_M \Sigma_N$. By $\mathcal{A} \models gu = gv$ and Lemma 5.3 (a), $\varrho_{u,v}$ is an equivalence on A, and the fact that $\mathcal{A} \models gxu = gv$ for each $x \in X$, implies by Lemma 3.1 of [5] that $\varrho_{u,v}$ is a direct sum congruence on $\mathcal{A}$. Now, by Lemma 5.3 (d) we have that $\mathcal{A} \models_M gu = hv$. Consider any identity $gu' = hv' \in \Sigma_N \setminus \{gu = hv\}$ and any $a \in A$, $b, c \in \langle a \rangle$. As we have proved, $\mathcal{A} \models_M gu = hv$, so $bu = cv$, and by $\mathcal{A} \models gu' = guu'$, $\mathcal{A} \models gu' = gv'$ and $\mathcal{A} \models gv' = gvv'$ it follows that $bu' = buu' = cvu' = cvv' = cv'$. Therefore, $\mathcal{A} \models_M gu' = hv'$, which finally gives $\mathcal{A} \models_M \Sigma$. □

The following corollary expresses in our terminology a result by Płonka [16], proved in a different way also by Graczyńska [10]. Let us first note that if $\mathbf{V}$ is an irregular variety, then by Lemma 5.2 there are words which are directing words of all members of $\mathbf{V}$; we call such words *directing words* of $\mathbf{V}$.

Corollary 5.2 *Let* $\mathbf{V}$ *be an irregular variety of* X*-algebras defined by a set of identities* Σ*, and let* w *be any directing word of* $\mathbf{V}$*. Then* $M(\mathbf{V}) = [\Sigma'']$ *for* $\Sigma'' = \Sigma_R \cup \Gamma_w \cup \Lambda_w$*, where*

$$\Gamma_w = \{gxw = gw \,|\, x \in X\}$$

and

$$\Lambda_w = \{gu = gwu, gu = gv, gv = gwv \,|\, gu = hv \in \Sigma_N \setminus \{gw = hw\}\}.$$

Example 5.1 It is obvious that $\mathbf{Dir_u} = [\{\mathbf{gu} = \mathbf{hu}\}]$ for any $u \in X^*$. Now the sets of identities defined in Theorem 5.7 are $\Gamma = \{gxu = gu \,|\, x \in X\}$, omitting the trivial identity $gu = gu$, and $\Sigma_R = \Lambda = \varnothing$, and so $M(\mathbf{Dir_u}) = [\{\mathbf{gxu} = \mathbf{gu} \,|\, \mathbf{x} \in \mathbf{X}\}]$.

It is not hard to see that $\mathbf{Nilp_n} = [\{\mathbf{gu} = \mathbf{hux} \,|\, \mathbf{u} \in \mathbf{X^n}, \mathbf{x} \in \mathbf{X}\}]$ and $\mathbf{RDef_n} = [\{\mathbf{gu} = \mathbf{gux} \,|\, \mathbf{u} \in \mathbf{X^n}, \mathbf{x} \in \mathbf{X}\}]$. According to Theorem 5.5 the variety $\mathbf{RDef_n}$ is M-closed. For $M(\mathbf{Nilp_n})$ Theorem 5.7 gives the set of identities $\Gamma \cup \Lambda$, where $\Gamma = \{gv = gvy\} \cup \{gxv = gvy \,|\, x \in X\}$ and $\Lambda = \{gu = gvu, gu = gux, gux = gvyux \,|\, u \in X^n, x \in X, (u,x) \neq (v,y)\}$ for any fixed $v \in X^n$, $y \in X$. Combining identities in Γ we get easily $gv = gxv$. Now by replacing g with gx in $gu = gvu$ and using the obtained identities, we get $gxu = gxvu = gvu = gu$. So, $\Gamma \cup \Lambda$ implies $\{gu = gux, gxu = gu \,|\, u \in X^n, x \in X\}$. The converse is obvious and hence $M(\mathbf{Nilp_n}) = [\{\mathbf{gux} = \mathbf{gu}, \mathbf{gu} = \mathbf{gxu} \,|\, \mathbf{u} \in \mathbf{X^n}, \mathbf{x} \in \mathbf{X}\}]$.

6 Generalized Varieties and the Local Closure Operators

Now we shall apply the operators L and M to generalized varieties of X-algebras. It turns out that they act quite differently on these than on varieties of X-algebras. In particular, they are not closure operators in this case. As one may expect, the effects of these operators on a generalized variety are also different depending on whether the generalized variety is regular or not.

The following lemma is quite obvious.

Lemma 6.4 *If $\{s_i = t_i\}_{i \in I}$ is a directed set of identities, then*

$$[s_i = t_i \,|\, i \in I]_u = [s_i = t_i \,|\, i \in F]_u$$

for each filter F of $(I, \preceq)$.

First we note an important negative fact about the operators L and M.

Theorem 6.8 *The operators L and M are not closure operators on the lattice of generalized varieties of X-algebras.*

Proof. We prove that the classes $M(\mathbf{G})$ and $L(\mathbf{G})$ are not always generalized varieties even when $\mathbf{G}$ is a generalized variety. For this we consider the generalized variety $\mathbf{Dir}$ of directable X-algebras.

For every $k \in \mathbb{N}$ we introduce the X-algebra $\mathcal{A}_k = (A_k, X)$, where $A_k = \{a_0^{(k)}, a_1^{(k)}, \ldots, a_k^{(k)}\}$ and the operations are given by $a_0^{(k)}x = a_1^{(k)}$, $a_1^{(k)}x = a_2^{(k)}$, $\ldots$, $a_{k-1}^{(k)}x = a_k^{(k)}x = a_k^{(k)}$ for every $x \in X$. Obviously $\mathcal{A}_k \in \mathbf{Nilp_k} \setminus \mathbf{Nilp_{k-1}}$. Let $\mathcal{A}$ be the direct sum of the X-algebras $\mathcal{A}_k$, $k \in \mathbb{N}$. Since every monogenic subalgebra of $\mathcal{A}$ is isomorphic to some $\mathcal{A}_k$ and each $\mathcal{A}_k$ is directable, $\mathcal{A} \in M(\mathbf{Dir})$. However, we are going to prove that $\mathcal{A}^{\mathbb{N}} \notin M(\mathbf{Dir})$.

Consider the element $a = (a_0^{(i)})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ and any word $u \in X^*$. If u is of length n, then in $\mathcal{A}_{n+1}$

$$a_0^{(n+1)}u = a_n^{(n+1)} \neq a_{n+1}^{(n+1)} = a_0^{(n+1)}xu$$

for any $x \in X$, and therefore $au \neq axu$ in $\mathcal{A}^{\mathbb{N}}$. Hence, $DW(\langle a\rangle) = \varnothing$ and $\mathcal{A}^{\mathbb{N}} \notin M(\mathbf{Dir})$. Furthermore, since $L(\mathbf{Dir}) \subseteq \mathbf{M}(\mathbf{Dir})$ it also follows that $\mathcal{A}^{\mathbb{N}} \notin L(\mathbf{Dir})$. Therefore, the classes $M(\mathbf{Dir})$ and $L(\mathbf{Dir})$ are not closed under direct powers, and hence they are not generalized varieties. □

Before continuing the study of the operators L and M, we present some useful characterizations of regular and irregular generalized varieties. A generalized variety $\mathbf{G}$ of X-algebras is *regular* if it is ultimately defined by a directed set of regular identities, otherwise it is *irregular*.

We call a non-empty subset $\mathcal{F}$ of a directed set of identities $\Sigma = \{s_i = t_i \mid i \in I\}$ a *filter* in Σ if there is a filter F of the directed set $(I, \preceq)$ such that $\mathcal{F} = \{s_i = t_i \mid i \in F\}$.

Theorem 6.9 *Let* $\mathbf{G}$ *be a generalized variety of* X*-algebras. Then the following conditions are equivalent:*

(1) $\mathbf{G}$ *is a regular generalized variety;*
(2) $\mathbf{G}$ *is a directed union of regular varieties;*
(3) $\mathbf{G}$ *contains a regular variety;*
(4) $\mathbf{D} \subseteq \mathbf{G}$*;*
(5) $\mathcal{D}_2 \in \mathbf{G}$*;*
(6) *every directed set of identities ultimately defining* $\mathbf{G}$ *contains a regular filter;*
(7) *there exists a filter* $\mathcal{F}$ *of the Boolean lattice* $\wp(\mathrm{Id}_R)$ *such that for any* X*-algebra* $\mathcal{A}$*,*

$$\mathcal{A} \in \mathbf{G} \;\Leftrightarrow\; \mathrm{Id}(\mathcal{A}) \in \mathcal{F}.$$

Proof. (1)⇒(2). Assume that $\mathbf{G}$ is ultimately defined by a directed set of regular identities $\{gu_i = gv_i \mid i \in I\}$. Then $\mathbf{G}$ is the union of the directed family of the regular varieties $\mathbf{V_k} = [\{\mathbf{gu_i} = \mathbf{gv_i} \mid \mathbf{i} \succeq \mathbf{k}\}]$, $k \in I$.

(2)⇒(3). This implication is trivial.

(3)⇒(4). If $\mathbf{V} \subseteq \mathbf{G}$ for a regular variety $\mathbf{V}$, then $\mathbf{D} \subseteq \mathbf{V} \subseteq \mathbf{G}$ by Theorem 5.5.

(4)⇒(5). This is obvious.

(5)⇒(6). Let $\Sigma = \{s_i = t_i \mid i \in I\}$ be any directed set of identities ultimately defining $\mathbf{G}$. Since $\mathcal{D}_2 \in \mathbf{G}$, there is a $k \in I$ such that $\mathcal{D}_2 \models s_i = t_i$ for every $i \succeq k$. As $\mathcal{D}_2$ satisfies regular identities only, $\{s_i = t_i \mid i \succeq k\}$ is a filter of regular identities in Σ.

(6)⇒(1). This implication is an immediate consequence of Lemma 6.4.

(2)⇒(7). Let $\mathbf{G}$ be the directed union of some regular varieties $\{\mathbf{V}_\alpha\}_{\alpha\in\mathbf{Y}}$ and let

$$\mathcal{F} = \Big\{\Sigma \in \wp(\mathrm{Id}_R) \;\Big|\; (\exists\alpha \in Y)\,\mathrm{Id}(\mathbf{V}_\alpha) \subseteq \Sigma\Big\}.$$

Then $\mathcal{F}$ is a filter of the Boolean lattice $\wp(\mathrm{Id}_R)$, and for any X-algebra $\mathcal{A}$

$$\begin{aligned}\mathrm{Id}(\mathcal{A}) \in \mathcal{F} &\Leftrightarrow (\exists \alpha \in Y)\ \mathrm{Id}(\mathbf{V}_\alpha) \subseteq \mathrm{Id}(\mathcal{A}) \\ &\Leftrightarrow (\exists \alpha \in Y)\ \mathcal{A} \in \mathbf{V}_\alpha \\ &\Leftrightarrow \mathcal{A} \in \mathbf{G}.\end{aligned}$$

(7)⇒(5). If $\mathcal{F}$ is any filter of the Boolean lattice $\wp(\mathrm{Id}_R)$ for which (7) holds, then $\mathcal{D}_2 \in \mathbf{G}$ as $\mathrm{Id}(\mathcal{D}_2) = \mathrm{Id}_R \in \mathcal{F}$. □

As a corollary we get the following characterization of irregular varieties of X-algebras.

Corollary 6.3 *Let* $\mathbf{G}$ *be a generalized variety of* X*-algebras. Then the following conditions are equivalent:*
(1) $\mathbf{G}$ *is an irregular generalized variety;*
(2) $\mathbf{G}$ *is a directed union of irregular varieties;*
(2) *every variety contained in* $\mathbf{G}$ *is irregular;*
(4) $\mathbf{D} \cap \mathbf{G} = \mathbf{O}$*;*
(5) $\mathbf{D} \nsubseteq \mathbf{G}$*;*
(6) $\mathcal{D}_2 \notin \mathbf{G}$*;*
(7) *every directed set of identities ultimately defining* $\mathbf{G}$ *has an irregular cofinal subset.*

By using Corollary 5.1 we get also the following conclusion.

Corollary 6.4 *The generalized variety* $\mathbf{Dir}$ *of directable* X*-algebras is the greatest irregular generalized variety of* X*-algebras.*

In the next theorem the operators L and M are applied separately to regular and irregular generalized varieties.

Theorem 6.10 *Let* $\mathbf{G}$ *be a generalized variety of* X*-algebras.*
(a) *If* $\mathbf{G}$ *is irregular, then* $L(\mathbf{G}) = \mathbf{M}(\mathbf{G}) \cap \mathbf{Conn}$.
(b) *If* $\mathbf{G}$ *is regular, then* $L(\mathbf{G}) = \mathbf{M}(\mathbf{G})$.

Proof. (a) If $\mathcal{A} \in L(\mathbf{G})$, then $\langle a, b\rangle \in \mathbf{G} \subseteq \mathbf{Dir} \subseteq \mathbf{Conn}$ for all $a, b \in A$, and hence $\mathcal{A} \in \mathbf{Conn}$. This fact and the obvious inclusion $L(\mathbf{G}) \subseteq \mathbf{M}(\mathbf{G})$ yield $L(\mathbf{G}) \subseteq \mathbf{M}(\mathbf{G}) \cap \mathbf{Conn}$.

Conversely, assume that $\mathcal{A} \in M(\mathbf{G}) \cap \mathbf{Conn}$. Let $\mathcal{B}$ be a finitely generated subalgebra of $\mathcal{A}$, i.e., $B = \langle a_1, \dots, a_n\rangle = \bigcup_{m=1}^{n} \langle a_m\rangle$, for some $n \in \mathbb{N}$ and $a_1, \dots, a_n \in B$. Let $\{s_i = t_i\}_{i\in I}$ be a directed set of identities ultimately defining $\mathbf{G}$. Since $\mathcal{A} \in M(\mathbf{G})$, there exists for each $m \in [1, n]$ a $k_m \in I$ such that $\langle a_m\rangle \models s_i = t_i$ for every $i \succeq k_m$. Since I is directed, there exists a $k \in I$ such that $k \succeq k_m$ for every $m \in [1, n]$. If $i \succeq k$, then $i \succeq k_m$, so $\langle a_m\rangle \models s_i = t_i$ for every $m \in [1, n]$. Let $\Sigma = \{s_i = t_i \mid i \succeq k\}$. If $s_i = t_i \in \Sigma_R$, then clearly $\mathcal{B} \models s_i = t_i$. Let $s_i = t_i \in \Sigma_N$ be an irregular identity of the form $gu_i = hv_i$, where $u_i, v_i \in X^*$. Since

$\langle a_m\rangle \models gu_i = hv_i$, Lemma 5.2 implies that $u_i, v_i \in DW(\langle a_m\rangle)$ for each $m \in [1,n]$. Now, let $b, c \in B$ be arbitrary elements. Then $b = a_l p$ and $c = a_m q$ for some $l, m \in [1,n]$ and $p, q \in X^*$. On the other hand, $\mathcal{A} \in \mathbf{Conn}$, so $a_l u = a_m v$ for some $u, v \in X^*$. Now $u_i \in DW(\langle a_l\rangle)$ and $v_i \in DW(\langle a_m\rangle)$ imply that $a_l p u_i = a_l u u_i$ and $a_m q v_i = a_m v v_i$. On the other hand, $a_l u = a_m v \in \langle a_l\rangle \cap \langle a_m\rangle$ and $\langle a_l\rangle, \langle a_m\rangle \models gu_i = hv_i$ yield $a_l u u_i = a_m v v_i$, and therefore

$$bu_i = a_l p u_i = a_l u u_i = a_m v v_i = a_m q v_i = c v_i,$$

and hence $\mathcal{B} \models gu_i = hv_i$. This shows that $\mathcal{B} \in \mathbf{G}$ as $\mathcal{B} \models s_i = t_i$ for each $i \succeq k$. Finally, this means that $\mathcal{A} \in L(\mathbf{G})$, and thus the proof of (a) has been completed.

(b) The proof of this assertion is contained in the proof of (a). □

Using the relations of the previous theorem, we can prove the following result valid for both regular and irregular generalized varieties.

Theorem 6.11 *Let* $\mathbf{G}$ *be a generalized variety of* X*-algebras. Then*

$$M(\mathbf{G}) = \mathbf{L}(\mathbf{G}) \circ \mathbf{D} = \mathbf{L}(\mathbf{G} \circ \mathbf{D}).$$

Proof. The relation $M(\mathbf{G}) \subseteq \mathbf{M}(\mathbf{G}) \circ \mathbf{D}$ obviously holds, and if $\mathbf{G}$ is regular then $M(\mathbf{G}) \subseteq \mathbf{L}(\mathbf{G}) \circ \mathbf{D}$ by Theorem 6.10. If $\mathbf{G}$ is irregular, then by Corollary 6.4, every $\mathcal{A} \in M(\mathbf{G})$ is monogenically directable, and hence monogenically connected. By Theorem 4.1, $\mathcal{A}$ is the direct sum of some connected X-algebras $\mathcal{A}_\alpha$, $\alpha \in Y$. Obviously $\mathcal{A}_\alpha \in M(\mathbf{G})$ for each $\alpha \in Y$. Therefore $\mathcal{A}_\alpha \in M(\mathbf{G}) \cap \mathbf{Conn} = \mathbf{L}(\mathbf{G})$ by Theorem 6.10 and so $\mathcal{A} \in L(\mathbf{G}) \circ \mathbf{D}$. Hence $M(\mathbf{G}) \subseteq \mathbf{L}(\mathbf{G}) \circ \mathbf{D}$ for any generalized variety $\mathbf{G}$.

Consider now any $\mathcal{A} \in L(\mathbf{G}) \circ \mathbf{D}$. Then $\mathcal{A}$ is the direct sum of some X-algebras $\mathcal{A}_\alpha \in L(\mathbf{G})$, $\alpha \in Y$. Let $\mathcal{B}$ be any finitely generated subalgebra of $\mathcal{A}$. Let $Z = \{\alpha \in Y \mid B \cap A_\alpha \neq \varnothing\}$, and for each $\alpha \in Z$, let $B_\alpha = B \cap A_\alpha$. Then $\mathcal{B}$ is the direct sum of the X-algebras $\mathcal{B}_\alpha$, $\alpha \in Z$, where for each $\alpha \in Z$, $\mathcal{B}_\alpha$ is a finitely generated subalgebra of $\mathcal{A}_\alpha$. Now $\mathcal{B}_\alpha \in \mathbf{G}$ for every $\alpha \in Z$ as $\mathcal{A}_\alpha \in L(\mathbf{G})$. Therefore, $\mathcal{B} \in \mathbf{G} \circ \mathbf{D}$, and hence $\mathcal{A} \in L(\mathbf{G} \circ \mathbf{D})$.

Finally, consider any $\mathcal{A} \in L(\mathbf{G} \circ \mathbf{D})$ and $a \in A$. Then $\langle a\rangle \in \mathbf{G} \circ \mathbf{D}$, and since every monogenic X-algebra is direct sum indecomposable, this means that $\langle a\rangle \in \mathbf{G}$. Hence, $\mathcal{A} \in M(\mathbf{G})$. This completes the proof of the theorem. □

Theorem 5.5 states that $M(\mathbf{V}) = \mathbf{V}$ if and only if $\mathbf{V}$ is a regular variety of X-algebras. To show that the same does not hold for regular generalized varieties, we consider the regular generalized variety of all reverse definite X-algebras $\mathbf{RDef} = \bigcup_{\mathbf{n} \in \mathbb{N}} \mathbf{RDef_n}$. Let $\mathcal{A}$ be the direct sum of some X-algebras $\mathcal{A}_n$, $n \in \mathbb{N}$, where $\mathcal{A}_n$ is a reverse n-definite X-algebra which is not reverse $n+1$-definite. Then $\mathcal{A} \in M(\mathbf{RDef}) \setminus \mathbf{RDef}$. A necessary and sufficient condition under which the equality holds for a generalized variety is presented in the next theorem.

Theorem 6.12 *For any generalized variety* $\mathbf{G}$ *of* X*-algebras,*

$$M(\mathbf{G}) = \mathbf{G} \;\Leftrightarrow\; \mathbf{G} \circ \mathbf{D} = \mathbf{G}.$$

Proof. Assume $M(\mathbf{G}) = \mathbf{G}$ and consider any X-algebra $\mathcal{A} \in \mathbf{G} \circ \mathbf{D}$. Then $\mathcal{A}$ is the direct sum of some X-algebras $\mathcal{A}_\alpha \in \mathbf{G}$, $\alpha \in Y$. For each $a \in A$ there exists an $\alpha \in Y$ such that $a \in A_\alpha$, and then $\langle a \rangle \subseteq A_\alpha$. Now $\mathcal{A}_\alpha \in \mathbf{G}$ implies $\langle a \rangle \in \mathbf{G}$, and therefore $\mathcal{A} \in M(\mathbf{G}) = \mathbf{G}$. Hence, $\mathbf{G} \circ \mathbf{D} \subseteq \mathbf{G}$. The reverse inclusion is obvious.

Conversely, suppose that $\mathbf{G} \circ \mathbf{D} = \mathbf{G}$ and let $\mathcal{A} \in M(\mathbf{G})$. Then $\langle a \rangle \in \mathbf{G}$ for every $a \in A$. To each $a \in A$ we associate an X-algebra $\mathcal{B}_a$ isomorphic to $\langle a \rangle$ so that $B_a \cap B_b = \varnothing$ for all $a \neq b$, $a, b \in A$, and let $\varphi_a : \mathcal{B}_a \to \langle a \rangle$ be an isomorphism. Denote by $\mathcal{B}$ the direct sum of the X-algebras $\mathcal{B}_a, a \in A$, and define the mapping $\varphi : B \to A$ so that if $c \in B_a$ for some $a \in A$, then $c\varphi = c\varphi_a$. It is not hard to check that φ is a homomorphism from $\mathcal{B}$ onto $\mathcal{A}$. Furthermore, $\mathcal{B} \in \mathbf{G}$ because $\mathcal{B}_a \cong \langle a \rangle \in \mathbf{G}$ for every $a \in A$ and $\mathbf{G} \circ \mathbf{D} = \mathbf{G}$, and as a homomorphic image of $\mathcal{B}$ also $\mathcal{A}$ belongs to $\mathbf{G}$. Therefore, $M(\mathbf{G}) \subseteq \mathbf{G}$, and the converse inclusion follows from Lemma 3.1. □

7 Monogenic Closures of Pseudovarieties

When finite X-algebras only are considered, for example in connection with pseudovarieties, it is natural to use the modified localization operators $\underline{M}$ and $\underline{L}$ defined so that $\underline{M}(\mathbf{K}) = \underline{\mathbf{M}(\mathbf{K})}$ and $\underline{L}(\mathbf{K}) = \underline{\mathbf{L}(\mathbf{K})}$ for any class $\mathbf{K}$ of X-algebras. Obviously, $\underline{M}(\mathbf{K}) = \underline{\mathbf{M}(\underline{\mathbf{K}})}$ and $\underline{L}(\mathbf{K}) = \underline{\mathbf{L}(\underline{\mathbf{K}})}$ for any $\mathbf{K}$, but in what follows, $\mathbf{K}$ will itself always be a class of finite X-algebras. Moreover, it is clear that $\underline{L}(\mathbf{K}) = \mathbf{K}$ whenever $\mathbf{K}$ is a class of finite X-algebras closed under subalgebras. Hence the operator $\underline{L}$ is of lesser interest here.

The following result can be proved similarly as Theorem 5.4.

Theorem 7.13 *The operators* $\underline{M}$ *and* $\underline{L}$ *are closure operators on the lattice of pseudovarieties of* X*-algebras, and* $\underline{L}(\mathbf{P}) = \mathbf{P} \subseteq \underline{\mathbf{M}}(\mathbf{P})$ *for any pseudovariety* $\mathbf{P}$*.*

Regular and irregular generalized varieties of X-algebras were defined in the previous section. Similarly, a pseudovariety of X-algebras $\mathbf{P}$ is *regular* if it is ultimately defined by a sequence of regular identities, otherwise it is called *irregular*.

Theorem 7.14 *If* X *is a finite alphabet, then the following conditions are equivalent for any pseudovariety* $\mathbf{P}$ *of* X*-algebras:*

(1) $\mathbf{P}$ *is a regular pseudovariety;*
(2) $\underline{\mathbf{D}} \subseteq \mathbf{P}$*;*
(3) $\mathcal{D}_2 \in \mathbf{P}$*;*
(4) *every sequence of identities ultimately defining* $\mathbf{P}$ *contains at most finitely many irregular identities;*
(5) $\mathbf{P} = \underline{\mathbf{G}}$ *for some regular generalized variety* $\mathbf{G}$ *of* X*-algebras.*

Proof. The implications (1)⇒(2) and (2)⇒(3) are obvious.

(3)⇒(4). Assume that $\mathbf{P} = [\mathbf{s_n} = \mathbf{t_n} \,|\, \mathbf{n} \in \mathbb{N}]_{\mathbf{u}}$. Then from $\mathcal{D}_2 \in \mathbf{P}$ follows the existence of a $k \in \mathbb{N}$ such that $\mathcal{D}_2 \models u_n = v_n$ for all $n \geq k$. This means that $\{u_n = v_n \,|\, n \geq k\}$ is a set of regular identities, and hence $\{u_n = v_n \,|\, n \in \mathbb{N}\}$ contains at most finitely many irregular identities.

(4)⇒(1). Let $\{s_n = t_n \,|\, n \in \mathbb{N}\}$ be any sequence of identities ultimately defining $\mathbf{P}$. By our assumption (4), there is a $k \geq 0$ such that the subsequence $\{s_n = t_n \,|\, n \geq k\}$ consists of regular identities only, and hence $\mathbf{P}$ is regular by Lemma 6.4.

(3)⇒(5). By a result due to Ash [1], we may write $\mathbf{P} = \underline{\mathbf{G}}$, where $\mathbf{G}$ is a generalized variety. On the other hand, $\mathcal{D}_2 \in \mathbf{P} \subseteq \mathbf{G}$ implies by Theorem 6.9 that any such $\mathbf{G}$ is regular.

(5)⇒(3). Since $\mathcal{D}_2$ is a finite X-algebra contained in every regular generalized variety, (5) implies that $\mathcal{D}_2 \in \mathbf{P}$. □

Corollary 7.5 *If X is a finite alphabet, then the following conditions are equivalent for any pseudovariety $\mathbf{P}$ of X-algebras:*

(1) *$\mathbf{P}$ is an irregular pseudovariety;*
(2) $\underline{\mathbf{D}} \cap \mathbf{P} = \underline{\mathbf{O}}$*;*
(3) $\underline{\mathbf{D}} \not\subseteq \mathbf{P}$*;*
(4) $\mathcal{D}_2 \notin \mathbf{P}$*;*
(5) *every sequence ultimately defining $\mathbf{P}$ contains infinitely many irregular identities;*
(6) *$\mathbf{P}$ is the set of all finite X-algebras of some irregular generalized variety of X-algebras.*

In [2] it is shown that any pseudovariety $\mathbf{P}$ is contained in a unique minimal regular pseudovariety $\underline{R}(\mathbf{P})$. Using this regularization operator we may formulate for pseudovarieties a result similar to Theorem 5.6.

Theorem 7.15 *For any pseudovariety $\mathbf{P}$ of X-algebras,*

$$\underline{M}(\mathbf{P}) = \underline{\mathbf{R}}(\mathbf{P}) = \mathbf{P} \circ \underline{\mathbf{D}} = \mathbf{P} \vee \underline{\mathbf{D}}.$$

Proof. According to Theorem 2 in [1], $\mathbf{P} = \underline{\mathbf{G}}$ for some generalized variety $\mathbf{G}$ of X-algebras, and hence by Theorem 6.11 we get

$$\begin{aligned}\underline{M}(\mathbf{P}) &= \underline{M}(\underline{\mathbf{G}}) = \underline{M(\underline{\mathbf{G}})} = \underline{M(\mathbf{G})} = \underline{L(\mathbf{G}) \circ \mathbf{D}} = \underline{L(\mathbf{G})} \circ \underline{\mathbf{D}} \\ &= \underline{L}(\underline{\mathbf{G}}) \circ \underline{\mathbf{D}} = \underline{L}(\mathbf{P}) \circ \underline{\mathbf{D}} = \mathbf{P} \circ \underline{\mathbf{D}}.\end{aligned}$$

Assume that $\underline{R}(\mathbf{P}) = [\mathbf{gu_n} = \mathbf{gv_n} \,|\, \mathbf{n} \in \mathbb{N}]_{\approx}$ and let $\mathcal{A} \in \underline{M}(\mathbf{P})$. For any $a \in A$, $\langle a \rangle \in \mathbf{P} \subseteq \underline{\mathbf{R}}(\mathbf{P})$ implies the existence of an $n_a \in \mathbb{N}$ such that $\langle a \rangle \models gu_n = gv_n$ for all $n \geq n_a$. Since $\mathcal{A}$ is finite, we may define $m = \max\{n_a \,|\, a \in A\}$, and then for every $a \in A$, $\langle a \rangle \models gu_n = gv_n$ for all $n \geq m$. Hence $\mathcal{A} \in \underline{R}(\mathbf{P})$, and therefore $\underline{M}(\mathbf{P}) \subseteq \underline{\mathbf{R}}(\mathbf{P})$.

On the other hand, by Theorem 7.13, $\underline{M}(\mathbf{P})$ is a pseudovariety, and since $\mathcal{D}_2 \in \underline{M}(\mathbf{P})$, it is regular. Hence $\underline{R}(\mathbf{P}) \subseteq \underline{\mathbf{M}}(\mathbf{P})$.

Furthermore, $\mathbf{P} \subseteq \underline{\mathbf{R}}(\mathbf{P})$ and $\underline{\mathbf{D}} \subseteq \underline{R}(\mathbf{P})$ imply $\mathbf{P} \vee \underline{\mathbf{D}} \subseteq \underline{\mathbf{R}}(\mathbf{P})$. Conversely, $\mathbf{P} \vee \underline{\mathbf{D}}$ is a regular variety containing $\mathbf{P}$, so $\underline{R}(\mathbf{P}) \subseteq \mathbf{P} \vee \underline{\mathbf{D}}$. This completes the proof of the theorem. □

References

[1] C. J. Ash, Pseudovarieties, generalized varieties and similarly described classes, *J. Algebra* **92** (1985), 104–115.

[2] S. Bogdanović, M. Ćirić and T. Petković, Generalized varieties of algebras, to appear.

[3] S. Bogdanović, B. Imreh, M. Ćirić and T. Petković, Directable automata and their generalizations – A survey, in: S. Crvenković and I. Dolinka (eds.), *Algebra and Logic* (Proc. VIII Int. Conf., Novi Sad, 1998), *Novi Sad J. Math.* **29** (1999), no. 2, pp. 31–74.

[4] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, New York, 1981.

[5] M. Ćirić and S. Bogdanović, Lattices of subautomata and direct sum decompositions of automata, *Algebra Colloq.* **6**:1 (1999), 71–88.

[6] S. Eilenberg, *Automata, Languages and Machines*, Academic Press, New York and London, Vol. A, 1974, Vol B, 1976.

[7] S. Eilenberg and M. P. Schützenberger, On pseudovarieties of monoids, *Adv. Math.* **19** (1976), 413–448.

[8] F. Gécseg and I. Peák, *Algebraic Theory of Automata*, Akadémiai Kiadó, Budapest, 1972.

[9] F. Gécseg and G. Thierrin, Characterizations of locally transitive semiautomata, *Papers on Automata and languages IX*, K. Marx Univ. of Economics, Dept. of Math., Budapest **87–2** (1987), 1–8.

[10] E. Graczyńska, Proofs of unary regular identities, *Demonstr. Math.* **16** (1983), no. 4, 925–929.

[11] J. M. Howie, *Automata and languages*, Clarendon Press, Oxford, 1991.

[12] B. Imreh and M. Steinby, Some remarks on directable automata, *Acta Cybern.* **12** (1995), no. 1, 23–35.

[13] J. Kovačević, M. Ćirić, T. Petković and S. Bogdanović, Decompositions of automata and reversible states, *Publ. Math.* Supplementum **60** (2002), 587–602.

[14] J. E. Pin, *Variétés de langages formels*, Masson, Paris, 1984.

[15] J. Płonka, On equational classes of abstract algebras defined by regular identities, *Fundam. Math.* **64** (1969), 241–247.

[16] J. Płonka, On the sum of a system of disjoint unary algebras corresponding to a given type, *Bull. Acad. Polon. Sci.* **30** (1982), no. 7–8, 305–309.

[17] J. Płonka, On the sum of a direct system of universal algebras with nullary polynomials, *Algebra Univers.* **19** (1984), 197–207.

[18] J. Płonka, On the lattice of varieties of unary algebras, *Simon Stevin* **59** (1985), no. 4, 353–364.

[19] M. Steinby, Classifying regular languages by their syntactic algebras, in: J. Karhumäki, H. Maurer and G. Rozenberg (eds.), *Results and Trends in Theoretical Computer Science* (Proc. Coll., Graz, Austria, June 1994), Lecture Notes in Computer Science **812**, Springer-Verlag, Berlin 1994, 396-409.

[20] G. Thierrin, Decompositions of locally transitive semiautomata, *Util. Math.* **2** (1972), 25–32.