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Rings satisfying some semigroup identities

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Abstract. Rings satisfying various identities of the form $x_1 \cdots x_n = w(x_1, \ldots, x_n)$, $|w| > n \geq 2$, have been considered many times. The purpose of this paper is to give a structural theorem for rings satisfying an arbitrary identity of this form. We introduce the notion of characteristic quadruplet (n, p, h, t) of such an identity, and using it we characterize rings satisfying this identity as ideal extensions of an n-nilpotent ring by a ring satisfying $x = x^{p+1}$ and satisfying also some additional conditions, determined by the numbers h and t, on nilpotents, idempotents and regular elements.

1. Introduction

Rings satisfying various semigroup identities have been considered many times. M. Petrich [13] described rings satisfying the identities $axy = axay$ and $xya = xaya$, called distributive rings. He proved that these rings are exactly direct sums of Boolean rings and of 3-nilpotent rings. A more general result was obtained by M. Ćirić and S. Bogdanović [4], for n-distributive rings, i.e. for rings satisfying the identities $ax_1 \cdots x_n = ax_1 \cdots ax_n$ and $x_1 \cdots x_n a = x_1 a \cdots x_n a$, $n \geq 2$. Left self distributive rings, i.e. rings in which the identity $axy = axay$ hold, were studied by G. F. Birkenmeier, H. Heatherly and T. Kepka [2]. Rings with $xy = (xy)^n$, $n \geq 2$, were considered by M. Ó Searcóid and D. McHale [12]. That these rings are direct sums of a J-ring (ring with the well-known Jacobson's property) and a null-ring was proved by S. Ligh and J. Luh [10] (see also S.-M. Lee [9]). Some sufficient conditions for rings to be direct sums of J-rings and of

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null-rings were given by H. E. Bell and S. Ligh [1]. Direct sums of nil-rings and of rings with Clifford's multiplicative semigroups and semigroup identities that give such sums were described by M. Ćirić and S. Bogdanović [4, 8]. Some properties of the commutator ideal of a ring satisfying the identity $x_1 \cdots x_n = w(x_1, \dots, x_n)$ were considered by M. S. Putcha and A. Yaqub [17]. For related results connected with such and other identities in semigroups we refer to M. Ćirić and S. Bogdanović [6, 7] and M. V. Sapir and E. V. Suhanov [19].

In the present paper we will give a structural theorem for rings satisfying an identity

$$x_1 \cdots x_n = w(x_1, \dots, x_n),$$

with $|w| > n \geq 2$. We will show that structure of such a ring is completely determined by a quadruplet (n, p, h, t) of natural numbers that characterize this identity, called its characteristic quadruplet. In Theorem 1 we prove that a ring satisfying the above identity with the characteristic quadruplet $(n, p, 0, 0)$ if and only if it is a direct sum of an n-nilpotent ring and a ring satisfying $x = x^{p+1}$. In Theorem 2 we give several structural characterizations for rings satisfying the above identity with an arbitrary characteristic quadruplet. We characterize such a ring as an ideal extension of an n-nilpotent ring by a ring satisfying $x = x^{p+1}$ and satisfying also some additional conditions, determined by the numbers h and t, on the nilpotents, on the idempotents and on the regular elements, and also as some special Everett sums of the above mentioned rings. At the end of the paper we give a consequent classification of semigroup identities over the two-element and the three-element alphabet.

Throughout this paper, $\mathbb{Z}^+$ will denote the set of all positive integers. If R is a ring, $\mathcal{M}R$ will denote the multiplicative semigroup of R, $E(R)$ will denote the set of all idempotents of R and $\mathrm{Reg}(R)$ will denote the set of all regular elements of R.

Let R be a ring with the zero 0. Throughout this paper we will write 0 instead of $\{0\}$. An element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}^+$, and R is a *nil-ring* if all of its elements are nilpotents. For $n \in \mathbb{Z}^+$, a ring R for which $R^n = 0$ will be called *n-nilpotent*. A 2-nilpotent ring will be called a *null-ring*. A regular ring whose idempotents are central will be called a *Clifford's ring*. These are also called *strongly regular rings*.

A semigroup S is *Archimedean* if for all $a, b \in S$ there exists $n \in \mathbb{Z}^+$ such that $a^n \in SbS$, and it is *completely Archimedean* if it is Archimedean and has a primitive idempotent.

A ring R is an *ideal extension* of a ring A by a ring B if R has an ideal I isomorphic to A and the factor ring R/I is isomorphic to B. Usually, we identify

A with I and B with R/I.

By A^+ we denote the *free semigroup* over an alphabet A and by A^* we denote the *free monoid* over A. For $n \in \mathbb{Z}^+$, $n \geq 4$, $A_n = \{x_1, x_2, \dots, x_n\}$, $A_3 = \{x, y, z\}$ and $A_2 = \{x, y\}$. By $|w|$ we denote the *length* of a word $w \in A^+$ and by $|x|_w$ we denote the number of appearances of a letter $x \in A$ in a word $w \in A^+$. A word $v \in A^+$ is a *left* (*right*) *cut* of a word $w \in A^+$ if $w = vu$ ($w = uv$), for some $u \in A^*$, and v is a *subword* of w if $w = u'vu''$, for some $u', u'' \in A^*$. By $h(w)$ ($t(w)$) we denote the first (last) letter of a word $w \in A_n^+$, called the *head* (*tail*) of w, and by $c(u)$ we denote the set of all letters which appear in w, called the *content* of w [16]. An expression $w(x_1, \dots, x_n)$ will mean that w is a word with $c(w) = \{x_1, \dots, x_n\}$. If $w \in A^+$ and $i \in \mathbb{Z}^+$, $i \leq |w|$, then $l_i(w)$ ($r_i(w)$) will denote the left (right) cut of w of the length i, $c_i(w)$ will denote the i-th letter of w and for $i, j \in \mathbb{Z}^+$, $i, j \leq |w|, i \leq j$, $m_i^j(w)$ will denote the subword w determined by: $w = l_{i-1}(w) m_i^j(w) r_{|w|-j}(w)$. For $n \in \mathbb{Z}^+$, Π_n will denote the word $x_1 x_2 \dots x_n \in A_n^+$. If $w \in A^+$ and $x \in A$, then $x \underset{l}{\parallel} w$ ($x \underset{r}{\parallel} w$) if $w = xv$ ($w = vx$), $v \in A^+$ and $x \notin c(v)$. Otherwise we will write $x \underset{l}{\nparallel} w$ ($x \underset{r}{\nparallel} w$).

Let $n \in \mathbb{Z}^+$, $w \in A_n^+$ and let S be a semigroup. By a *value* of the word w in S, for the *valuation* $a = (a_1, a_2, \dots, a_n)$, $a_i \in S$, $i \in \{1, 2, \dots, n\}$, in notation $w(a)$ or $w(a_1, a_2, \dots, a_n)$, we mean the element $w\varphi \in S$, where $\varphi : A_n^+ \to S$ is the homomorphism determined by $x_i \varphi = a_i$, $i \in \{1, 2, \dots, n\}$. Also, we then say that for $i \in \{1, 2, \dots, n\}$, the letter x_i *assumes a value* a_i in S, in notation $x_i := a_i$.

Let $u = v$ be a semigroup identity over an alphabet A_n, $n \in \mathbb{Z}^+$, and for $i \in \mathbb{Z}^+$, $1 \leq i \leq n$, let $p_i = ||x_i|_u - |x_i|_v|$. If some of the numbers $p_1, \dots, p_n$ are greater than 0, then we say that this identity is *periodic* and the number $p = \gcd(p_1, \dots, p_n)$ is the *period* of this identity [6]. Clearly, every semigroup (ring) satisfying a periodic identity is periodic.

For undefined notions and notations we refer to [11], [14], [15], [16], [18].

2. Everett sums of rings

A very important tool that will be used in the proof of the main result of this paper is well-known Everett's theorem that gives a way to construct all ideal extensions of a ring by another. In this section we give a definition of an Everett sum of rings and we consider more complicated multiplications in Everett sums of rings.

Let R be a ring. An endomorphism λ (ϱ) of the additive group of R, written on the left (right), is a *left* (*right*) *translation* of R if $\lambda(xy) = (\lambda x)y$ ($(xy)\varrho = x(y\varrho)$), for all $x, y \in R$. A left translation λ and a right translation ϱ of R are *linked* if $x(\lambda y) = (x\varrho)y$, for all $x, y \in R$, and in such a case the pair (λ, ϱ) is called a *bitranslation* of R. It is sometimes convenient to consider a bitranslation (λ, ϱ) as a bioperator denoted by a single letter, say π, which acts as λ, if it is written on the left, and as ϱ, if it is written on the right, i.e. $\pi x = \lambda x$ and $x\pi = x\varrho$, for $x \in R$. For any $a \in R$, the *inner left* (*right*) *translation* induced by a is the mapping λ_a (ϱ_a) of R into itself defined by $\lambda_a x = ax$ ($x\varrho_a = xa$), for $x \in R$, and the pair $\pi_a = (\lambda_a, \varrho_a)$ is called the *inner bitranslation* of R induced by a.

A left translation λ and a right translation ϱ of a ring R are *permutable* if $(\lambda x)\varrho = \lambda(x\varrho)$, for all $x \in R$, and a set T of bitranslations of R is *permutable* if for all $(\lambda, \varrho), (\lambda', \varrho') \in T$, λ and ϱ' are permutable.

The set $\Lambda(R)$ ($\mathrm{P}(R)$) of all left (right) translations of a ring R is a ring under the addition and the multiplication defined by:

$$(\lambda + \lambda')x = \lambda x + \lambda' x \qquad \big(\, x(\varrho + \varrho') = x\varrho + x\varrho' \,\big),$$
$$(\lambda\lambda')x = \lambda(\lambda' x) \qquad \big(\, x(\varrho\varrho') = (x\varrho)\varrho' \,\big),$$

for $\lambda, \lambda' \in \Lambda(R)$ ($\varrho, \varrho' \in \mathrm{P}(R)$) and $x \in R$. The subring $\Omega(R)$ of the direct sum of rings $\Lambda(R)$ and $\mathrm{P}(R)$, consisting of all bitranslations of R, is called the *translational hull* of R. Much information about the translational hulls of rings and semigroups can be found in [14] and [15].

Everett's theorem. *Let A and B be disjoint rings. Let θ be a function of B onto a set of permutable bitranslations of A, in notation $\theta : a \mapsto \theta^a \in \Omega(A)$, $a \in B$, and let $[,], \ \langle,\rangle : B \times B \to A$ be functions such that for all $a, b, c \in B$ the following conditions hold:*

(E1) $\theta^a + \theta^b - \theta^{a+b} = \pi_{[a,b]}$;
(E2) $\theta^a \cdot \theta^b - \theta^{ab} = \pi_{\langle a,b\rangle}$;
(E3) $\langle ab, c\rangle + \langle a, b\rangle\,\theta^c = \langle a, bc\rangle + \theta^a\,\langle b, c\rangle$;
(E4) $[0, 0] = 0$;
(E5) $[a, b] = [b, a]$;
(E6) $[a, b] + [a + b, c] = [a, b + c] + [b, c]$;
(E7) $[a, b]\theta^c + \langle a + b, c\rangle = [ac, bc] + \langle a, c\rangle + \langle b, c\rangle$;
(E8) $\theta^a[b, c] + \langle a, b + c\rangle = [ab, ac] + \langle a, b\rangle + \langle a, c\rangle$.

Define an addition and a multiplication on $R = A \times B$ by:

(E9) $(\alpha, a) + (\beta, b) = (\alpha + \beta + [a, b], a + b)$;
(E10) $(\alpha, a) \cdot (\beta, b) = (\alpha\beta + \langle a, b\rangle + \theta^a\beta + \alpha\theta^b, ab)$,

$\alpha, \beta \in A$, $a, b \in B$. Then $(R, +, \cdot)$ is a ring isomorphic to an ideal extension of A by B.

Conversely, every ideal extension of A by B can be so constructed.

The ring constructed as in Everett's theorem is called the *Everett sum* of rings A and B *by the triplet* $(\theta; [,]; \langle,\rangle)$ of functions and we denote it by $E(A, B; \theta; [,]; \langle,\rangle)$. The representation of a ring R as an Everett sum of some rings is called an *Everett representation* of R.

Much information about the Everett's theorem can be found in [14] and [18]. There we can see that an Everett representation $E(A, B; \theta; [,]; \langle,\rangle)$ of some ring R is determined by the choice of a set of representatives of the cosets of A in R. Namely, if for every coset $a \in B$ we choose a representative, in notation a', then the set $\{a' \mid a \in B\}$ determines the triplet $(\theta; [,]; \langle,\rangle)$ in the following way:

(E11) $\alpha\theta^a = \alpha \cdot a', \theta^a\alpha = a' \cdot \alpha, \quad \alpha \in A,\ a \in B;$

(E12) $[a, b] = a' + b' - (a + b)', \quad a, b \in B;$

(E13) $\langle a, b\rangle = a' \cdot b' - (a \cdot b)', \quad a, b \in B.$

Let $n \in \mathbb{Z}^+$ and let $w \in A_n^+$. If $X_1, X_2, \ldots, X_n$ are sets, then $w(X_1, X_2, \ldots, X_n)$ will denote the set obtained by substituting the sets $X_1, X_2, \ldots, X_n$ for the letters $x_1, x_2, \ldots, x_n$ in w, respectively, where juxtaposition in w is meant as Cartesian multiplication of sets. Let R be a ring, let P be a set of permutable bitranslations of R and let μ be an element of the Cartesian n-th power of $R \cup P$. If at least one projection of μ is in R, then $w(\mu)$ will denote the element of R obtained as follows: substitute the i-th projection of μ for the letter x_i, $i \in \{1, 2, \ldots, n\}$, and let juxtaposition in w mean either multiplication in $\mathcal{M}R$ and $\mathcal{M}\Omega(R)$, or the application of bitranslation from P on an element of R. Otherwise, if all the projections of μ are in P, then $w(\mu)$ will denote the value of w in the semigroup $\mathcal{M}\Omega(R)$, for the valuation μ.

Proposition 1. *Let $R = E(N, Q; \theta; [,]; \langle,\rangle)$, let $n \in \mathbb{Z}^+$, $n \geq 2$, $w = w(x_1, \ldots, x_n) \in A_n^+$, $|w| = k$, $a = (a_1, \ldots, a_n) \in Q^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$, $\xi_i = (\alpha_i, a_i)$, $i \in \{1, \ldots, n\}$, $\xi = (\xi_1, \ldots, \xi_n)$, and $\theta^a = (\theta^{a_1}, \ldots, \theta^{a_n})$. Then for*

$$\beta = \sum_{j=1}^{k-2} \langle (l_j(w))(a), (c_{j+1}(w))(a)\rangle\, (r_{k-j-1}(w))(\theta^a) + \langle (l_{k-1}(w))(a), (t(w))(a)\rangle\,, \tag{1}$$

the following statements hold:

$$w(\theta^a) = \theta^{w(a)} + \pi_\beta, \tag{2}$$

$$w(\xi) = \left(\sum_{\mu \in M_w} \Pi_k(\mu) + \beta, w(a) \right), \tag{3}$$

where $M_w = w(X_1,\dots,X_n) - \{\theta^a\}$, $X_i = \{\alpha_i, a_i\}$, $i \in \{1,\dots,n\}$.

Furthermore, if $\theta^b N \theta^c = 0$, *for all* $b, c \in Q$ *and if* $k \geq 3$, *then*

$$\beta = \big\langle (h(w))(a), (m_2^{k-1}(w))(a) \big\rangle\, \theta^{(t(w))(a)} + \langle (l_{k-1}(w))(a), (t(w))(a) \rangle\,. \tag{4}$$

Proof. All the statements (2)–(4) will be proved by induction on the length of w. Let $u = l_{k-1}(w)$, $v = t(w)$.

First we will prove (2). By (E2), this holds for all words of length 2. Assume that (2) hold for all words of length smaller than k. Then this holds for u, i.e., $u(\theta^a) = \theta^{u(a)} + \pi_{\beta'}$, where

$$\begin{aligned}\beta' = \sum_{j=1}^{k-3} \langle (l_j(u))(a), (c_{j+1}(u))(a) \rangle\, (r_{k-j-2}(u))(\theta^a) + \\ + \langle (l_{k-1}(u))(a), (t(u))(a) \rangle\,,\end{aligned} \tag{5}$$

so by (E2),

$$\begin{aligned} w(\theta^a) &= u(\theta^a) v(\theta^a) = \left(\theta^{u(a)} + \pi_{\beta'}\right) \theta^{v(a)} = \theta^{u(a)}\theta^{v(a)} + \pi_{\beta'}\theta^{v(a)} \\ &= \theta^{u(a)v(a)} + \pi_{\langle u(a), v(a)\rangle} + \pi_{\beta'}\theta^{v(a)} = \theta^{w(a)} + \pi_\beta, \end{aligned}$$

where $\beta = \beta'\theta^{v(a)} + \langle u(a), v(a)\rangle$. It is easy to verify that this β can be expressed as in (1). Thus, (2) holds.

Further, let we prove (3). By (E10), this holds for all words of length 2. Assume that (3) hold for all words of length smaller than k. Then this holds for u, i.e.,

$$u(\xi) = \left(\sum_{\mu \in M_u} \Pi_{k-1}(\mu) + \beta', u(a) \right),$$

with β as in (1). Then

$$\begin{aligned} w(\xi) = u(\xi)v(\xi) &= \left(\sum_{\mu \in M_u} \Pi_{k-1}(\mu) + \beta', u(a) \right) \cdot \Big(v(\alpha), v(a) \Big) \\ &= \left(\sum_{\mu \in M_u} \Pi_{k-1}(\mu) v(\alpha) + \beta' v(\alpha) + \theta^{u(a)} v(\alpha) \right. \\ &\qquad \left. + \sum_{\mu \in M_u} \Pi_{k-1}(\mu)\theta^{v(a)} + \beta'\theta^{v(a)} + \langle u(a), v(a)\rangle\,, u(a)v(a) \right). \end{aligned} \tag{6}$$

By (2), $u(\theta^a) = \theta^{u(a)} + \pi_{\beta'}$, so

$$\theta^{u(a)}v(\alpha) + \beta' v(\alpha) = u(\theta^a)v(\alpha). \tag{7}$$

On the other hand,

$$\beta' v(\alpha) + \langle u(a), v(a)\rangle = \beta, \tag{8}$$

where β is determined by (1). Finally,

$$\sum_{\mu\in M_u} \Pi_{k-1}(\mu)v(\alpha) + \sum_{\mu\in M_u} \Pi_{k-1}(\mu)\theta^{v(a)} + u(\theta^a)v(\alpha) = \sum_{\mu\in M_w} \Pi_k(\mu), \tag{9}$$

so by (6)–(9) we obtain (3).

Finally, let us prove (4), under the assumption $\theta^b N\theta^c = 0$, for all $b, c \in Q$. Clearly, this holds for all words of length 3. Assume that (4) hold for all words of length smaller than k. Then this holds for u, so

$$\begin{aligned}\beta' &= \left\langle (h(u))(a), (m_2^{k-2}(u))(a)\right\rangle \theta^{(t(u))(a)} + \langle (l_{k-2}(u))(a), (t(u))(a)\rangle \\ &= \left\langle (h(w))(a), (m_2^{k-2}(w))(a)\right\rangle \theta^{(c_{k-1}(w))(a)} + \langle (l_{k-2}(w))(a), (c_{k-1}(w))(a)\rangle \\ &= \theta^{(h(w))(a)} \left\langle (m_2^{k-2}(w))(a), (c_{k-1}(w))(a)\right\rangle + \left\langle (h(w))(a), (m_2^{k-1}(w))(a)\right\rangle,\end{aligned}$$

by (E3), where β' is determined by (1). Now

$$\begin{aligned}\beta &= \beta'\theta^{(t(w))(a)} + \langle (l_{k-1}(w))(a), (t(w))(a)\rangle \\ &= \left\langle (h(w))(a), (m_2^{k-1}(w))(a)\right\rangle \theta^{(t(w))(a)} + \langle (l_{k-1}(w))(a), (t(w))(a)\rangle,\end{aligned}$$

since $\theta^b N\theta^c = 0$, for all $b, c \in Q$. Therefore, (4) holds.

■

3. The main theorems

The subject of this paper is studying of rings satisfying an identity

$$x_1 \cdots x_n = w(x_1, \dots, x_n), \tag{10}$$

where $n \in \mathbb{Z}^+$, $n \geq 2$ and $|w| \geq n+1$. Let p be the period of the identity (10) and let

$$\begin{aligned} h &= \max\Big(\{i \mid w = x_1 \cdots x_i u(x_{i+1}, \dots, x_n)\} \cup 0\Big), \\ t &= \max\Big(\{l \mid w = u'(x_1, \dots, x_{n-l})x_{n-l+1} \cdots x_n\} \cup 0\Big). \end{aligned}$$

The quadruplet (n,p,h,t) will be called the *characteristic quadruplet* of the identity (10). Clearly, $h+t \le n-1$,

$$x_1 \cdots x_n = x_1 \cdots x_h u(x_{h+1}, \cdots, x_n), \tag{11}$$

with $x_{h+1} \underset{l}{\nparallel} u$, if $h \ge 1$,

$$x_1 \cdots x_n = u'(x_1, \dots, x_{n-t}) x_{n-t+1} \dots x_n, \tag{12}$$

with $x_{n-t} \underset{r}{\nparallel} u'$, if $t \ge 1$, and

$$x_1 \cdots x_n = x_1 \cdots x_h v(x_{h+1}, \cdots, x_{n-t}) x_{n-t+1} \cdots x_n, \tag{13}$$

with $x_{h+1} \underset{l}{\nparallel} v$, $x_{n-t} \underset{r}{\nparallel} v$, if $h \ge 1$ and $t \ge 1$.

In the special case of rings satisfying an identity with characteristic quadruplet of the form $(n,p,0,0)$, their structure can be immediately described, using Theorem 3 [7] and Theorem 2 [8].

Theorem 1. *Let* (10) *be an identity with characteristic quadruplet* $(n,p,0,0)$. *Then a ring* R *satisfies* (10) *if and only if* R *is a direct sum of an* n*-nilpotent ring and a ring satisfying the identity* $x = x^{p+1}$.

Further we will consider the general case.

Theorem 2. *Let* (10) *be an identity with characteristic quadruplet* (n,p,h,t). *Then the following conditions for a ring* R *are equivalent:*

(i) R *satisfies* (10)*;*

(ii) R *is an ideal extension of an* n*-nilpotent ring* N *by a ring satisfying the identity* $x = x^{p+1}$ *and*

$$N^{h+1} \cdot E(R) = E(R) \cdot N^{t+1} = E(R) \cdot N \cdot E(R) = 0; \tag{14}$$

(iii) R *is an ideal extension of an* n*-nilpotent ring* N *by a ring satisfying the identity* $x = x^{p+1}$ *and*

$$N^{h+1} \cdot Reg(R) = Reg(R) \cdot N^{t+1} = Reg(R) \cdot N \cdot Reg(R) = 0; \tag{15}$$

(iv) $R = E(N, Q; \theta; [\,,\,]; \langle\,,\,\rangle)$, *where* N *is an* n*-nilpotent ring,* Q *is a ring satisfying the identity* $x = x^{p+1}$, *and*

$$\theta^b N \theta^c = 0, \qquad \text{for all } b, c \in Q. \tag{16}$$

$$N^{h+1} \theta^b = \theta^b N^{t+1} = 0, \qquad \text{for each } b \in Q. \tag{17}$$

Proof. (i) $\Longrightarrow$ (ii). By Theorem 2 [19], $\mathcal{M}R$ is a semilattice of completely Archimedean semigroups, so by Theorem 1 [5], R is an ideal extension of a nil-ring N by a Clifford's ring Q. Clearly, N and Q satisfy (10), so by Lemma 1 [19], N is n-nilpotent, and by Lemma 4 [8], Q satisfies the identity $x = x^{p+1}$.

If $h = 0$, then by the proof of Theorem 1 [7], $N \cdot E(R) = 0$, and also $E(R) \cdot N \cdot E(R) = 0$. Let $h \neq 0$. Then (10) may be written as (11), with $x_{h+1} \underset{l}{\nparallel} u$.

Assume $e, f \in E(R)$, $\alpha \in N$. If $h(u) = x_{h+1}$, then $l = |x_{h+1}|_u \geq 2$, and for $x_i := e$, $1 \leq i \leq h$, $x_{h+1} := \alpha f$, $x_i := f$, $h+2 \leq i \leq n$, $e\alpha f = e(\alpha f)^l$, whence $e\alpha f = 0$, since $l \geq 2$ and $\alpha f \in N$. Let $h(u) \neq x_{h+1}$. Since $|w| \geq n+1$, then $l = |x_j|_w = |x_j|_u \geq 2$, for some $j \in \{h+1, \dots, n\}$. Now, for $x_i := e$, $1 \leq i \leq h-1$, $x_h := ef$, $x_j := \alpha f$, $x_i := f$, $h+1 \leq i \leq n$, $i \neq j$, $ef\alpha f = ef(\alpha f)^l$, so $ef\alpha f = 0$, since $l \geq 2$ and $\alpha f \in N$. Further, for $x_i := e$, $1 \leq i \leq h$, $x_{h+1} := \alpha f$, $x_i := e$, $h+2 \leq i \leq n$, we obtain $e\alpha f = ef(\alpha f)^{|x_{h+1}|_u} = 0$, since $efxf = 0$. This completes the proof for $E(S) \cdot N \cdot E(R) = 0$.

Assume $\alpha_1, \dots, \alpha_{h+1} \in N$, $e \in E(R)$. Let $l = |x_{h+1}|_u$ and let $x_i := \alpha_i$, $1 \leq i \leq h$, $x_{h+1} := \alpha_{h+1}e$, $x_i := e$, $h+2 \leq i \leq n$. If $h(u) = x_{h+1}$, then $l \geq 2$ and $\alpha_1 \cdots \alpha_h \alpha_{h+1} e = \alpha_1 \cdots \alpha_h (\alpha_{h+1}e)^l = 0$, since $E(R) \cdot N \cdot E(R) = 0$. If $h(u) \neq x_{h+1}$, then $\alpha_1 \cdots \alpha_h \alpha_{h+1} e = \alpha_1 \cdots \alpha_h e(\alpha_{h+1}e)^l = 0$, again by $E(R) \cdot N \cdot E(R) = 0$. Therefore, $N^{h+1} \cdot E(R) = 0$. Similarly we prove that $E(R) \cdot N^{t+1} = 0$.

(ii) $\Longrightarrow$ (iii). This follows immediately.

(iii) $\Longrightarrow$ (iv). By Everett's theorem, $R = E(N, Q; \theta; [,]; \langle,\rangle)$, where the triplet $(\theta; [,]; \langle,\rangle)$ can be determined by (E11)–(E13). By Lemma 2 [3], the set $\{a' \mid a \in Q\}$ of representatives can be chosen to be in $\mathrm{Reg}(R)$. By this and by (15) we obtain (iv).

(iv) $\Longrightarrow$ (i). First we note that by the well-known "Jacobson's $a^n = a$ theorem", Q is commutative, and by Lemma 4 [8], Q satisfies every semigroup identity of the period p.

Let $\xi \in R^n$ be as in Proposition 1. Then

$$\Pi_n(\xi) = \left(\sum_{\mu \in M_{\Pi_n}} \Pi_n(\mu) + \langle a_1, a_2 \cdots a_{n-1} \rangle \, \theta^{a_n} + \langle a_1 \cdots a_{n-1}, a_n \rangle \, , \Pi_n(a) \right)$$

$$w(\xi) = \left(\sum_{\mu \in M_w} \Pi_k(\mu) + \big\langle (h(w))(a), (m_2^{k-1}(w))(a) \big\rangle \, \theta^{(t(w))(a)} + \right.$$
$$\left. + \langle (l_{k-1}(w))(a), (t(w))(a) \rangle \, , w(a) \right),$$

where $k = |w|$. We distinguish the following four cases: 1. $h = t = 0$; 2. $h \neq 0$, $t \neq 0$; 3. $h \neq 0$, $t = 0$; 4. $h = 0$, $t \neq 0$.

Case 1. By Theorem 1 [4] and Theorem 1 [8], R is a direct sum of N and Q, so R satisfies (10).

Case 2. Here (10) can be written as (13), with $x_{h+1} \underset{l}{\nparallel} v$ and $x_{n-t} \underset{r}{\nparallel} v$, and the identity $x_{h+1} \cdots x_{n-t} = v$ is of period p. Since $(h(w))(a) = a_1$, $(t(w))(a) = a_n$ and since Q satisfies the identities $x_2 \cdots x_{n-1} = m_2^{k-1}(w)$ and $x_1 \cdots x_{k-1} = l_{k-1}(w)$, then

$$(18) \quad \langle a_1, a_2 \cdots a_{n-1} \rangle \, \theta^{a_n} + \langle a_1 \cdots a_{n-1}, a_n \rangle = \big\langle (h(w))(a), (m_2^{k-1}(w))(a) \big\rangle \, \theta^{(t(w))(a)} + \langle (l_{k-1}(w))(a), (t(w))(a) \rangle \, .$$

Further, by (16) and (17) we obtain that

$$\sum_{\mu \in M_{\Pi_n}} \Pi_n(\mu) = \Sigma_1 + \Sigma_2 + \Sigma_3, \qquad \sum_{\mu \in M_w} \Pi_k(\mu) = \Sigma_1' + \Sigma_2' + \Sigma_3',$$

where

$$\Sigma_1 = \sum_{i=1}^{h} \sum_{l=1}^{t} \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

$$\Sigma_2 = \sum_{i=1}^{h} \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_n},$$

$$\Sigma_3 = \sum_{l=1}^{t} \theta^{a_1} \cdots \theta^{a_h} \theta_{a_{h+1}} \cdots \theta^{a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta_{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

$$\Sigma_1' = \sum_{i=1}^{h} \sum_{l=1}^{t} \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta_{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

$$\Sigma_2' = \sum_{i=1}^{h} \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta_{a_n},$$

$$\Sigma_3' = \sum_{l=1}^{t} \theta^{a_1} \cdots \theta^{a_h} v(\theta_a) \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n.$$

By (2),

$$(19) \qquad v(\theta^a) = \theta^{v(a)} + \pi_\beta, \qquad \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} = \theta^{a_{h+1} \cdots a_{n-t}} + \pi_{\beta'},$$

with $\beta, \beta' \in N$ determined as in (1).

Assume $i \in \{1, \dots, h-1\}$, $l \in \{1, \dots, t-1\}$. Then

$$\alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n = \\ = \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{v(a)} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

by (19) and (16). Similarly

$$\alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_{n-l}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n = \\ = \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1} \cdots a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n.$$

Since Q satisfies the identity $x_{h+1} \cdots x_{n-t} = v$, then

$$\text{(20)} \quad \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n = \\ = \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_{n-l}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

for $1 \leq i \leq h-1,\ 1 \leq l \leq t-1$. Similarly we prove

$$\text{(21)} \quad \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta^{a_n} = \\ = \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_n},$$

for $1 \leq i \leq h-1$, and

$$\text{(22)} \quad \theta^{a_1} \cdots \theta^{a_h} v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n = \\ = \theta^{a_1} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_{n-l}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

for $1 \leq l \leq t-1$.

Further, for $1 \leq l \leq t-1$,

$$\alpha_1 \cdots \alpha_h v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n = \\ = \alpha_1 \cdots \alpha_h \theta^{v(a)} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

$$\alpha_1 \cdots \alpha_h \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n = \\ = \alpha_1 \cdots \alpha_h \theta^{a_{h+1} \cdots a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

by (19) and (17). Thus

$$\text{(23)} \quad \alpha_1 \cdots \alpha_h v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n = \\ = \alpha_1 \cdots \alpha_h \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_{n-l}} \alpha_{n-l+1} \cdots \alpha_n,$$

for $1 \leq l \leq t-1$. Similarly we prove

$$\text{(24)} \quad \alpha_1 \cdots \alpha_h v(\theta^a) \theta^{a_{n-t+1}} \cdots \theta^{a_n} = \alpha_1 \cdots \alpha_h \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} \theta^{a_{n-t+1}} \cdots \theta^{a_n},$$

$$\text{(25)} \quad \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} v(\theta^a) \alpha_{n-t+1} \cdots \alpha_n = \\ \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_{n-t}} \alpha_{n-t+1} \cdots \alpha_n$$

for $1 \le i \le h-1$, and

(26) $$\theta^{a_1}\cdots\theta^{a_h}v(\theta^a)\alpha_{n-t+1}\cdots\alpha_n = \theta^{a_1}\cdots\theta^{a_h}\theta^{a_{h+1}}\cdots\theta^{a_{n-t}}\alpha_{n-t+1}\cdots\alpha_n.$$

Finally, by (19) and (17),

$$\begin{aligned}
&\alpha_1\cdots\alpha_h v(\theta^a)\alpha_{n-t+1}\cdots\alpha_n\\
&\qquad = \alpha_1\cdots\alpha_h\theta^{v(a)}\alpha_{n-t+1}\cdots\alpha_n +\\
&\qquad\qquad + \alpha_1\cdots\alpha_h\left\langle (l_{s-1}(v))(a), (t(v))(a)\right\rangle\alpha_{n-t+1}\cdots\alpha_n,\\
&\alpha_1\cdots\alpha_h\theta^{a_{h+1}}\cdots\theta^{a_{n-t}}\alpha_{n-t+1}\cdots\alpha_n\\
&\qquad = \alpha_1\cdots\alpha_h\theta^{a_{h+1}\cdots a_{n-t}}\alpha_{n-t+1}\cdots\alpha_n +\\
&\qquad\qquad + \alpha_1\cdots\alpha_h\left\langle a_{h+1}\cdots a_{n-t-1}, a_{n-t}\right\rangle\alpha_{n-t+1}\cdots\alpha_n,
\end{aligned}$$

where $s = |v|$. As in the previous case we obtain

(27) $$\alpha_1\cdots\alpha_h\theta^{v(a)}\alpha_{n-t+1}\cdots\alpha_n = \alpha_1\cdots\alpha_h\theta^{a_{h+1}\cdots a_{n-t}}\alpha_{n-t+1}\cdots\alpha_n.$$

Further, if $t(v) = x_{n-t}$, then by the commutativity in Q and by (E3),

$$\begin{aligned}
\left\langle (l_{s-1}(v))(a), (t(v))(a)\right\rangle &= \left\langle a' a_{n-t}^p, a_{n-t}\right\rangle\\
&= \left\langle a', a_{n-t}^{p+1}\right\rangle + \theta^{a'}\left\langle a_{n-t}^p, a_{n-t}\right\rangle - \left\langle a', a_{n-t}^p\right\rangle\theta^{a_{n-t}},
\end{aligned}$$

where $a' = a_{h+1}\cdots a_{n-t-1}$, so by (17),

(28) $$\begin{aligned}
&\alpha_1\cdots\alpha_h\left\langle (l_{s-1}(v))(a), (t(v))(a)\right\rangle\alpha_{n-t+1}\cdots\alpha_n\\
&\qquad = \alpha_1\cdots\alpha_h\left\langle a_{h+1}\cdots a_{n-t-1}, a_{n-t}\right\rangle\alpha_{n-t+1}\cdots\alpha_n.
\end{aligned}$$

Otherwise, if $t(v) = x_j,\ h+1 \le j \le n-t-1$, then again by the commutativity in Q and by (E3),

$$\begin{aligned}
&\left\langle (l_{s-1}(v))(a), (t(v))(a)\right\rangle\\
&\quad = \left\langle a'' a_{n-t}, a_j\right\rangle\\
&\quad = \left\langle a'', a_{n-t}a_j\right\rangle + \theta^{a''}\left\langle a_{n-t}, a_j\right\rangle - \left\langle a'', a_{n-t}\right\rangle\theta^{a_j}\\
&\quad = \left\langle a'', a_j a_{n-t}\right\rangle + \theta^{a''}\left\langle a_{n-t}, a_j\right\rangle - \left\langle a'', a_{n-t}\right\rangle\theta^{a_j}\\
&\quad = \left\langle a'' a_j, a_{n-t}\right\rangle + \left\langle a'', a_j\right\rangle\theta^{a_{n-t}} - \theta^{a''}\left\langle a_j, a_{n-t}\right\rangle + \theta^{a''}\left\langle a_{n-t}, a_j\right\rangle - \left\langle a'', a_{n-t}\right\rangle\theta^{a_j},
\end{aligned}$$

where $a'' = a_{h+1}^{q_{h+1}}\cdots a_{n-t-1}^{q_{n-t-1}},\ q_j = p$ and $q_r = 1$, for $h+1 \le r \le n-t-1,\ r \ne j$. Now by $a''a_j = a_{h+1}\cdots a_{n-t-1}$ and by (17), (28) follows. Thus, by (27) and (28) we obtain

(29) $$\alpha_1\cdots\alpha_h v(\theta^a)\alpha_{n-t+1}\cdots\alpha_n = \alpha_1\cdots\alpha_h\theta^{a_{h+1}}\cdots\theta^{a_{n-t}}\alpha_{n-t+1}\cdots\alpha_n.$$

Hence, by (20)–(26) and (29) we obtain that $\Sigma_j = \Sigma'_j$, for all $j \in \{1,2,3\}$, and by this and by (18), R satisfies (10).

Case 3. Here (10) can be written as (11), with $x_{h+1} \underset{l}{\nparallel} u$, and the identity $x_{h+1} \cdots x_n = u$ is of the period p. Now

$$\sum_{\mu \in M_{\Pi_n}} \Pi_n(\mu) = \sum_{i=1}^{h} \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_n},$$

$$\sum_{\mu \in M_w} \Pi_k(\mu) = \sum_{i=1}^{h} \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} u(\theta^a),$$

and we have

$$\alpha_1 \cdots \alpha_h u(\theta^a) = \alpha_1 \cdots \alpha_h \theta^{a_{h+1}} \cdots \theta^{a_n}, \tag{30}$$

$$\alpha_1 \cdots \alpha_i \theta^{a_{a+1}} \cdots \theta^{a_h} u(\theta^a) = \alpha_1 \cdots \alpha_i \theta^{a_{i+1}} \cdots \theta^{a_h} \theta^{a_{h+1}} \cdots \theta^{a_n}, \tag{31}$$

for $1 \le i \le h-1$, where (30) can be proved as (23), and (31) as (20). Further, by (E3) and (17),

$$\begin{aligned} &\big\langle (h(w))(a), (m_2^{k-1}(w))(a) \big\rangle\, \theta^{(t(w))(a)} + \langle (l_{k-1}(w))(a), (t(w))(a) \rangle \\ &= \langle (h(w))(a), (r_{k-1}(w))(a) \rangle = \langle a_1, a_2 \cdots a_n \rangle \\ &= \langle a_1, a_2 \cdots a_{n-1} \rangle\, \theta^{a_n} + \langle a_1 \cdots a_{n-1}, a_n \rangle\,, \end{aligned} \tag{32}$$

since $h(w) = x_1$ and Q satisfies the identity $x_2 \cdots x_n = r_{k-1}(w)$. Therefore, by (30)–(32) we obtain that R satisfies (10).

We treat Case 4 similarly.

■

Using the previous theorem we give examples in which the identities of the form (10) with two and three letters are classified.

Example 1. For the identity

$$xy = w(x,y) \tag{33}$$

with $w \in A_2^+$, $|w| \ge 3$, there are exactly three possibilities:

(i) The characteristic quadruplet of (33) is $(2,p,0,0)$. Then a ring satisfies (33) if and only if it is a direct sum of a ring satisfying $x = x^{p+1}$ and a null-ring, and consequently these rings are commutative.

(ii) The characteristic quadruplet of (33) is $(2,p,1,0)$. This holds if and only if (33) is of the form $xy = xy^{p+1}$.

(iii) The characteristic quadruplet of (33) is $(2,p,0,1)$. This holds if and only if (33) is of the form $xy = x^{p+1}y$.

Example 2. For the identity

$$xyz = w(x, y, z) \tag{34}$$

with $w \in A_3^+$, $|w| \geq 4$, there are exactly six possibilities:

(i) The characteristic quadruplet of (34) is $(3, p, 0, 0)$. Then a ring satisfies (34) if and only if it is a direct sum of a ring satisfying $x = x^{p+1}$ and a 3-nilpotent ring.

(ii) The characteristic quadruplet of (34) is $(3, p, 1, 0)$. This holds if and only if (34) is of the form $xyz = xu(y, z)$, $|u| \geq 3$.

(iii) The characteristic quadruplet of (34) is $(3, p, 0, 1)$. This holds if and only if (34) is of the form $xyz = v(x, y)z$, $|v| \geq 3$.

(iv) The characteristic quadruplet of (34) is $(3, p, 2, 0)$. This holds if and only if (34) is of the form $xyz = xyz^{p+1}$.

(v) The characteristic quadruplet of (34) is $(3, p, 0, 2)$. This holds if and only if (34) is of the form $xyz = x^{p+1}yz$.

(vi) The characteristic quadruplet of (34) is $(3, p, 1, 1)$. This holds if and only if (34) is of the form $xyz = xy^{p+1}z$.

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