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QUASI-SEMILATTICE DECOMPOSITIONS OF SEMIGROUPS WITH ZERO

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ABSTRACT. In the present paper we introduce a new type of decompositions of semigroups with zero. These are decompositions carried by a partially ordered set, called quasi-semilattice decompositions. We prove that such decompositions form a complete lattice and we characterize this lattice in terms of quasi-orders and ideals. We show that quasi-semilattice decompositions generalize two important kinds of decompositions: semilattice decompositions of semigroups and orthogonal decompositions of semigroups with zero, and that some well-known results concerning these decompositions can be obtained from the related results concerning quasi-semilattice decompositions.

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1. Preliminaries

T. Tamura in [15], 1975, started a new way in studying semilattice decompositions of semigroups, based on usage of quasi-orders on a semigroup. The starting point in such an investigation was the well-known result given by G. Birkhoff in [1], by which to any quasi-order ξ on a set X we can associate an equivalence relation on X, called its *natural equivalence*, such that the related factor set is partially ordered by a partial ordering which is induced in a natural way by ξ. In the above mentioned paper, T. Tamura found several conditions on a quasi-order on a semigroup under which its natural equivalence is a semilattice congruence, i.e. the related partially ordered set is a semilattice. This investigation was recently continued and developed by the authors in [5], where semilattice decompositions and corresponding quasi-orders were studied through certain sublattices of the lattice of ideals of a semigroup.

In the present paper we apply a similar methodology to studying quasi-semilattice decompositions of semigroups with zero, which we introduce here. We prove that these decompositions form a complete lattice and we characterize this lattice in through certain lattices of quasi-orders and certain sublattices of the lattice of ideals of a semigroup. Note that various properties of quasi-orders on a semigroup with zero, which will be used here, were studied in the previous paper of the authors [6].

Throughout this paper, $S = S^0$ will mean that S is a semigroup with zero 0, and if $S = S^0$, we will write 0 instead $\{0\}$, and if A is a subset of S, then $A^\bullet = A - 0$, $A^0 = A \cup 0$ and $A' = (S - A)^0$.

A subset A of a semigroup S is: *consistent*, if for $x, y \in S$, $xy \in A$ implies $x, y \in A$, *completely semiprime*, if for $x \in S$, $n \in \mathbf{Z}^+$, $x^n \in A$ implies $x \in A$, and it is *completely prime*, if for $x, y \in S$, $xy \in A$ implies $x \in A$ or $y \in A$. For semigroups with zero the authors in [6] introduced the following more general notions: A subset A of a semigroup $S = S^0$ is: *0-consistent*, if $A^\bullet$ is consistent, *completely 0-semiprime*, if $A^\bullet$ is completely semiprime, and it is *completely 0-prime*, if $A^\bullet$ is completely prime.

If ξ is a binary relation on a set A, ξ^{-1} will denote the relation defined by: $a \, \xi^{-1} \, b \Leftrightarrow b \, \xi \, a$, for $a \in A$, $a\xi = \{x \in A \mid a \, \xi \, x\}$, $\xi a = \{x \in A \mid x \, \xi \, a\}$. A relation ξ on a semigroup $S = S^0$ is *left 0-restricted* (*right 0-restricted*) if $0\xi = 0$ ($\xi 0 = 0$), and it is *0-restricted* if it is both left and right 0-restricted, i.e. if $0\xi = \xi 0 = 0$.

By a *quasi-order* we mean a reflexive and transitive binary relation. For a quasi-order ξ on a set A, $\widetilde{\xi}$ will denote the *natural equivalence* of ξ, i.e. an equivalence relation on A defined by: $\widetilde{\xi} = \xi \cap \xi^{-1}$. By Lemma II 1.1 of [1], if X and Y be two $\widetilde{\xi}$-classes of A, then $x \ \xi \ y$ either for no $x \in X$, $y \in Y$ or for all $x \in X$, $y \in Y$, and a relation $\leq$ on the quotient set $A/\widetilde{\xi}$ defined by: $X \leq Y$ if and only if $x \ \xi \ y$ for some (hence all) $x \in X$, $y \in Y$, is a partial order on $A/\widetilde{\xi}$, called the *natural partial order* of ξ.

A binary relation ξ on a semigroup S is *positive*, if $a \ \xi \ ab$ and $b \ \xi \ ab$, for all $a, b \in S$, and it has the *cm-property* (*common multiple property*) if for $a, b, c \in S$, $a \ \xi \ c$ and $b \ \xi \ c$ implies $ab \ \xi \ c$, [14]. If $S = S^0$, then ξ is: *0-positive*, if for $a, b \in S$, $ab \neq 0$ implies $a \ \xi \ ab$ and $b \ \xi \ ab$, and it has the *0-cm-property*, if for $a, b, c \in S$, $ab \neq 0$, $a \ \xi \ c$ and $b \ \xi \ c$ implies $ab \ \xi \ c$ [6].

Let L be a complete lattice. A sublattice K of L is a *complete sublattice* of L if it contains the meet and the join of any its nonempty subset, it is a *1-sublattice* (*0-sublattice*) if it contains the unity (zero) of L, and it is a *0,1-sublattice* if it is both 1-sublattice and 0-sublattice of L. A subset A of a lattice L is *meet-dense* in L if any element of L can be represented as the meet of some subset of A.

For undefined notions and notations we refer to [1], [7], [8] and [14].

The following lemma proved in [6] will be used in the proof of the main theorem of this paper.

Lemma 1. *A quasi-order ξ on a semigroup $S = S^0$ is 0-positive and it satisfies the 0-cm-property if and only if for all $a, b \in S$, $ab \neq 0$ implies $a\xi \cap b\xi = (ab)\xi$.*

2. The Main Results

Semigroups with zero have a specific structure and many authors studied various types of decompositions more suitable for semigroups with zero than other types of decompositions. For example, E. S. Lyapin in [11] and [12], and Š. Schwarz in [13] introduced orthogonal decompositions of semigroups with zero, which were investigated in details by S. Bogdanović and

M. Ćirić in [2] and [3]. The authors in [3] also studied decompositions of semigroups with zero into a left, right and matrix sum of semigroups. In the Shevrin's text [14] we meet so-called 0-band decompositions, which by A. V. Kelarev [9] were attributed to O. B. Kozhevnikov [10].

The starting idea in all mentioned decompositions is representation of a semigroup $S = S^0$ in the form

$$(1) \qquad S = \bigcup_{\alpha \in Y} S_\alpha, \qquad \text{where } S_\alpha \cap S_\beta = 0, \quad \text{for } \alpha \neq \beta,\ \alpha, \beta \in Y,$$

and $S_\alpha \neq 0,\ \alpha \in Y$, are subsemigroups of S. When S is so represented, we will say that S is a *0-sum* of semigroups S_α, $\alpha \in Y$, and the partition of S whose components are 0 and $S_\alpha^\bullet$, $\alpha \in Y$, will be called a *0-decomposition* of S, and S_α, $\alpha \in Y$, will be called *summands* of this 0-decomposition. The set of all 0-decompositions of a semigroup $S = S^0$, considered as a subset of the partition lattice of S, is a poset under the usual ordering of partitions, and it can be easily proved that this poset is a complete lattice dually isomorphic to the lattice of 0-restricted equivalence relations on S satisfying the 0-*cm*-property.

Various special types of 0-decompositions one can obtain assuming a special index set Y and requiring that the multiplication on S is carried by the properties of Y. So one can assume that Y is a partial groupoid and that the multiplication on S is carried by Y according to the following rule:

$$(2) \qquad \begin{cases} S_\alpha S_\beta \subseteq S_{\alpha\beta}, & \text{if } \alpha\beta \text{ is defined,} \\ S_\alpha S_\beta = 0, & \text{otherwise.} \end{cases}$$

$(\alpha, \beta \in Y)$. In this case Y is called the *carrier* of this 0-decomposition.

In this paper we will consider 0-decompositions carried by an arbitrary partially ordered set. Namely, we will assume that Y is a partially ordered set and the partial multiplication on Y is defined as the meet of a couple of elements of Y, whenever it exists, and so carried 0-decompositions will be called *quasi-semilattice decompositions* and related 0-sums will be called *quasi-semilattice sums*, since they are an analogue of semilattice decompositions, as we will see latter.

The authors in [6] investigated 0-restricted 0-positive quasi-orders with the 0-cm-property on a semigroup $S = S^0$, and they proved that such quasi-orders form a complete lattice. Using this lattice we here characterize the poset of quasi-semilattice decompositions of a semigroup with zero.

Theorem 1. *The poset of quasi-semilattice decompositions of a semigroup $S = S^0$ is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property.*

Proof. Let L denote the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property. To prove this theorem, it is enough to establish an order-isomorphism between L and the poset of quasi-semilattice decompositions of S. For $\xi \in L$ let

$$\mathcal{D}_\xi = \{(a\widetilde{\xi})^0 \mid a \in S^\bullet\}.$$

Let us prove that $\mathcal{D}_\xi$ is a quasi-semilattice decomposition of S. Let Z denote the quotient set of S, with respect to $\widetilde{\xi}$, let the $\widetilde{\xi}$-class of 0 be denoted also by 0, let $Y = Z - 0$, and for $\alpha \in Y$, let S_α denote the related $\widetilde{\xi}$-class of S with 0 adjoined. Clearly, $\mathcal{D}_\xi = \{S_\alpha \mid \alpha \in Y\}$.

Assume $\alpha \in Y$ and $a, b \in S_\alpha$. If $ab = 0$, then $ab \in S_\alpha$. Assume that $ab \neq 0$. Then $a, b \in S_\alpha^\bullet$, whence $a \;\widetilde{\xi}\; b$, i.e. $a\xi = b\xi$. By Lemma 1, $(ab)\xi = a\xi \cap b\xi = a\xi$, whence $ab \;\widetilde{\xi}\; a$, so $ab \in S_\alpha^\bullet$. Therefore, S_α is a subsemigroup of S, for any $\alpha \in Y$, so $\mathcal{D}_\xi$ is a 0-decomposition of S.

Let $\leq$ denote the natural partial order of ξ^{-1}. Then Y becomes a poset and Z can be assumed to be a quasi-semilattice determined by the poset Y. Assume $a, b \in S$ such that $ab \neq 0$. Then $a \in S_\alpha^\bullet$, $b \in S_\beta^\bullet$ and $ab \in S_\gamma^\bullet$, for some $\alpha, \beta, \gamma \in Y$. By the 0-positivity of ξ, γ is a lower bound of α and β. Let δ be any lower bound of α and β and let $c \in S_\delta^\bullet$. Then $a \;\xi\; c$ and $b \;\xi\; c$, whence $ab \;\xi\; c$, by the 0-cm-property, whence $\delta \leq \gamma$. Therefore, γ is the meet of α and β. By this it follows that (2) holds, so $\mathcal{D}_\xi$ is a quasi-semilattice decomposition of S.

Conversely, let $\mathcal{D} = \{S_\alpha \mid \alpha \in Y\}$ be a quasi-semilattice decomposition of S. Define a relation $\xi_{\mathcal{D}}$ on S by

$$(3) \qquad a \;\xi_{\mathcal{D}}\; b \Leftrightarrow a = b = 0 \;\text{ or }\; \Big((\exists \alpha, \beta \in Y)\; a \in S_\alpha^\bullet,\; b \in S_\beta^\bullet,\; \beta \leq \alpha \Big),$$

for $a, b \in S$. Clearly, $\xi_{\mathcal{D}}$ is a 0-restricted quasi-order on S. Assume $a, b \in S$ such that $ab \neq 0$. Then $a \in S^{\bullet}_{\alpha}$, $b \in S^{\bullet}_{\beta}$, for some $\alpha, \beta \in Y$, $\alpha\beta \neq 0$, and $ab \in S^{\bullet}_{\alpha\beta}$. Since $\alpha\beta$ is a lower bound of α and β, then $\xi_{\mathcal{D}}$ is 0-positive. Further, assume $a, b, c \in S$ such that $a \; \xi_{\mathcal{D}} \; c$, $b \; \xi_{\mathcal{D}} \; c$ and $ab \neq 0$. Then $a \in S^{\bullet}_{\alpha}$, $b \in S^{\bullet}_{\beta}$, $c \in S^{\bullet}_{\gamma}$, for some $\alpha, \beta, \gamma \in Y$, and γ is a lower bound of α and β. Now, $\gamma \leq \alpha\beta$, whence $ab \; \xi_{\mathcal{D}} \; c$. Therefore, $\xi_{\mathcal{D}}$ satisfies the 0-*cm*-property. Further, if $\alpha \in Y$ and $a \in S^{\bullet}_{\alpha}$, then $S^{\bullet}_{\alpha} = a\widetilde{\xi_{\mathcal{D}}}$, whence $\mathcal{D} = \mathcal{D}_{\xi_{\mathcal{D}}}$. Hence, the mapping $\xi \mapsto \mathcal{D}_{\xi}$ maps L onto the poset of quasi-semilattice decompositions of S.

Finally, assume $\xi, \eta \in L$. Then $\xi \subset \eta$ if and only if $\widetilde{\xi} \subseteq \widetilde{\eta}$, and also, $\widetilde{\xi} \subseteq \widetilde{\eta}$ if and only if $\mathcal{D}_{\eta} \leq \mathcal{D}_{\xi}$. Therefore, the mapping $\xi \mapsto \mathcal{D}_{\xi}$ is an order isomorphism of L onto the poset of all quasi-semilattice decompositions of S. This completes the proof of the theorem. $\square$

The authors in [6] also proved that the lattice of 0-restricted 0-positive quasi-orders on a semigroup $S = S^0$ satisfying the 0-*cm*-property is isomorphic to the lattice of left 0-restricted positive quasi-orders satisfying the 0-*cm*-property on S. By this and by Theorem 1 we obtain the following corollary:

Corollary 1. *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ is dually isomorphic to the lattice of left 0-restricted positive quasi-orders on S satisfying the 0-cm-property.*

For a semigroup $S = S^0$, the set of completely 0-semiprime ideals of S is a complete 0,1-sublalttice of the lattice of ideals of S, and it will be denoted by $\mathcal{I}d^{\mathbf{c0s}}(S)$. For a sublattice K of $\mathcal{I}d^{\mathbf{c0s}}(S)$ we say that it satisfies the *completely 0-prime ideal-property*, briefly the *c-0-pi-property*, if the set of completely 0-prime ideals from K is meet-dense in K, i.e. if any completely 0-semiprime ideal from K can be represented as the intersection of some family of completely 0-prime ideals from K [6]. Note that this property is an analogue of the *completely prime ideal property* (see [5]), which appears in many algebraic theories and is closely related to the well-known Prime Ideal theorems.

The authors in [6] proved that the lattice of 0-restricted 0-positive quasi-orders with the 0-*cm*-property on a semigroup $S = S^0$ is also dually isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi-property. By this and by Theorem 1 we obtain the following:

Corollary 2. *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ is isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$, satisfying the c-0-pi-property.*

Note that the above used methodology comes from the methodology developed by T. Tamura in [15] and the authors in [5] for studying semilattice decompositions of semigroups. However, quasi-semilattice decompositions of semigroups with zero can be treated as a generalization of semilattice decompositions of semigroups, since the following holds:

Proposition 1. *The lattice of semilattice decompositions of a semigroup S is isomorphic to the lattice of quasi-semilattice decompositions of a semigroup S^0 obtained from S by adjoining the zero.*

Proof. This is an immediate consequence of the fact that S^0 has no zero divisors. □

By the previous proposition, many results of T. Tamura [15] and the authors [5] concerning semilattice decompositions of a semigroup S can be obtained from the related results concerning quasi-semilattice decompositions of the semigroup S^0 obtained from S by adjoining the zero.

On the other hand, quasi-semilattice decompositions are an immediate generalization of orthogonal decompositions of semigroups with zero. Namely, orthogonal decompositions are exactly quasi-semilattice decompositions carried by discrete partially ordered sets, by which we mean partially ordered sets in which distinct elements are incomparable. Many results from [2] and [3] concerning orthogonal decompositions can be obtained from some results concerning quasi-semilattice decompositions. For example, when $\mathcal{D}$ is an orthogonal decomposition, then the quasi-order $\xi_{\mathcal{D}}$ defined in (3) is exactly a 0-restricted 0-consistent equivalence relation used in [3] for studying orthogonal decompositions.

Let us note that semilattice decompositions and orthogonal decompositions of semigroups with zero are among the rare kinds of decompositions having the so-called *atomic property*. This means that the components in the greatest semilattice decompositions are semilattice indecomposable and similarly, that the summands of the greatest orthogonal decomposition of a semigroup with zero are orthogonally indecomposable. Similarly we will define a *quasi-semilattice indecomposable* semigroup as a semigroup $S = S^0$ having the property that $\{S\}$ is the unique quasi-semilattice decomposition of S, and we will state the following problem:

Problem. Are the summands of the greatest quasi-semilattice decomposition of a semigroup with zero quasi-semilattice indecomposable?

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