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On Nil-Extensions of Rectangular Groups*

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Abstract. Putcha [8] and Ren, Shum and Guo [9] gave very interesting descriptions of nil-extensions of rectangular groups in terms of subdirect and spined products. Since subdirect and spined product decompositions can be characterized in terms of congruences, we state a natural question: What are the systems of congruences which determine these decompositions? In the present paper, we give a complete answer on this question. We define two systems of congruence relations on a completely Archimedean semigroup and make certain subdirect product decompositions of this semigroup. In the case when this semigroup is a nil-extension of a rectangular group, we give structure descriptions of the corresponding factor semigroups. By the obtained results, we deduce more constructive proofs of the theorems of [8, 9].

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1 Introduction

There are various structural characterizations of semigroups that are nil-extensions of rectangular groups (they were called quasi-rectangular groups in [9]). For example, they were characterized as π-regular semigroups whose

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idempotents form a rectangular band, as completely Archimedean semigroups whose idempotents form a subsemigroup etc. Among the most interesting ones are the characterizations given in terms of subdirect products by Putcha [8] in 1973 and by Ren, Shum and Guo [9] in 1997.

Putcha proved a theorem that characterizes the mentioned semigroups in two ways: as subdirect products of a group and a nil-extension of a rectangular band; and as subdirect products of a group, a nil-extension of a left zero band and a nil-extension of a right zero band. He proved this theorem using the famous Birkhoff representation theorem, by which every algebra can be represented as a subdirect product of subdirectly irreducible algebras, and the fact that every subdirectly irreducible semigroup in the class of nil-extensions of rectangular groups is a group, or a nil-extension of a left zero band, or a nil-extension of a right zero band.

Shum and Ren [11] gave a general method for construction of completely Archimedean semigroups, generalizing the famous Rees-Sushkevitsch method for construction of completely simple semigroups. As a particular case of this construction, Ren, Shum and Guo [9] constructed all nil-extensions of rectangular groups and proved another theorem that characterizes nil-extensions of rectangular groups as spined products of a nil-extension of a left group and a nil-extension of a right group with respect to a nil-extension of a group.

These results motivate us to consider these subdirect decompositions again but from another point of view. As is well known, another Birkhoff theorem [1] applied to semigroups says that a semigroup S is a subdirect product of semigroups S_i $(i \in I)$ if and only if there exists a family $\{\varrho_i \mid i \in I\}$ of congruences on S such that $\cap_{i\in I}\varrho_i = \Delta_S$ and $S/\varrho_i \cong S_i$ for each $i \in I$. We will call $\{\varrho_i \mid i \in I\}$ a family of *factor congruences*. The first problem that we state is: What are the families of factor congruences which realize Putcha's decompositions?

There are also theorems that characterize spined products in terms of congruences. Spined products, known as *pullback products* in universal algebra, first appeared in [4]. In the semigroup theory, their intensive investigation was initiated by Kimura [6] in 1958. The first theorem by which spined products were characterized in terms of congruences was proved by Fleischer [3] in 1955. In the case of semigroups, this theorem asserts that a semigroup S can be represented as a spined product of semigroups P and Q with respect to their common homomorphic image H if and only if there exists a pair ϱ, ϱ' of factor congruences on S such that ϱ and ϱ' commute, $S/\varrho \cong P$, $S/\varrho' \cong Q$, and $S/\varrho\varrho' \cong H$. For spined products with more than two factors, a similar theorem was proved by Wenzel [12] in 1968. The factor congruences that appear in this theorem are congruences satisfying some conditions which can be viewed as a generalization of the conditions of the Chinese Remainder Theorem. Naturally, we also state the following question: What is the pair of factor congruences which determines the spined product decomposition given by [9]?

We give the answers to both of the above stated questions. First, in terms of products of elements with idempotents, we define two systems of congruence relations on a completely Archimedean semigroup. Further, using these systems of congruences we form several systems of factor congruences on the given semigroup, and in the case when this semigroup is a nil-extension of a rectangular group, we give structure descriptions of the corresponding factors. By the obtained results we deduce constructive proofs of the theorems of Putcha and Ren, Shum and Guo.

2 Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. For a set X, Δ_X denotes the equality relation on X, and for relations ϱ and ϱ' on X, $\varrho \cdot \varrho'$, or briefly $\varrho\varrho'$, denotes the product of these relations in the semigroup of binary relations on X.

Let S be a semigroup. Then $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ denote the Green's relations on S, $E(S)$ denotes the set of all idempotents of S and $Reg(S)$ denotes the set of all regular elements of S. As is well known, $a \in Reg(S)$ means that there exists $x \in S$ such that $a = axa$ and $x = xax$. Such an element x will be called an *inverse* of a, and the set of all inverses of a will be denoted by $V(a)$. For $e \in E(S)$, we denote by G_e the maximal subgroup of S having e as its identity. Note that the inverse of a regular element of S and the (*group*) *inverse* in G_e of an element $a \in G_e$, are different notions.

A semigroup S is called a *nil-semigroup* if it has a zero and, for every $a \in S$, there exists $n \in \mathbb{N}$ such that $a^n = 0$. We say that a semigroup S is a *nil-extension* of a semigroup T if it is an ideal extension of T by a nil-semigroup. A semigroup S is called *Archimedean* if, for all $a, b \in S$, there exists $n \in \mathbb{N}$ such that $b^n \in SaS$. S is called *π-regular* if, for each $a \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in Reg(S)$.

Let P and Q be semigroups, and H their common homomorphic image with respect to homomorphisms φ of P onto H and ψ of Q onto H. Then $S = \{(p, q) \in P \times Q \,|\, p\varphi = q\psi\}$ is a subdirect product of P and Q, called the *spined product* of P and Q with respect to H, φ and ψ.

For undefined notions and notations, we refer to [2, 5].

Let X be a non-empty subset of a semigroup S. Then $L(X) = XS^1$, $R(X) = S^1X$ and $J(X) = S^1XS^1$ denote respectively the left, right and two-sided ideals of S generated by X. X is said to be a *duo subset* of S if $L(X) = R(X)$. Evidently, $L(X) = R(X) = J(X)$ for each duo subset X of S. On the other hand, if K is an ideal of S, then $L(K) = R(K) = K$, so every ideal is a duo subset.

Let X be a non-empty subset of a semigroup S. Define relations $\overline{\mathcal{L}}_X$,

$\overline{\mathcal{R}}_X$ and $\overline{\mathcal{D}}_X$ on S by:

$$\begin{aligned}(a,b)\in\overline{\mathcal{L}}_X &\iff (\forall x\in X)\ xa=xb,\\ (a,b)\in\overline{\mathcal{R}}_X &\iff (\forall y\in X)\ ay=by,\\ (a,b)\in\overline{\mathcal{D}}_X &\iff (\forall x,y\in X)\ xay=xby\end{aligned}$$

It is easy to check that these are equvalence relations on S. Moreover, $\overline{\mathcal{L}}_X$ is a right congruence, $\overline{\mathcal{R}}_X$ is a left congruence, and both of them are contained in $\overline{\mathcal{D}}_X$. It is natural to investigate some conditions on the set X under which the relations $\overline{\mathcal{L}}_X$, $\overline{\mathcal{R}}_X$ and $\overline{\mathcal{D}}_X$ are congruence relations on S.

Lemma 2.1. *If X is a duo subset of a semigroup S, then $\overline{\mathcal{L}}_X$, $\overline{\mathcal{R}}_X$ and $\overline{\mathcal{D}}_X$ are congruence relations on S. Furthermore, $\overline{\mathcal{G}}_X=\overline{\mathcal{G}}_K$ for each $\mathcal{G}\in\{\mathcal{L},\mathcal{R},\mathcal{D}\}$ where $K=J(X)$.*

Proof. We will only prove the assertions concerning $\overline{\mathcal{D}}_X$. The assertion concerning $\overline{\mathcal{L}}_X$ and $\overline{\mathcal{R}}_X$ can be proved similarly.

Assume $(a,b)\in\overline{\mathcal{D}}_X$, $c\in S$, and $x,y\in X$. Since X is a duo subset, we have $xc=sz$ for some $z\in X$ and $s\in S^1$, hence $x(ca)y=(xc)ay=(sz)ay=s(zay)=s(zby)=(sz)by=(xc)by=x(cb)y$. Similarly, we have $x(ac)y=x(bc)y$. Therefore, $\overline{\mathcal{D}}_X$ is a congruence relation on S.

Since $X\subseteq K$, it follows that $\overline{\mathcal{D}}_K\subseteq\overline{\mathcal{D}}_X$. Conversely, assume $(a,b)\in\overline{\mathcal{D}}_X$ and $p,q\in K$. Since $K=XS^1=S^1X$, we see $p=sx$ and $q=yt$ for some $x,y\in X$ and $s,t\in S^1$. Now we have $paq=(sx)a(yt)=s(xay)t=s(xby)t=(sx)b(yt)=pbq$. Therefore, $(a,b)\in\overline{\mathcal{D}}_K$, i.e., $\overline{\mathcal{D}}_X\subseteq\overline{\mathcal{D}}_K$, which completes the proof of the lemma. □

Remark. The congruences $\overline{\mathcal{L}}_S$, $\overline{\mathcal{R}}_S$ and $\overline{\mathcal{D}}_S$ on a semigroup S were investigated by Kopamu [7].

Let X be a duo subset of a semigroup S, $K=J(X)$, and ϱ_K the *Rees congruence* on S determined by K. Then we set

$$\widehat{\mathcal{L}}_X=\overline{\mathcal{L}}_X\cap\varrho_K,\quad \widehat{\mathcal{R}}_X=\overline{\mathcal{R}}_X\cap\varrho_K\quad\text{and}\quad \widehat{\mathcal{D}}_X=\overline{\mathcal{D}}_X\cap\varrho_K.$$

3 The Main Results

In further work, the subject of our interest will be completely Archimedean semigroups. Recall that a semigroup S is *completely Archimedean* if it is Archimedean and has a primitive idempotent, or equivalently, it is a nil-extension of a completely simple semigroup. In this case, this completely simple ideal of S is the kernel of S.

If S is a completely Archimedean semigroup with the kernel K and $E=E(S)$, then $L(E)=R(E)=K$, i.e., E is a duo subset of S. By Lemma 2.1, we have $\overline{\mathcal{G}}_E=\overline{\mathcal{G}}_K$ for every $\mathcal{G}\in\{\mathcal{L},\mathcal{R},\mathcal{D}\}$. On the other hand, we have the following.

Lemma 3.1. *Let S be a completely Archimedean semigroup with the kernel K. Then*

(i) *$\mathcal{D}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are congruence relations on S;*

(ii) *$\mathcal{D} = \varrho_K$ and $S/\mathcal{D}$ is a nil-semigroup;*

(iii) *$S/\mathcal{L}$ is a nil-extension of a right zero band, $S/\mathcal{R}$ is a nil-extension of a left zero band, and $S/\mathcal{H}$ is a nil-extension of a rectangular band.*

Proof. The assertions (i) and (ii) were proved by Shevrin [10], and (iii) is an immediate consequence of (i) and (ii). □

By the next result, we prove that the above-considered congruence relations on a completely Archimedean semigroup form several factor congruence pairs.

Theorem 3.2. *Let S be a completely Archimedean semigroup with the kernel K. Then*

$$\overline{\mathcal{L}}_K \cap \mathcal{R} = \overline{\mathcal{R}}_K \cap \mathcal{L} = \overline{\mathcal{D}}_K \cap \mathcal{H} = \widehat{\mathcal{L}}_K \cap \widehat{\mathcal{R}}_K = \Delta_S.$$

Proof. Assume $(a,b) \in \overline{\mathcal{L}}_K \cap \mathcal{R}$. If $a, b \notin K$, then $(a,b) \in \mathcal{R} \subseteq \varrho_K$ yields $a = b$. Suppose $a, b \in K$. Since a and b are $\mathcal{R}$-related in S, they are also $\mathcal{R}$-related in K, i.e., $a, b \in R$ where R is some $\mathcal{R}$-class of K. But R is a right group and e is a left identity in R for each $e \in E(R)$ by Lemma VI.3.1.2 in [2]. Now $ea = a$ and $eb = b$. Since $(a,b) \in \overline{\mathcal{L}}_K$, it follows that $ea = eb$. Therefore, $a = b$, which is to be proved. Hence, we have proved $\overline{\mathcal{L}}_K \cap \mathcal{R} = \Delta_S$. Similarly, we can prove $\overline{\mathcal{R}}_K \cap \mathcal{L} = \Delta_S$.

Now assume $(a,b) \in \overline{\mathcal{D}}_K \cap \mathcal{H}$. As in the previous case, $a, b \notin K$ yields $a = b$. Let $a, b \in K$. Then $(a,b) \in \mathcal{H}$ implies $a, b \in G_e$ for some $e \in E(S)$, hence $a = eae$ and $b = ebe$. But $(a,b) \in \overline{\mathcal{D}}$ yields $eae = ebe$, so we have $a = b$. Therefore, $\overline{\mathcal{D}}_K \cap \mathcal{H} = \Delta_S$.

Let $(a,b) \in \widehat{\mathcal{L}}_K \cap \widehat{\mathcal{R}}_K = \mathcal{D} \cap \overline{\mathcal{L}}_K \cap \overline{\mathcal{R}}_K$. If $a, b \notin K$, then $(a,b) \in \mathcal{D}$ implies $a = b$. Now suppose $a, b \in K$. Let L be the $\mathcal{L}$-class of K containing a and R the $\mathcal{R}$-class of K containing b. Then L is a left group and R is a right group. By Lemma VI.3.1.2 in [2], for arbitrary $e \in E(L)$ and $f \in E(R)$, e is a right identity in L and f is a left identity in R. Therefore, $ae = a$ and $fb = b$. On the other hand, $(a,b) \in \overline{\mathcal{L}}_K \cap \overline{\mathcal{R}}_K$ yields $ae = be$ and $fa = fb$, i.e., $a = be$ and $fa = b$. Now we have $a = be = (fb)e = f(be) = fa = b$, which is to be proved. Therefore, $\widehat{\mathcal{L}}_K \cap \widehat{\mathcal{R}}_K = \Delta_S$. This completes the proof of the theorem. □

Using the above theorem we immediately obtain the following.

Corollary 3.3. *Let S be a completely Archimedean semigroup with the kernel K. Then S is a subdirect product of the following semigroups:*

(1) *$S/\overline{\mathcal{L}}_K$ and $S/\mathcal{R}$;*

(2) *$S/\overline{\mathcal{R}}_K$ and $S/\mathcal{L}$;*

(3) *$S/\overline{\mathcal{D}}_K$ and $S/\mathcal{H}$;*

(4) *$S/\widehat{\mathcal{D}}_K$, $S/\mathcal{L}$ and $S/\mathcal{R}$;*

(5) *$S/\widehat{\mathcal{L}}_K$ and $S/\widehat{\mathcal{R}}_K$;*

(6) *$S/\mathcal{D}$, $S/\widehat{\mathcal{L}}_K$ and $S/\widehat{\mathcal{R}}_K$.*

Unfortunately, in the general case, when S is a completely Archimedean semigroup, we have not much information about the structure of the semigroups $S/\overline{\mathcal{G}}_K$ and $S/\widehat{\mathcal{G}}_K$, for $\mathcal{G} \in \{\mathcal{L}, \mathcal{R}, \mathcal{D}\}$. However, in the further text, we will determine the structure of these semigroups in the case when the idempotents of S form a subsemigroup, i.e., when S is a nil-extension of a rectangular group.

First, we prove the following lemma.

Lemma 3.4. *Let S be a nil-extension of a rectangular group K. Then $eaf = efaef$, for all $a \in S$ and $e, f \in E(S)$.*

Proof. Assume $a \in S$ and $e, f \in E(S)$. Since $ea \in K$, we see $ea \in G_g$ for some $g \in E(S)$, hence $eg = g$ and $gf = egf = ef$ since $E(S)$ is a rectangular band. Now we have $eaf = eagf = eaef$. Further, $aef \in G_h$ for some $h \in E(S)$, so $hef = h$ and $eh = ehef = ef$. Therefore, $eaf = eaef = ehaef = efaef$, which is to be proved. □

The next theorem is one of the most important results of the present paper. It gives some new characterizations of semigroups which are nil-extensions of rectangular groups in terms of the congruence relations introduced here.

Theorem 3.5. *Let S be a completely Archimedean semigroup with the kernel K. Then the following conditions are equivalent:*

(i) *S is a nil-extension of a rectangular group;*

(ii) *$S/\overline{\mathcal{D}}_K$ is a group;*

(iii) *$S/\overline{\mathcal{D}}_K$ is a nil-extension of a group;*

(iv) *$S/\overline{\mathcal{L}}_K$ is a nil-extension of a right group;*

(v) *$S/\overline{\mathcal{R}}_K$ is a nil-extension of a left group;*

(vi) *$S/\widehat{\mathcal{D}}_K$ is a nil-extension of a group;*

(vii) *$S/\widehat{\mathcal{L}}_K$ is a nil-extension of a right group;*

(viii) *$S/\widehat{\mathcal{R}}_K$ is a nil-extension of a left group.*

Proof. (i) ⇒ (ii). Let (i) hold, i.e., K is a rectangular group. Assume $a \in S$ and $e, f, g \in E(S)$. By Lemma 3.4, we have $e(ag)f = ea(gf) = (egf)a(egf) = efaef = eaf$, so $(a, ag) \in \overline{\mathcal{D}}_K$. Since $ag \in K = Reg(S)$, each $\overline{\mathcal{D}}_K$-class of S contains a regular element, and hence, $S/\overline{\mathcal{D}}_K$ is a regular semigroup.

On the other hand, for arbitrary $e, f, g, h \in E(S)$, we have $egf = ef = ehf$, hence $(g, h) \in \overline{\mathcal{D}}_K$. Since every $\overline{\mathcal{D}}_K$-class of S, which is an idempotent in $S/\overline{\mathcal{D}}_K$, contains an idempotent from S, by Corollary IV.3 in [2], we conclude that $S/\overline{\mathcal{D}}_K$ is a regular semigroup containing exactly one idempotent, so it is a group, which is to be proved.

(i) ⇒ (iii)–(viii). As is well known, a factor semigroup of a π-regular semigroup is also π-regular, so $S/\overline{\mathcal{D}}_K$, $S/\overline{\mathcal{L}}_K$ and $S/\overline{\mathcal{R}}_K$, $S/\widehat{\mathcal{D}}_K$, $S/\widehat{\mathcal{L}}_K$, and $S/\widehat{\mathcal{R}}_K$ are all π-regular semigroups. By the proof of the first implication,

$S/\widehat{\mathcal{D}}_K$ has exactly one idempotent, which yields (iii) and (vi). On the other hand, for arbitrary $e, f, g \in E(S)$, we have $efg = eg$ and $fge = fe$, which means $(fg, g) \in \widehat{\mathcal{L}}_K$ and $(fg, f) \in \widehat{\mathcal{R}}_K$ for all $f, g \in E(S)$. Therefore, $E(S/\widehat{\mathcal{L}}_K)$ is a right zero band and $E(S/\widehat{\mathcal{R}}_K)$ is a left zero band. So (iv), (vii), (v), and (viii) hold.

(ii) $\Rightarrow$ (i). If $S/\overline{\mathcal{D}}_K$ is a group, then $(g, h) \in \overline{\mathcal{D}}_K$ for all $g, h \in E(S)$, i.e., $egf = ehf$ for all $e, f, g, h \in E(S)$. If we set $e = g = f$, then $g = ghg$. Hence, $E(S)$ is a rectangular band, so we have proved (i).

(iii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i). These implications can be proved similarly as the previous one.

(iv) $\Rightarrow$ (i). Since the $\overline{\mathcal{L}}_K$-classes of idempotents of S are idempotents in $S/\overline{\mathcal{L}}_K$ and $E(S/\overline{\mathcal{L}}_K)$ is a right zero band, we have $(gh, h) \in \overline{\mathcal{L}}_K$ for all $g, h \in E(S)$, i.e., $egh = eh$ for all $e, g, h \in E(S)$. Letting $e = h$, we have $h = hgh$, so $E(S)$ is a rectangular band, which is to be proved.

Similarly, we can prove (v) $\Rightarrow$ (i), (vii) $\Rightarrow$ (i), and (viii) $\Rightarrow$ (i). □

The equivalence of the conditions (ii) and (iii) in Theorem 3.5 motivates us to state the following problem.

Problem. If S is a nil-extension of a rectangular group K, are $S/\overline{\mathcal{L}}_K$ and $S/\overline{\mathcal{R}}_K$ a right group and a left group, respectively?

When a semigroup S is a nil-extension of a rectangular group K, the relations $\widehat{\mathcal{L}}_K$, $\widehat{\mathcal{R}}_K$, and $\widehat{\mathcal{D}}_K$ have another nice property as shown by the following theorem.

Theorem 3.6. *Let S be a nil-extension of a rectangular group K. Then $\widehat{\mathcal{L}}_K \cdot \widehat{\mathcal{R}}_K = \widehat{\mathcal{R}}_K \cdot \widehat{\mathcal{L}}_K = \widehat{\mathcal{D}}_K$.*

Proof. Since $\overline{\mathcal{L}}_K \subseteq \overline{\mathcal{D}}_K$ and $\overline{\mathcal{R}}_K \subseteq \overline{\mathcal{D}}_K$, we have $\widehat{\mathcal{L}}_K \subseteq \widehat{\mathcal{D}}_K$ and $\widehat{\mathcal{R}}_K \subseteq \widehat{\mathcal{D}}_K$, hence $\widehat{\mathcal{L}}_K \cdot \widehat{\mathcal{R}}_K \subseteq \widehat{\mathcal{D}}_K$ and $\widehat{\mathcal{R}}_K \cdot \widehat{\mathcal{L}}_K \subseteq \widehat{\mathcal{D}}_K$. Therefore, it remains to prove $\widehat{\mathcal{D}}_K \subseteq \widehat{\mathcal{L}}_K \cdot \widehat{\mathcal{R}}_K$ and $\widehat{\mathcal{D}}_K \subseteq \widehat{\mathcal{R}}_K \cdot \widehat{\mathcal{L}}_K$.

Assume $(a, b) \in \widehat{\mathcal{D}}_K$. If $a = b$, then $(a, b) \in \widehat{\mathcal{L}}_K \cdot \widehat{\mathcal{R}}_K$ and $(a, b) \in \widehat{\mathcal{R}}_K \cdot \widehat{\mathcal{L}}_K$, which is to be proved. Otherwise, if $a \neq b$, then $a, b \in K$, so $a \in G_e$ and $b \in G_f$ for some $e, f \in E(K)$. From this and by $(a, b) \in \widehat{\mathcal{D}}_K$, we have $af = eb$ and $fa = be$. Let $x = fa = be$ and $y = af = eb$. Then for arbitrary $g, h \in E$, we have $gx = gfa = gfea = gea = ga$ and $xh = beh = bfeh = bfh = bf$, hence $(a, x) \in \widehat{\mathcal{L}}_K$ and $(x, b) \in \widehat{\mathcal{R}}_K$, so $(a, b) \in \widehat{\mathcal{L}}_K \cdot \widehat{\mathcal{R}}_K$. On the other hand, for arbitrary $g, h \in E(K)$, we also have $yg = afg = aefg = aeg = ag$ and $hy = heb = hefb = hfb = hb$, hence $(a, y) \in \widehat{\mathcal{R}}_K$ and $(y, b) \in \widehat{\mathcal{L}}_K$, so $(a, b) \in \widehat{\mathcal{R}}_K \cdot \widehat{\mathcal{L}}_K$. This completes the proof of the theorem. □

Finally, we give a new proof of the following decomposition theorem due to Putcha [8] and Ren, Shum and Guo [9].

Theorem 3.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a rectangular group;*

(ii) *S is a subdirect product of a group and a nil-extension of a rectangular band;*

(iii) *S is a subdirect product of a group, a nil-extension of a left zero band and a nil-extension of a right zero band;*

(iv) *S is a spined product of a nil-extension of a left group and a nil-extension of a right group (with respect to a nil-extension of a group).*

Proof. The equivalence of the conditions (i)–(iii) was proved by Putcha [8], and (i)$\Leftrightarrow$(iv) was proved by Ren et al. [9]. But our results above give another proof of these equivalences. Namely, the implication (i) $\Rightarrow$ (ii) is an immediate consequence of Theorem 3.5 and Corollary 3.3(3), the implication (i) $\Rightarrow$ (iii) is a consequence of Theorem 3.5, Lemma 3.1 and Corollary 3.3(4), and the implication (i) $\Rightarrow$ (iv) follows by Theorems 3.5 and 3.6.

On the other hand, the proofs of the reverse implications were omitted in [8] and [9] as easier parts of the proofs. But these parts of the proofs are not so evident because the π-regularity and regularity are not preserved under subdirect products in general. We prove the reverse implications in order to show the reason for which the π-regularity is preserved in this case.

(ii) $\Rightarrow$ (i). Let $S \subseteq G \times T$ be a subdirect product of a group G and a semigroup T that is a nil-extension of a rectangular band. Then $E(S) \subseteq E(G) \times E(T)$, and if $E(S)$ is non-empty, it is a rectangular band. Therefore, it remains to prove the π-regularity of S. Let $u \in S$ and $u = (a, x)$ for some $a \in G$ and $x \in T$. Let b denote the inverse of a in G. Then there exists $y \in T$ such that $(b, y) \in S$ and there exists $n \in \mathbb{N}$ such that $x^n, y^n \in Reg(T) = E(T)$, Since $E(T)$ is a rectangular band, we see $u^n v^n u^n = (a^n b^n a^n, x^n y^n x^n) = (a^n, x^n) = u^n$ where $v = (b, y)$. Therefore, $u^n \in Reg(S)$, which is to be proved.

Similarly, we can prove the implication (iii) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (i). Let $S = \{(u, v) \in P \times Q \,|\, u\varphi = v\psi\}$ be a spined product of semigroups P and Q with respect to a semigroup H and homomorphisms φ of P onto H and ψ of Q onto H, where P is a nil-extension of a left group, Q is a nil-extension of a right group and H is a nil-extension of a group. Then $E(S) \subseteq E(P) \times E(Q)$, and if $E(S)$ is non-empty, it is a rectangular band. Therefore, it remains to prove the π-regularity of S.

Assume $a \in S$. Then $a = (u, v) \in P \times Q$ with $u\varphi = v\psi$ and there exists $n \in \mathbb{N}$ such that u^n and v^n are regular. Let $x \in V(u^n)$ and $y \in V(v^n)$. Then $x\varphi$ is an inverse of $(u\varphi)^n$, $y\psi$ is an inverse of $(v\psi)^n$, and $(u\varphi)^n = (v\psi)^n$. But a regular element of a nil-extension of a group can have at most one inverse, so $x\varphi = y\psi$. Now we have $(x, y) \in S$ and it is an inverse of a^n. Thus, S is π-regular. $\square$

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