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Lattices of Subautomata and Direct Sum Decompositions of Automata*

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Abstract. The subject of this paper are general properties of direct sum decompositions of automata. Using certain properties of the lattice $\mathrm{Sub}(A)$ of subautomata of an automaton A and its Boolean part, lattices of direct sum congruences and direct sum decompositions of A are characterized. We show that every automaton A can be represented as a direct sum of direct sum indecomposable automata, and that the lattice $\mathrm{Sub}(A)$ can be represented as a direct product of directly indecomposable lattices. Some special types of direct sum decompositions of automata are also investigated.

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1 Introduction and Preliminaries

Direct sum decompositions of automata were first defined and studied by Huzino [15] and have been investigated in [1, 9–11, 20, 22, 24] and others. In this paper we investigate some general properties of direct sum decompositions of automata, using the methodology developed by the authors in [2, 3, 5] for studying some decompositions of semigroups. We especially study these decompositions through the properties of lattices of subautomata and their Boolean parts. In Sec. 2, we describe the Boolean part of the lattice

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Sub(A) of subautomata of an automaton A. We prove that it is a complete atomic Boolean algebra and give two algorithms for computing the atoms of $\mathrm{F}(A)$. Using these results, in Sec. 3, we characterize the lattices of direct sum congruences and direct sum decompositions of A and show that the summands in the greatest direct sum decomposition of A are direct sum indecomposable.

In Sec. 4, we give a relationship between direct sum decompositions of an automaton A and direct product decompositions of Sub(A) and prove that Sub(A) can be represented as a direct product of directly indecomposable lattices. Some special direct sum decompositions are investigated in Sec. 5. In Sec. 6, we establish some connections between direct sum congruences, Rees congruences and principal congruences on an automaton.

All automata considered here will be automata without outputs in the sense of [9] but similar results can be also obtained for automata with arbitrary (not necessarily free) input monoids, Mealy-type automata and unary algebras.

The considered automata will be treated as unary algebras. So the notions such as *congruence*, *subautomaton*, *generating set* etc., will have their usual algebraic meanings. An automaton and its set of states will be denoted by the same letter. All automata will have the same input alphabet X and the free monoid over X will be denoted by X^*. Under the action of an input word $u \in X^*$, the automaton A goes from a state $a \in A$ into the state au.

An automaton A is a *direct sum* of its subautomata A_α $(\alpha \in Y)$, in notation $A = \sum_{\alpha\in Y} A_\alpha$, if $A = \bigcup_{\alpha\in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \varnothing$, for $\alpha \neq \beta$. The equivalence relation on A, whose classes are different A_α, is called a *direct sum equivalence* on A, the related partition of A is called a *direct sum decomposition* of A, and the automata A_α are called *direct summands* of A. As we will see later, any direct sum equivalence on an automaton is a congruence, and the name *direct sum congruence* will be also used. An automaton A is called *direct sum indecomposable* if the universal equivalence on A is the unique direct sum equivalence on A.

Throughout the paper, $\mathbb{Z}$ denotes the set of all integers and $\mathbb{N}$ the set of all positive integers. For a binary relation ξ on a set T and $n \in \mathbb{N}$, ξ^n denotes the n-th power of ξ in the semigroup of binary relations on T. The identity and the universal relation on T are denoted by Δ_T and ∇_T, respectively, or briefly Δ and ∇. By a *quasi-order*, we mean a reflexive and transitive binary relation. The notion *poset* is used as a synonym for the notion "partially ordered set". The two-element Boolean algebra will be denoted by $\mathbf{2}$.

For a non-empty set T, $\mathrm{P}(T)$ will denote the *lattice of subsets* of T. Let L be a sublattice of $\mathrm{P}(T)$ containing its unity and having the property that any non-empty intersection of elements of L is also in L. Then for any $a \in T$ there exists the smallest element of L containing a, which will be called the *principal element* of L generated by a and usually denoted by

$L(a)$. The set of all principal elements of L is called the *principal part* of L.

For a non-empty set T, the *lattice of equivalence relations* on T is denoted by $\mathrm{Eq}\,(A)$. Its dual lattice, the *lattice of partitions* of T, is denoted by $\mathrm{Part}\,(T)$. For an automaton A, the *lattice of subautomata* of A (including the *empty subautomaton*) is denoted by $\mathrm{Sub}(A)$ and the *lattice of congruences* on A is denoted by $\mathrm{Con}\,(A)$. As known, $\mathrm{Con}\,(A)$ and $\mathrm{Sub}(A)$ are complete sublattices of $\mathrm{Eq}\,(A)$ and $\mathrm{P}\,(A)$, respectively.

For undefined notions and notations concerning automata we refer to [9, 14, 17, 21], and for those concerning lattices and universal algebras, we refer to [3–5, 13, 23].

2 The Boolean Part of the Lattice of Subautomata

Much information about a distributive lattice L with zero and unity can be obtained through information concerning its *Boolean part* (or *center*) defined as the set of all elements of L which are complemented in L. In [2, 3, 5], we studied some decompositions of semigroups using the Boolean parts of lattices of ideals. Here we apply this methodology to the lattice of subautomata and direct sum decompositionos of an automaton.

A subset P of an automaton A is called *consistent* if for each $a \in A$ and $u \in X^*$, $au \in P$ implies $a \in P$ (this notion was introduced in [8] for semigroups and in [12], under the name "*isolated subset*" for universal algebras). A consistent subautomaton of A is called a *filter* of A. The empty subautomaton of A is also a filter of A. A filter of A different than $\varnothing$ and A is called a *proper filter* of A.

Let us observe that a subset P of an automaton A is a consistent subset of A if and only if its set-theoretical complement in A is a subautomaton of A. Using this we obtain the following lemma:

Lemma 2.1 *The following conditions for a subautomaton B of an automaton A are equivalent:*

(i) *B is a filter of A.*

(ii) *the set-theoretical complement of B in A is a filter of A.*

(iii) *B is a direct summand of A.*

Lemma 2.2 *Let B be a filter of an automaton A and C a filter of B. Then C is a filter of A.*

The set of all filters of an automaton A will be denoted by $\mathrm{F}\,(A)$. Now we are ready to prove the main theorem of this section.

Theorem 2.3 *For any automaton A, $\mathrm{F}\,(A)$ is the Boolean part of $\mathrm{Sub}(A)$ and it is a complete atomic Boolean algebra. Furthermore, any complete atomic Boolean algebra can be represented as the Boolean algebra of filters of some automaton.*

Proof. By Lemma 2.1, $\mathrm{F}(A)$ is the Boolean part of $\mathrm{Sub}(A)$. Recall from [3] that the Boolean part of a complete Brouwerian lattice is its complete sublattice if and only if it is a complete atomic Boolean algebra. Since $\mathrm{Sub}(A)$ is a complete Brouwerian lattice and $\mathrm{F}(A)$ is a complete sublattice of $\mathrm{Sub}(A)$, we have that $\mathrm{F}(A)$ is a complete atomic Boolean algebra.

Furthermore, let B be an arbitrary complete atomic Boolean algebra and Y the set of all its atoms. To any $\alpha \in Y$, we associate a direct sum indecomposable automaton A_α such that different A_α have disjoint sets of states. For example, we can assume A_α $(\alpha \in Y)$ are connected or strongly connected automata. Let A be the direct sum of automata A_α $(\alpha \in Y)$. Then $\{A_\alpha \,|\, \alpha \in Y\}$ is the set of all atoms in $\mathrm{F}(A)$ in view of Lemma 2.1. It is well known that B is isomorphic to the Boolean algebra of all subsets of Y, and since the set of all atoms of $\mathrm{F}(A)$ and Y have the same cardinallity, B and $\mathrm{F}(A)$ are isomorphic. This completes the proof of the theorem. □

Our next goal is to characterize the atoms in $\mathrm{F}(A)$. For an automaton A and $a \in A$, let $F(a)$ denote the principal element of $\mathrm{F}(A)$ generated by a. The filter $F(a)$ will be called the *principal filter* of A generated by a. Evidently, the atoms in $\mathrm{F}(A)$ are exactly the principal filters of A.

Given a subset P of an automaton A. The *subautomaton* of A generated by P will be denoted by $S(P)$. In other words,

$$S(P) = \{b \in A \,|\, (\exists a \in P)(\exists u \in X^*)\, b = au\} = \{au \,|\, a \in P,\, u \in X^*\}.$$

The subautomaton generated by a single state $a \in A$ will be denoted by $S(a)$. Further, $C(P)$ will denote the smallest consistent subset of A containing P, called the *consistent subset* of A generated by P. In other words,

$$C(P) = \{a \in A \,|\, (\exists u \in X^*)\, au \in P\}.$$

Now we are ready to prove the following theorem which gives an algorithm for finding principal filters of an automaton.

Theorem 2.4 *Let A be an automaton and $a \in A$. Let a sequence $\{U_n(a)\}_{n\in\mathbb{N}}$ of subsets of A be defined by $U_1(a) = C\big(S(a)\big)$ and $U_{n+1}(a) = C\big(S\big(U_n(a)\big)\big)$ for $n \in \mathbb{N}$. Then $\{U_n(a)\}_{n\in\mathbb{N}}$ is an increasing sequence of sets and $F(a) = \bigcup_{n\in\mathbb{N}} U_n(a)$.*

Proof. For any $n \in \mathbb{N}$, we have $U_n(a) \subseteq S\big(U_n(a)\big) \subseteq C\big(S\big(U_n(a)\big)\big) = U_{n+1}(a)$ and so $\{U_n(a)\}_{n\in\mathbb{N}}$ is an increasing sequence. Further, let $U = \cup_{n\in\mathbb{N}} U_n(a)$. Since each $U_n(a)$ is a consistent subsets of A,U is also consistent. If $b \in U$, $u \in X^*$, then $b \in U_n(a)$ for some $n \in \mathbb{N}$ and $bu \in S\big(U_n(a)\big) \subseteq C\big(S\big(U_n(a)\big)\big) = U_{n+1}(a)$. So U is a subautomaton of A. Therefore, U is a filter of A containing a, hence $F(a) \subseteq U$.

To prove the opposite inclusion, it is enough to prove $U_n(a) \subseteq F(a)$ for any $n \in \mathbb{N}$. This will be proved by induction. First, we observe $S(a) \subseteq F(a)$ since $F(a)$ is a subautomaton of A containing a, and now $U_1(a) = C\big(S(a)\big) \subseteq$

$F(a)$, since $F(a)$ is consistent. Suppose $U_n(a) \subseteq F(a)$ for some $n \in \mathbb{N}$. Then $S\big(U_n(a)\big) \subseteq F(a)$ since $F(a)$ is a subautomaton of A containing $U_n(a)$ and $U_{n+1}(a) = C\big(S\big(U_n(a)\big)\big) \subseteq F(a)$, since $F(a)$ is consistent. Hence, $U_n(a) \subseteq F(a)$ for any $n \in \mathbb{N}$. This completes the proof of the theorem. $\square$

It can be also proved that, for any automaton A, $\mathrm{F}\,(A)$ is the Boolean part of the lattice of consistent subsets of A.

For finite automata we have the following:

Corollary 2.5 *Let A be a finite automaton and*

$$n = \min\{k \in \mathbb{N} \mid (\forall a \in A)\, U_k(a) = U_{k+1}(a)\}.$$

Then $n \leq |A|$ and $F(a) = U_n(a)$ for any $a \in A$.

By the previous theorem, the principal filter $F(a)$ of an automaton A generated by a state $a \in A$ can be computed as a union of the sequence $\{U_n(a)\}_{n\in\mathbb{N}}$ of sets obtained by successive application of the operators $P \mapsto S(P)$ (generating a subautomaton by P) and $P \mapsto C(P)$ (generating a consistent subset by P) starting from a. In the following theorem, we prove that a permutation of these operators does not change the final result of this procedure.

Theorem 2.6 *Let A be an automaton, $a \in A$, $L_1(a) = S\big(C(a)\big)$ and $L_{n+1}(a) = S\big(C\big(L_n(a)\big)\big)$ for $n \in \mathbb{N}$. Then $\{L_n(a)\}_{n\in\mathbb{N}}$ is an increasing sequence of sets and $F(a) = \bigcup_{n\in\mathbb{N}} L_n(a)$.*

Proof. We can prove that $\{L_n(a)\}_{n\in\mathbb{N}}$ is an increasing sequence similarly as the related part of Theorem 2.4. To simplify the notations, set $L(a) = \bigcup_{n\in\mathbb{N}} L_n(a)$.

Since $a \in S(a)$, it follows that $C(a) \subseteq C\big(S(a)\big) = U_1(a)$, hence

$$L_1(a) = S\big(C(a)\big) \subseteq S\big(U_1(a)\big) \subseteq C\big(S\big(U_1(a)\big)\big) = U_2(a).$$

Now suppose $L_n(a) \subseteq U_{n+1}(a)$ for some $n \in \mathbb{N}$. Then we have $C\big(L_n(a)\big) \subseteq C\big(U_{n+1}(a)\big) = U_{n+1}(a)$ since $U_{n+1}(a)$ is a consistent subset of A. So

$$L_{n+1}(a) = S\big(C\big(L_n(a)\big)\big) \subseteq S\big(U_{n+1}(a)\big) \subseteq C\big(S\big(U_{n+1}(a)\big)\big) = U_{n+2}(a).$$

By induction, we obtain $L_n(a) \subseteq U_{n+1}(a)$ for any $n \in \mathbb{N}$. By Theorem 2.4 it follows that $L(a) \subseteq F(a)$.

On the other hand, since $S(a) \subseteq S\big(C(a)\big) = L_1(a)$, we obtain

$$U_1(a) = C\big(S(a)\big) \subseteq C\big(L_1(a)\big) \subseteq S\big(C\big(L_1(a)\big)\big) = L_2(a).$$

Suppose $U_n(a) \subseteq L_{n+1}(a)$ for some $n \in \mathbb{N}$. Then $S\big(U_n(a)\big) \subseteq S\big(L_{n+1}(a)\big) = L_{n+1}(a)$ since $L_{n+1}(a)$ is a subautomaton of A, hence

$$U_{n+1}(a) = C\big(S\big(U_n(a)\big)\big) \subseteq C\big(L_{n+1}(a)\big) \subseteq S\big(C\big(L_{n+1}(a)\big)\big) = L_{n+2}(a).$$

By induction, we obtain $U_n(a) \subseteq L_{n+1}(a)$ for any $n \in \mathbb{N}$, and by Theorem 2.4 we have $F(a) \subseteq L(a)$. Therefore, $F(a) = L(a)$. $\square$

Corollary 2.7 *Let A be a finite automaton and*

$$n = \min\{k \in \mathbb{N} \,|\, (\forall a \in A)\, L_k(a) = L_{k+1}(a)\}.$$

Then $n \leq |A|$ and $F(a) = L_n(a)$ for any $a \in A$.

3 Direct Sum Decompositions of an Automaton

In this section we return to the direct sum decompositions of automata. Recall that an equivalence relation θ on an automaton A was called a *direct sum equivalence* if every θ-class of A is a subautomaton of A. An automaton A is called an *identity automaton* if $au = a$, for all $a \in A$ and $u \in X^*$. In some other origins, such automata have been called *discrete automata*. By the following lemma we characterize direct sum equivalences on an automaton:

Lemma 3.1 *The following conditions for an equivalence relation θ on an automaton A are equivalent:*

(i) *θ is a direct sum equivalence on A.*

(ii) $(\forall a \in A)(\forall u \in X^*)$ $au\,\theta\,a$.

(iii) *θ is a congruence on A and A/θ is an identity automaton.*

Proof. The implications (i) ⇒ (ii) and (iii) ⇒ (i) are obvious.

(ii) ⇒ (iii). If $(a, b) \in \theta$ and $u \in X^*$, then $au\,\theta\,a\,\theta\,b\,\theta\,bu$, hence $au\,\theta\,bu$. Therefore, θ is a congruence. Clearly, A/θ is an identity automaton. $\square$

In view of the previous lemma, when we deal with automata without outputs, the name "*direct sum congruence*" will be used as a synonym for "direct sum equivalence". In the general case, when we deal with Mealy-type automata, a direct sum equivalence is not necessary a congruence of a Mealy-type automaton (see [9]).

Using the variety of all identity automata and a result obtained by the authors in [7], we can prove that direct sum congruences on an automaton A form a principal dual ideal of $\mathrm{Con}\,(A)$. A direct proof of this assertion will be given, followed by a construction of the generating element of this principal dual ideal.

Theorem 3.2 *If σ denote the transitive closure of the relation $-$ on an automaton A defined by*

$$a - b \iff S(a) \cap S(b) \neq \varnothing \qquad (a, b \in A),$$

then the set of all direct sum congruences on A is the principal dual ideal of $\mathrm{Eq}\,(A)$ generated by σ.

Proof. The relation $-$ is obviously reflexive and symmetric, so σ is an equivalence relation on A. For $a \in A$ and $u \in X^*$, by $au - a$, we obtain $au\,\sigma\,a$, so by Lemma 3.1, σ is a direct sum congruence on A.

Let $[\sigma)$ denote the principal dual ideal of $\mathrm{Eq}\,(A)$ generated by σ. If $\theta \in [\sigma)$, then for arbitrary $a \in A$ and $u \in X^*$, we have $(au, a) \in \sigma \subseteq \theta$. By Lemma 3.1, θ is a direct sum congruence on A. Conversely, let θ be an arbitrary direct sum congruence on A. Assume $a, b \in A$ such that $a - b$, i.e. $au = bv$, for some $u, v \in X^*$. Then $a\,\theta\,au = bv\,\theta\,b$. Therefore, $- \subseteq \theta$. So $\sigma \subseteq \theta$, proving that σ is the transitive closure of $-$. This completes the proof of the theorem. $\square$

Theorem 3.3 *The smallest direct sum congruence σ on an automaton A equals the transitive closure of the relation $\_$ on A defined by:*

$$a \_ b \iff C(a) \cap C(b) \neq \varnothing \qquad (a, b \in A).$$

Proof. Assume $a, b \in A$ such that $a \_ b$. Then there exists $c \in C(a) \cap C(b)$, i.e., $a = cu$ and $b = cv$ for some $u, v \in X^*$ and so $a = cu\,\sigma\,c\,\sigma\,cv = b$, which yields $a\,\sigma\,b$. This means that $\_$ is contained in σ and so its transitive closure is also contained in σ. But the transitive closure of $\_$ is a direct sum congruence on A since $au \_ a$ for all $a \in A$ and $u \in X^*$. By Theorem 3.2, we have that σ equals the transitive closure of $\_$. $\square$

Corollary 3.4 *The smallest direct sum congruence σ on an automaton A equals the transitive closure of the relation --- on A defined by $\text{---} = - \cap \_$.*

Let $\mid$ denote the *division relation* on an automaton A, i.e., the quasi-order on A defined by:

$$a \mid b \iff (\exists u \in X^*)\ b = au.$$

Then the relations $-$ and $\_$ on A can also be defined in the following way:

$$\begin{aligned} a - b &\iff (\exists c \in A)\ a \mid c \ \&\ b \mid c, \\ a \_ b &\iff (\exists c \in A)\ c \mid a \ \&\ c \mid b. \end{aligned}$$

For the poset of direct sum decompositions, by Theorem 3.2, we obtain the following.

Theorem 3.5 *The set of all direct sum decompositions of an automaton A is a principal ideal of the partition lattice* $\mathrm{Part}\,(A)$.

By the previous theorem, we found that direct sum decompositions of an automaton A form a complete lattice that is dually isomorphic to the lattice of direct sum equivalences on A which are characterized by Lemma 3.1. Another characterization of the lattice of direct sum decompositions of A, through the Boolean algebra $\mathrm{F}\,(A)$, is given by the following theorem.

Theorem 3.6 *The lattice of direct sum decompositions of an automaton A is isomorphic to the lattice of complete Boolean subalgebras of* $\mathrm{F}(A)$.

Proof. In order to establish the desired lattice isomorphism, it is enough to find an order isomorphism between these lattices. Let B be a complete Boolean subalgebra of $\mathrm{F}(A)$. For $a \in A$, let $B(a)$ denote the principal element of B generated by a. Since B is a complete sublattice of $\mathrm{F}(A)$, by [3, Theorem 10], B is atomic, and the atoms of B are exactly its principal elements. Set $\mathcal{D}_B = \{B(a) \,|\, a \in A\}$. It is clear that $\mathcal{D}_B$ is a direct sum decomposition of A whose summands are exactly the atoms of B. We will prove that the mapping $B \mapsto \mathcal{D}_B$ is an order isomorphism of the lattice of complete Boolean subalgebras of $\mathrm{F}(A)$ onto the lattice of direct sum decompositions of A.

Let B and E be two complete Boolean subalgebras of $\mathrm{F}(A)$. If $B \subseteq E$, then $E(a) \subseteq B(a)$ for any $a \in A$, hence $\mathcal{D}_B \leq \mathcal{D}_E$ in $\mathrm{Part}(A)$. Conversely, let $\mathcal{D}_B \leq \mathcal{D}_E$ in $\mathrm{Part}(A)$. Then for any $a \in A$, there exists $b \in A$ such that $E(a) \subseteq B(b)$, and $a \in E(a)$ implies $a \in B(b)$. Hence, $B(a) \subseteq B(b)$, hence $B(a) = B(b)$ since $B(b)$ is an atom in B. Therefore, $E(a) \subseteq B(a)$ for any $a \in A$, which means that $B \subseteq E$. Hence, $B \subseteq E$ if and only if $\mathcal{D}_B \subseteq \mathcal{D}_E$.

It remains to prove that the mapping $B \mapsto \mathcal{D}_B$ is onto. Let $\mathcal{D} = \{A_\alpha \,|\, \alpha \in Y\}$ be an arbitrary direct sum decomposition of A. By Lemma 2.1, A_α $(\alpha \in Y)$ are filters of A. Set

$$B = \left\{ F \in \mathrm{F}(A) \,\middle|\, (\exists Z \subseteq Y)\, F = \textstyle\bigcup_{\alpha \in Z} A_\alpha \right\}.$$

Note that $\varnothing \in B$ since we can assume $Z = \varnothing$. Then B is a complete Boolean subalgebra of $\mathrm{F}(A)$, and so is a complete atomic Boolean algebra whose atoms are exactly A_α $(\alpha \in Y)$. In other words, for $\alpha \in Y$ and $a \in A_\alpha$, $A_\alpha = B(a)$. Now we have $\mathcal{D} = \mathcal{D}_B$, which proves that $B \mapsto \mathcal{D}_B$ is onto. This ends the proof of the theorem. □

Remark 3.7 The previous theorem can be also formulated and proved in terms of direct sum congruences on an automaton A, namely, the lattice of complete Boolean subalgebras of $\mathrm{F}(A)$ is dually isomorphic to the lattice of direct sum congruences on A and a dual isomorphism between these lattices can be given by $B \mapsto \sigma_B$. Here, for a complete Boolean subalgebra B of $\mathrm{F}(A)$, the relation σ_B on A is defined by $(a, b) \in \sigma_B \Leftrightarrow B(a) = B(b)$. Clearly, σ_B is the direct sum congruence on A which corresponds to the direct sum decomposition $\mathcal{D}_B$ of A. Recall that, for $a \in A$, $B(a)$ denotes the principal element of B generated by a.

The smallest direct sum congruence on A is given by $\sigma = \sigma_{\mathrm{F}(A)}$, i.e., $(a, b) \in \sigma \Leftrightarrow F(a) = F(b)$.

An automaton A will be called *σ-connected* if it satisfies one of the following equivalent conditions: (1) $\sigma = \nabla$, (2) $(\forall a, b \in A)(\exists n \in \mathbb{N})\; a \overset{n}{-} b$, (3) $(\forall a, b \in A)(\exists n \in \mathbb{N})\; a \overset{n}{\_} b$, (4) $(\forall a, b \in A)(\exists n \in \mathbb{N})\; a \overset{n}{—} b$. Here $\overset{n}{-}$, $\overset{n}{\_}$ and $\overset{n}{—}$ denote the nth powers of relations $-$, $\_$ and $—$, respectively.

Theorem 3.8 *The following conditions on an automaton A are equivalent:*

(i) *A is a direct sum indecomposable automaton.*
(ii) *A has no proper filters.*
(iii) *A is σ-connected.*
(iv) $\mathrm{F}(A) \cong \mathbf{2}$.
(v) $\mathrm{Sub}(A)$ *is a directly indecomposable lattice.*

Proof. The equivalence of statements (i)–(iv) is an immediate consequence of Lemma 2.1 and Theorems 2.3 and 2.4. We have (iv)⇔(v) because of [3, Lemma 4]. □

Theorem 3.9 *Any automaton A can be represented as a direct sum of direct sum indecomposable automata. This is the greatest direct sum decomposition of A and its summands are the atoms of* $\mathrm{F}(A)$.

Proof. The existence of the greatest direct sum decomposition of A follows by the previous two theorems. By Theorem 3.6, this decomposition corresponds to the greatest complete Boolean subalgebra of $\mathrm{F}(A)$, i.e., to the whole Boolean algebra $\mathrm{F}(A)$, and its summands are exactly the atoms of $\mathrm{F}(A)$.

Let B be an arbitrary summand in the greatest direct sum decomposition of A. If B is not direct sum indecomposable, by Lemma 2.1, B has a proper filter C. By Lemma 2.2, C is also a filter of A, which contradicts the fact that B is an atom of $\mathrm{F}(A)$. Therefore, any summand in the greatest direct sum decomposition of A must be direct sum indecomposable. □

It may seem that the indecomposability of the summands in the greatest direct sum decomposition of an automaton A is a natural consequence of the atomicity of $\mathrm{F}(A)$. But this is not true, namely, in the proof of the indecomposability of these summands, Lemma 2.2 plays a crucial role. The authors in [3] studied decomposition of semigroups with zero into a so-called right sum of semigroups, where they used the Boolean part of the lattice of left ideals of this semigroup, which is also a complete atomic Boolean algebra. For such decompositions, an example has been given where the summands in the greatest decomposition may be decomposable in such a sum (see [3]).

4 Direct Product Decompositions of the Lattice of Subautomata

Other connections between the direct sum decompositions of an automaton A and the lattice $\mathrm{Sub}(A)$ are given by the following theorem:

Theorem 4.1 *The lattice* $\mathrm{Sub}(A)$ *of subautomata of an automaton A is a direct product of lattices L_α ($\alpha \in Y$) if and only if A is a direct sum of automata A_α ($\alpha \in Y$) and $L_\alpha \cong \mathrm{Sub}(A_\alpha)$ for any $\alpha \in Y$.*

Proof. Let $\mathrm{Sub}(A)$ be a direct product of lattices L_α ($\alpha \in Y$). For any $\alpha \in Y$, L_α is a homomorphic image of $\mathrm{Sub}(A)$ with respect to the projection homomorphism π_α of $\mathrm{Sub}(A)$ onto L_α, so L_α has a zero 0_α and a unity 1_α. Let $A_\alpha \in \mathrm{Sub}(A)$ be an element satisfying the condition

$$A_\alpha \pi_\beta = \begin{cases} 1_\alpha & \text{for } \beta = \alpha, \\ 0_\alpha & \text{for } \beta \neq \alpha. \end{cases}$$

By a straightforward verification, we obtain that L_α is isomorphic to the principal ideal of $\mathrm{Sub}(A)$ generated by A_α. On the other hand, the principal ideal of $\mathrm{Sub}(A)$ generated by A_α is isomorphic to $\mathrm{Sub}\,(A_\alpha)$, since any subautomaton of A_α is also a subautomaton of A. Therefore, $L_\alpha \cong \mathrm{Sub}\,(A_\alpha)$ for any $\alpha \in Y$. We obtain immediately that A is a direct sum of automata A_α $(\alpha \in Y)$. This completes the proof of the direct part of the theorem.

To prove the converse, assume A is a direct sum of automata A_α $(\alpha \in Y)$. Let L_α denote the principal ideal of $\mathrm{Sub}(A)$ generated by A_α, $L = \prod_{\alpha\in Y} L_\alpha$ and π_α the projection homomorphism of L onto L_α. Then the mapping $\phi : \mathrm{Sub}(A) \to L$ defined by $(B\phi)\pi_\alpha = B \cap A_\alpha$ for $B \in \mathrm{Sub}(A)$ and $\alpha \in Y$ is an isomorphism, proving that $\mathrm{Sub}(A)$ is a complete Brouwerian lattice. Hence, it is infinitely distributive for meets. Finally, $L_\alpha = \mathrm{Sub}\,(A_\alpha)$, which ends the proof of the theorem. □

By the previous theorem we obtain the following consequence on representation of the lattice of subautomata of an automaton:

Corollary 4.2 *Let A be an arbitrary automaton. Then the lattice* $\mathrm{Sub}(A)$ *can be represented as a direct product of directly indecomposable lattices and* $\mathrm{Sub}(A) \cong \prod_{\alpha\in Y} \mathrm{Sub}(A_\alpha)$, *where $A = \sum_{\alpha\in Y} A_\alpha$ is a representation of A as a direct sum of direct sum indecomposable automata.*

Using the dualism between subautomata and consistent subsets of an automaton, we can obtain similar results concerning direct product decompositions of the lattice of consistent subsets of an automaton A.

5 Direct Sums of υ_n-, λ_n-, μ_n- and ι_n-connected Automata

Using sequences $\{U_n\}_{n\in\mathbb{N}}$ and $\{L_n\}_{n\in\mathbb{N}}$ in Sec. 2, we can define many new equivalence relations on an automaton as follows. For $n \in \mathbb{N}$, define equivalence relations υ_n and λ_n on an automaton A by

$$a\,\upsilon_n\, b \iff U_n(a) = U_n(b),$$
$$a\,\lambda_n\, b \iff L_n(a) = L_n(b).$$

Moreover, for $a \in A$ set $I_n(a) = U_n(a) \cap L_n(a)$ and $M_n(a) = \{b \in A \mid a \overset{n}{-\!\!-} b\}$. Define equivalence relations ι_n and μ_n on A by

$$a\,\iota_n\, b \iff I_n(a) = I_n(b),$$
$$a\,\mu_n\, b \iff M_n(a) = M_n(b).$$

Clearly, $M_1(a) = I_1(a)$ and $M_n(a) \subseteq I_n(a)$ for all $a \in A$ and $n \in \mathbb{N}$ with $n \geq 2$. Some useful properties of the sets $U_n(a)$ and the equivalence relations υ_n are given by the following lemma.

Lemma 5.1 *Let A be an automaton, $a \in A$, and $n \in \mathbb{N}$. Then*

(i) $U_n(a) = \{b \in A \,|\, a \overset{n}{-} b\}$,

(ii) $U_n(au) \subseteq U_n(a)$ *for any* $u \in X^*$,

(iii) $\upsilon_n \subseteq \overset{n}{-}$.

Proof. (i) This will be proved by induction. First, we have the following sequence of equivalences:

$$\begin{aligned} b \in U_1(a) &\iff (\exists v \in X^*)\, bv \in S(a) \\ &\iff (\exists v \in X^*)\, bv \in S(a) \cap S(b) \\ &\iff a - b. \end{aligned}$$

Suppose (i) holds for some $n \in \mathbb{N}$. Let us prove that this also holds for $n + 1$. We have

$$\begin{aligned} b \in U_{n+1}(a) &\iff (\exists v \in X^*)\, bv \in S\big(U_n(a)\big) \\ &\iff (\exists v \in X^*)(\exists c \in U_n(a))(\exists u \in X^*)\, bv = cu \\ &\iff (\exists c \in U_n(a))\, c - b \\ &\iff (\exists c \in A)\, a \overset{n}{-} c - b \qquad \text{(by the induction hypothesis)} \\ &\iff a \overset{n+1}{-} b. \end{aligned}$$

(ii) This follows by (i) and the fact that $bu - c$ implies $b - c$ for all $b, c \in A$ and $u \in X^*$.

(iii) If $b\, \upsilon_n\, c$, then $b \in U_n(b) = U_n(c)$, hence $b \overset{n}{-} c$ by (i). Hence, (iii) holds. $\square$

The assertions of the same form can be also proved for sets $L_n(a)$, $M_n(a)$ and $I_n(a)$, i.e., for relations λ_n, μ_n and ι_n.

Define the equivalence relation γ on A by $a\, \gamma\, b \Leftrightarrow S(a) = S(b)$, or equivalently, $a\, \gamma\, b \Leftrightarrow C(a) = C(b)$.

The next lemma establishes a hierarchy between the above defined relations:

Lemma 5.2 *On any automaton A, the following hierarchy holds:*

$$\begin{aligned} \gamma \subseteq \upsilon_1 \subseteq \cdots \subseteq \upsilon_n \subseteq \upsilon_{n+1} \subseteq \cdots \subseteq \sigma, \\ \gamma \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_n \subseteq \lambda_{n+1} \subseteq \cdots \subseteq \sigma, \\ \gamma \subseteq \mu_1 \subseteq \cdots \subseteq \mu_n \subseteq \mu_{n+1} \subseteq \cdots \subseteq \sigma. \end{aligned}$$

Moreover, $\upsilon_n \cap \lambda_n \subseteq \iota_n$ for any $n \in \mathbb{N}$.

Proof. Assume $n \in \mathbb{N}$ and $(a,b) \in \upsilon_n$. Then $U_n(a) = U_n(b)$, hence

$$U_{n+1}(a) = C\big(S\big(U_n(a)\big)\big) = C\big(S\big(U_n(b)\big)\big) = U_{n+1}(b)$$

and so $(a,b) \in \upsilon_{n+1}$. Therefore, $\upsilon_n \subseteq \upsilon_{n+1}$ for any $n \in \mathbb{N}$. Similarly, $\gamma \subseteq \upsilon_1$, and by Lemma 5.1, we have $\upsilon_n \subseteq \overset{n}{-} \subseteq \sigma$ for any $n \in \mathbb{N}$. We can similarly prove the inclusions in the second and third rows. The inclusion $\upsilon_n \cap \lambda_n \subseteq \iota_n$ is obvious. □

In Sec. 3, we defined a σ-connected automaton. Here, we introduce the following more special notions. For $n \in \mathbb{N}$, an automaton A will be called an υ_n-*connected automaton* if $\upsilon_n = \nabla$ on A, or equivalently, $a \overset{n}{-} b$ for all $a, b \in A$, i.e., $U_n(a) = A$ for any $a \in A$. Similarly we define λ_n-*connected*, ι_n-*connected* and μ_n-*connected automata*. Note that υ_1-connected automata are known as *connected automata*.

By the proof of Theorem 2.4, any υ_n-connected automaton is λ_{n+1}-connected and any λ_n-connected automaton is υ_{n+1}-connected. Any μ_n-connected automaton is ι_n-connected since $M_n(a) \subseteq I_n(a)$ for all $a \in A$ and $n \in \mathbb{N}$. By Lemma 5.1, an automaton A is ι_n-connected if and only if it is both υ_n-connected and λ_n-connected.

Example 5.3 Define an automaton A in the following way: The set of states of A is $\mathbb{Z}$, the input alphabet is $X = \{x, y\}$, and the transition function is defined by

$$kx = \begin{cases} k+1 & \text{if } k \text{ is even} \\ k & \text{if } k \text{ is odd} \end{cases} \qquad \text{and} \qquad ky = \begin{cases} k-1 & \text{if } k \text{ is even} \\ k & \text{if } k \text{ is odd} \end{cases},$$

for $k \in \mathbb{Z}$, or by the transition graph shown in Fig. 1.

For $i, j \in \mathbb{Z}$ with $i < j$, let $[i,j] = \{m \in \mathbb{Z} \mid i \leq m \leq j\}$. Then

$$S(k) = \begin{cases} [k-1, k+1] & \text{if } k \text{ is even,} \\ \{k\} & \text{if } k \text{ is odd,} \end{cases}$$

$$C(k) = \begin{cases} \{k\} & \text{if } k \text{ is even,} \\ [k-1, k+1] & \text{if } k \text{ is odd,} \end{cases}$$

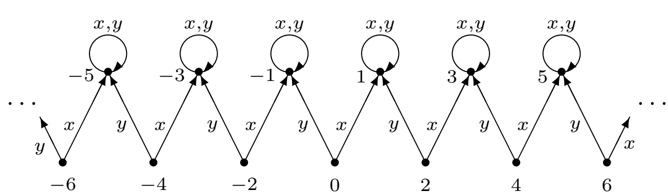

Fig. 1. Transition graph

$$U_n(k) = \begin{cases} [k-2n, k+2n] & \text{if } k \text{ is even,} \\ [k-(2n-1), k+(2n-1)] & \text{if } k \text{ is odd,} \end{cases}$$

$$L_n(k) = \begin{cases} [k-(2n-1), k+(2n-1)] & \text{if } k \text{ is even,} \\ [k-2n, k+2n] & \text{if } k \text{ is odd,} \end{cases}$$

$$M_n(k) = [k-n, k+n]\,, \qquad I_n(k) = [k-(2n-1), k+(2n-1)]\,.$$

Therefore, $F(k) = A$ for any $k \in \mathbb{Z}$ and so A is a σ-connected automaton. But there does not exists $n \in \mathbb{N}$ such that A is υ_n-connected since $U_n(k) \neq U_{n+1}(k)$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Similarly, there is no $n \in \mathbb{N}$ such that A is λ_n-, μ_n- or ι_n-connected. We also have $\gamma = \upsilon_n = \lambda_n = \mu_n = \iota_n = \Delta$ for any $n \in \mathbb{N}$ and $\sigma = \nabla$ on A.

Consider the subautomaton B of A with the state set $[-5, 5]$. The relations $-$, $\_$ and $—$ on B are given by the following graphs.

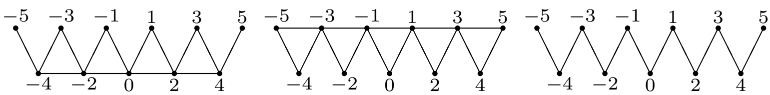

Fig. 2. Graphs of $-$, $\_$ and $—$

B is υ_6-connected, λ_5-connected, ι_6-connected and μ_{10}-connected.

Now we are ready to prove a theorem which gives several characterizations of direct sums of υ_n-connected automata.

Theorem 5.4 *Let $n \in \mathbb{N}$. Then the following conditions on an automaton A are equivalent:*

(i) *A is a direct sum of υ_n-connected automata.*

(ii) $(\forall a, b \in A)(\forall u \in X^*)\, a \overset{n}{-} b \;\Rightarrow\; au \overset{n}{-} b$.

(iii) *υ_n is a direct sum congruence on A.*

(iv) *$U_n(a)$ is a subautomaton of A, for any $a \in A$.*

(v) *$\overset{n}{-}$ is a transitive relation.*

(vi) *$U_n(a) = U_{n+1}(a)$, for any $a \in A$.*

(vii) *$U_n(a) = a\upsilon_n$, for any $a \in A$.*

Proof. (i) $\Rightarrow$ (ii). Let A be a direct sum of υ_n-connected automata A_α $(\alpha \in Y)$ and θ the corresponding direct sum congruence on A. Assume $a, b \in A$ such that $a \overset{n}{-} b$ and $u \in X^*$. By Theorem 3.2, $(a, b) \in \overset{n}{-} \subseteq \sigma \subseteq \theta$ and so $a, b \in A_\alpha$ for some $\alpha \in Y$. Now $au, b \in A_\alpha$, hence $au \overset{n}{-} b$ since A_α is an υ_n-connected automaton. Therefore, (ii) holds.

(ii) $\Rightarrow$ (iii). By (ii) and Lemma 5.1(ii), we have $U_n(au) = U_n(a)$, i.e., $au\, \upsilon_n\, a$ for all $a \in A$ and $u \in X^*$, which yields (iii) by Lemma 3.1.

(iii) $\Rightarrow$ (i). Let υ_n be a direct sum congruence on A. By Theorem 3.2 and Lemma 5.2, $\upsilon_n = \sigma$. Let A_α $(\alpha \in Y)$ be the summands in the direct sum decomposition of A which corresponds to υ_n. Assume $\alpha \in Y$ and $a, b \in A_\alpha$. Then $a\, \upsilon_n\, b$, i.e., $U_n(a) = U_n(b)$. By Lemma 5.1, $a \overset{n}{-} b$ in A, and hence, $a \overset{n}{-} b$ in A_α. Therefore, A_α is a υ_n-connected automaton.

(ii) $\Rightarrow$ (iv). Assume $a \in A$. If $b \in U_n(a)$, then $a \overset{n}{-} b$ by Lemma 5.1 and $b \overset{n}{-} a$ since $\overset{n}{-}$ is a symmetric relation. By (ii), $bu \overset{n}{-} a$, and thus, $a \overset{n}{-} bu$, which means $bu \in U_n(a)$. Therefore, $U_n(a)$ is a subautomaton of A.

(iv) $\Rightarrow$ (v). If (iv) holds, then clearly $U_n(a) = F(a)$ for any $a \in A$. By Lemma 5.1, we have $\overset{n}{-} = \sigma$.

(v) $\Rightarrow$ (vi). This follows by Lemma 5.1.

(vi) $\Rightarrow$ (vii). If (vi) holds, then $U_n(a) = F(a)$ for any $a \in A$ and $\upsilon_n = \sigma$, hence we obtain (vii).

(vii) $\Rightarrow$ (ii). By Lemma 5.1, $\overset{n}{-} = \upsilon_n$. So $\overset{n}{-}$ is transitive and $\overset{n}{-} = \sigma$. Since σ is a direct sum congruence on A, we obtain (ii). $\square$

Theorems of the same form can be also proved for direct sums of λ_n-, ι_n- and μ_n-connected automata.

Recall that an automaton A is *strongly connected* if one of the following equivalent conditions holds: (1) $(\forall a \in A)\, S(a) = A$, (2) $(\forall a \in A)\, C(a) = A$, (3) $\gamma = \nabla$ on A. This notion was introduced by Moore [18]. These automata are also known under other names such as *transitive automata* (cf. [10, 16]) and *simple automata* (cf. [11]). But the name "strongly connected" is used most frequently.

Following the terminology of [10], an automaton A will be called *locally transitive* if, for all $a \in A$ and $u \in X^*$, there exists $v \in X^*$ such that $auv = a$ (by [11], these automata are called *invertible*). Note that locally transitive automata are exactly the automata on which the quasi-order introduced in Sec. 3 is symmetric (i.e., an equivalence relation).

Direct sums of strongly connected automata have been investigated by [11, 15] and its complete characterization was given by [10, 24]. Here, we give another proof of the Thierrin Theorem [24] and some new results concerning these automata.

Theorem 5.5 *The following conditions on an automaton A are equivalent:*

(i) *A is a direct sum of strongly connected automata.*

(ii) *γ is a direct sum congruence on A.*

(iii) *any subautomaton of A is consistent.*

(iv) *any consistent subset of A is a subautomaton.*

(v) *$S(a) = C(a)$, for any $a \in A$.*

(vi) *$\mathrm{Sub}(A)$ is a Boolean algebra.*

(vii) *A is a locally transitive automaton.*

Proof. (i)$\Leftrightarrow$(ii). This is obvious.

(ii) $\Rightarrow$ (iii). Let B be an arbitrary subautomaton of A and $au \in B$ for some $a \in A$ and $u \in X^*$. By $au\,\gamma\, a$, it follows that $S(au) = S(a)$. So $S(au) \subseteq B$, hence $S(a) \subseteq B$, and hence $a \in B$.

(iii)$\Rightarrow$(vi). If (iii) holds, then $\mathrm{Sub}(A) = \mathrm{F}\,(A)$. By Theorem 2.3, $\mathrm{Sub}(A)$ is a Boolean algebra.

(vi)$\Rightarrow$(ii). If $\mathrm{Sub}(A)$ is a Boolean algebra, then $\mathrm{Sub}(A) = \mathrm{F}\,(A)$. Hence $F(a) = S(a)$ for any $a \in A$ and $\gamma = \sigma$, which yields (ii).

(ii) $\Rightarrow$ (iv). This can be proved similarly as (ii) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (ii). If (iv) holds, then $F(a) = C(a)$ for any $a \in A$. So $\sigma = \gamma$, hence we obtain (ii).

(ii) $\Rightarrow$ (v). Since (ii)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iv) are already proved, using (iii) and (iv), we obtain (v).

(v)$\Rightarrow$(ii). If (v) holds, by Theorem 2.4, we have $F(a) = S(a) = C(a)$ for any $a \in A$. So $\sigma = \gamma$ and (ii) holds.

(ii)$\Leftrightarrow$(vii). The condition (ii) is equivalent to the condition $S(au) = S(a)$ for all $a \in A$ and $u \in X^*$, which is clearly equivalent to the definition of locally transitive automata. □

By Theorems 5.5 and 3.8 we obtain the following.

Corollary 5.6 [21]) *An automaton A is strongly connected if and only if* $\mathrm{Sub}(A) \cong \mathbf{2}$.

6 Direct Sum Congruences Viewed from Rees Congruences and Principal Congruences

In Remark 3.7, we characterized direct sum congruences on an automaton A through complete Boolean subalgebras of $\mathrm{F}\,(A)$. In this section these congruences will be characterized through two important types of congruences on an automaton, i.e., Rees congruences and principal congruences on an automaton.

Let B be a subautomaton of an automaton A. Then the relation ϱ_B on A defined by

$$(a,b) \in \varrho_B \iff a, b \in B \text{ or } a = b, \qquad (a, b \in A),$$

is a congruence relation on A, called the *Rees congruence* on A determined by B. The corresponding factor automaton, denoted by A/B, is called the *Rees factor* of A with respect to B. The Rees congruence determined by the empty subautomaton of A is clearly the equality relation on A.

Before we characterize direct sum congruences on an automaton in terms of Rees congruences, we prove the following.

Theorem 6.1 *Let $\{A_\alpha \,|\, \alpha \in Y\}$ be a family of pairwise disjoint subautomata of an automaton A.*

(i) *If $Y = \{1, 2, \ldots, n\}$, then* $\bigcup_{i=1}^n \varrho_{A_i} = \varrho_{A_1}\varrho_{A_2}\cdots\varrho_{A_n}$.

(ii) *If Y is an arbitrary set, then* $\bigvee_{\alpha\in Y}\varrho_{A_\alpha}=\bigcup_{\alpha\in Y}\varrho_{A_\alpha}$.

Proof. Let $(a,b)\in\varrho_{A_1}\varrho_{A_2}\cdots\varrho_{A_n}$. Then there exist $c_0,c_1,\dots,c_{n-1},c_n\in A$ such that $c_0=a$, $c_n=b$, and $(c_{i-1},c_i)\in\varrho_{A_i}=A_i\times A_i\cup\Delta$ for each $i\in\{1,\dots,n\}$. If $c_{i-1}=c_i$ for each $i\in\{1,\dots,n\}$, then $a=b$ so $(a,b)\in\varrho_{A_i}$, for any $i\in\{1,\dots,n\}$. Otherwise, there exists $k\in\{1,\dots,n\}$ such that $c_{k-1}\neq c_k$. Then $c_{k-1},c_k\in A_k$. Since $A_1,\dots,A_n$ are pairwise disjoint, we have $a=c_0=\dots=c_{k-1}$ and $c_k=\dots=c_n=b$, hence $(a,b)\in\varrho_{A_k}$. Therefore, we have proved (i).

The assertion (ii) is an immediate consequence of (i). □

By the following theorem, we establish a connection between a direct sum congruence on an automaton A and the Rees congruences on A determined by the summands in the corresponding direct sum decomposition of A.

Theorem 6.2 *Let A be a direct sum of automata A_α ($\alpha\in Y$) and θ the corresponding direct sum congruence on A. Then* $\theta=\bigvee_{\alpha\in Y}\varrho_{A_\alpha}=\bigcup_{\alpha\in Y}\varrho_{A_\alpha}$.

Proof. By Theorem 6.1, it is sufficient to prove $\theta=\varrho$, where $\varrho=\bigcup_{\alpha\in Y}\varrho_{A_\alpha}$. Assume $(a,b)\in\theta$. Then $a,b\in A_\alpha$ for some $\alpha\in Y$, hence $(a,b)\in\varrho_{A_\alpha}$. Therefore, $\theta\subseteq\varrho$. Conversely, assume $(a,b)\in\varrho$. Then $(a,b)\in\varrho_{A_\alpha}$ for some $\alpha\in Y$, which means $a,b\in A_\alpha$ or $a=b$. In both cases, we have $(a,b)\in\theta$. Hence, $\varrho\subseteq\theta$. This completes the proof of the theorem. □

A state a of an automaton A is called a *trap* if $ax=a$ for any $x\in X$, or equivalently, $au=a$ for any $u\in X^*$. If A is an arbitrary automaton and $t\notin A$, then we define an automaton A^t to be a direct sum of A and the automaton having only one state t. In other words, the automaton A^t is obtained from A by adjoining a trap.

Let $\{B_\alpha\,|\,\alpha\in Y\}$ be an arbitrary family of subautomata of A. It is easy to prove $\bigcap_{\alpha\in Y}\varrho_{B_\alpha}=\varrho_B$ where $B=\bigcap_{\alpha\in Y}B_\alpha$. Using this property, we obtain the following.

Theorem 6.3 *Let A be a direct sum of automata A_α ($\alpha\in Y$). Then A is a subdirect product of automata A_α^t ($\alpha\in Y$).*

Proof. For any $\alpha\in Y$, let A'_α denote the set-theoretical complement of A_α in A and $A'=\varnothing$. Then $\cap_{\alpha\in Y}A'_\alpha=(\cup_{\alpha\in Y}A_\alpha)'=A'=\varnothing$, hence we obtain $\cap_{\alpha\in Y}\varrho_{A'_\alpha}=\Delta$, where Δ denotes the equality relation on A. Therefore, A is a subdirect product of automata A/A'_α and $A/A'_\alpha\cong A_\alpha^t$ for any $\alpha\in Y$. □

We say that an equivalence relation θ on a set S *saturates* a subset T of S if T is the union of some θ-classes of S. The equivalence relation θ_T, having only two equivalence classes (T and its set-theoretical complement in S), is the greatest equivalence relation on S which saturates T, called the *principal equivalence* on S determined by T.

Let A be an automaton. Following [14], for $T \subseteq A$ and $a \in A$, let $T.a = \{u \in X^* \mid au \in T\}$. Then $T.a$ is a language in X^*, called the *quotient* of T with respect to a. Recall that, for a language $L \subseteq X^*$ and $u \in X^*$, the *right quotient* of L with respect to u is defined by $L.u = \{v \in X^* \mid uv \in L\}$. It is easy to verify $(T.a).u = T.au$ for all $a \in A$ and $u \in X^*$, hence the relation P_T on A defined by

$$(a, b) \in P_T \iff T.a = T.b \qquad (a, b \in A)$$

is a congruence relation on A. Moreover, this is the greatest congruence on A which saturates T and is called the *principal congruence* on A determined by T.

If F is a filter of an automaton A, then θ_F is a direct sum congruence on A. So $\theta_F = P_F$ for any filter F of A. Using this, we obtain the next theorem, by which we characterize direct sum equivalences on an automaton in terms of principal congruences determined by filters.

Theorem 6.4 *Let A be an automaton, B a complete Boolean subalgebra of* $\mathrm{F}(A)$*, and σ_B the direct sum congruence on A which corresponds to B defined as in Remark* 3.7*. Then $\sigma_B = \bigcap_{F \in B} P_F = \bigcap_{c \in A} P_{B(c)}$.*

Proof. Assume $(a, b) \in \sigma_B$ and $F \in B$. Then $B(a) = B(b)$ is an atom in B, hence $B(a) \subseteq F$ or $B(a) \subseteq F'$ where F' denotes the set-theoretical complement of F in A. Therefore, $a, b \in B(a) \subseteq F$ or $a, b \in B(a) \subseteq F'$, hence $(a, b) \in \theta_F = P_F$. This proves $\sigma_B \subseteq \cap_{F \in B} P_F$. The inclusion $\cap_{F \in B} P_F \subseteq \cap_{c \in A} P_{B(c)}$ is obvious. Finally, assume $(a, b) \in \cap_{c \in A} P_{B(c)}$. Then $(a, b) \in P_{B(a)}$. Since $a \in B(a)$, we have $b \in B(a)$, hence $B(b) = B(a)$, which yields $(a, b) \in \sigma_B$. Hence, $\cap_{c \in A} P_{B(c)} \subseteq \sigma_B$, which completes the proof of the theorem. □

By the previous theorem we immediately obtain the following.

Corollary 6.5 *Let A be the direct sum of automata A_α $(\alpha \in Y)$ and θ the corresponding direct sum congruence on A. Then $\theta = \bigcap_{\alpha \in Y} P_{A_\alpha}$.*

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