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# $U_{n+1}$-SEMIGROUPS

In this paper we consider semigroups with the following condition:

$$(\forall x_1)(\forall x_2)\ldots(\forall x_{n+1})\ x_1x_2\ldots x_{n+1} \in \langle x_1\rangle \cup \langle x_2\rangle \cup \ldots \cup \langle x_{n+1}\rangle. \qquad (1)$$

## 1.

In conection with a study of a lattice of subsemigroups of some semigroup the important place is captured by $U$-semigroups. A semigroup $S$ is a *U-semigroup* if the union of every two subsemigroups of $S$ is a subsemigroup of $S$, which is equivalent with $xy \in \langle x\rangle \cup \langle y\rangle$ for all $x, y \in S$. These semigroups have been considered more recently, predominantly in special cases. A more detailed description can be found in: *Lectures in semigroups,* M. Petrich, Akad.-Verlag, Berlin, 1977. E. G. Shutov [18], and independently of him, T. Tamura, N. Kimura and R. Merkel [9], considered semigroups in which every subsemigroup is a left ideal, i.e. semigroups in which $xy \in \langle y\rangle$ for all $x, y$. E. S. Ljapin and A. E. Evseev [10] considered semigroups in which $xy \in \{x, y\} \cup (\langle x\rangle \cap \langle y\rangle)$ for all $x, y$. A. E. Evseev [8] considered semigroups in which $xy \in \{x, x^2, y, y^2\}$ for all $x, y$. In this paper we consider $U_{n+1}$*-semigroups,* i.e. semigroups in which (1) holds. $U_2$*-semigroups,* in fact, are $U$-semigroups. In section 2. we study the general properties of $U_{n+1}$-semigroups. So we describe cyclic $U_{n+1}$-semigroups, $U_{n+1}$-groups and regular $U_{n+1}$-semigroups. By Theorem 2.4. we describe $U_{n+1}$-semigroups in which Reg($S$) (the set of all regular elements of a semigroup $S$) is an ideal of $S$. The main result is Theorem 2.3., which characterize $U_{n+1}$-semigroups in a general case. In section 3. we consider some subclasses of the class of $U_{n+1}$-semigroups and we give constructions of some unipotent $U_{n+1}$-semigroups. By Theorem 3.6., we describe the $n$-inflation of an ordinal sum of singular bands. Some special cases of $U$-semigroups have been considered by: B. Trpenovski [16], S. Bogdanović, P. Kržovski, P. Protić and B. Trpenovski [1], J. Pelikán [11], B. Pondělíček

[15], L. Rédei [19], S. Bogdanović and B. Stamenković [6], B. Trpenovski and N. Celakoski [17].

By $\mathbf{Z}^+$ we denote the set of all positive integers. By Reg($S$) (Gr($S$), E($S$)) we denote the set of all regular (completely regular, *idempotent)* elements of a semigroups $S$. A semigroup $S$ is *π-regular* if some power of every element from $S$ is regular. A semigroups $S$ is a *GV-semigroup* if $S$ is π-regular and Reg($S$) = Gr($S$), [3]. An ideal extension $S$ of a semigroup $T$ is a retract extension if there exists a homomorphism φ of $S$ onto $T$ such that $\varphi(t) = t$ for all $t \in T$. In this case we say that φ is a *retraction.* A semigroup $S$ is an *n-inflation* of a semigroup $T$ if $S^{n+1} \subseteq T$ and there exists a retraction of $S$ onto $T$, [5]. For nondefined notions and notations we refer to [3, 7, 13, 14].

## 2.

In this section a characterization for $U_{n+1}$-semigroups will be given (Theorem 2.3.).

LEMMA 2.1. Every subsemigroup and every homomorphic image of a $U_{n+1}$-semigroup is a $U_{n+1}$-semigroup. □

A group $G$ is a $U_{n+1}$-group if $G$ is a $U_{n+1}$-semigroup.

LEMMA 2.2. $G$ is a $U_{n+1}$-group if and only if $G$ is a $U$-group.

**Proof.** Let $G$ be a $U_{n+1}$-group with the identity $e$ and let $x \in G$. If $n+1=3k$ then we have $x^{6k+1} = x^3(x^2)^{3k-1} \in \langle x^3\rangle \cup \langle x^2\rangle$; if $n+1 = 3k+1$ then $x^{6k+5} = x^5(x^2)^{3k} \in \langle x^5\rangle \cup \langle x^2\rangle$ and if $n+1 = 3k+2$ then $x^{6k+5} = x^3(x^2)^{3k+1} \in \langle x^3\rangle \cup \langle x^2\rangle$. Hence, $\langle x\rangle$ is a periodic subsemigroup of $G$, so $e \in \langle x\rangle$ for all $x \in G$. Let $x,y \in G$, then $xy = xye\ldots e \in \langle x\rangle \cup \langle y\rangle \cup \{e\} = \langle x\rangle \cup \langle y\rangle$. Thus $G$ is a $U$-group.

The converse follows immediately. □

LEMMA 2.3. [13]. Let $G$ be a group. Then $G$ is a $U$-group if and only if $G$ is a cyclic group of order $p^k$ or $Z_{p^\infty}$ for some prime $p$. □

THEOREM 2.1. Let $S$ be a cyclic semigroup, then:

(A) $S$ is a $U$-semigroup if and only if $S$ is an ideal extension of a cyclic group of prime power order by a 5-nilpotent cyclic semigroup;

(B) $S$ is a $U_{3k}$-semigroup if and only if $S$ is an ideal extension of a cyclic group of prime power order by a $(6k+1)$-nilpotent cyclic semigroup;

(C) $S$ is a $U_{3k+1}$-semigroup if and only if $S$ is an ideal extension of a cyclic group of prime power order by a $(6k+5)$-nilpotent cyclic semigroup;

(D) $S$ is a $U_{3k+2}$-semigroup if and only if $S$ is an ideal extension of a cyclic group of prime power order by a $(6k+5)$-nilpotent cyclic semigroup.

**Proof.** (A) See [13] pp. 130.

(B) Let $\langle x\rangle$ be a $U_{3k}$-semigroup. Then $x^{6k+1} \in \langle x^3\rangle \cup \langle x^2\rangle$, so $\langle x\rangle$ is periodic, so by Lemma 2.2. $\langle x\rangle$ is an ideal extension of a cyclic $U$-group by a $(6k + 1)$-nilpotent cyclic semigroup. The converse follows by Lemma 2.2.

(C) If $\langle x \rangle$ is a $U_{3k+1}$-semigroup, then $x^{6k+5} \in \langle x^5 \rangle \cup \langle x^2 \rangle$, so $\langle x \rangle$ is an ideal extension of a cyclic $U$-group by a cyclic $(6k + 5)$-nilpotent semigroup. The converse follows by Lemma 2.2.

(D) If $\langle x \rangle$ is a $U_{3k+2}$-semigroup then $x^{6k+5} \in \langle x^3 \rangle \cup \langle x^2 \rangle$, so $\langle x \rangle$ is an ideal extension of a cyclic $U$-group by a cyclic $(6k + 5)$-nilpotent semigroup. The converse follows by Lemma 2.2. □

COROLLARY 2.1. A $U_{n+1}$-semigroup is periodic. □

COROLLARY 2.2. Let $S$ be a cyclic semigroup with zero. Then $S$ is a $U_{3k+1}$-semigroup if and only if $S$ is a $U_{3k+2}$-semigroup, $k \in \mathbf{Z}^+$. □

A semigroup $S$ is a *Rédei's band* if $xy \in \{x, y\}$ for all $x,y \in S$. A semigroup $S = \bigcup_{\alpha \in Y} S_\alpha$ is an *ordinal sum* of semigroups $S_\alpha$, $\alpha \in Y$ if $Y$ is a chain, $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$ and for any $a \in S_\alpha$, $b \in S_\beta$, $\alpha < \beta$ implies $ab = ba = a$ (for this concept, the terminology "successively annihilating band" is also used [13]). A *singular band* is a semigroup which is either a left or a right zero semigroup.

LEMMA 2.4. [19]. $S$ is a Rédei's band if and only if $S$ is an ordinal sum of singular bands. □

LEMMA 2.5. Let $S$ be a $U_{n+1}$-semigroup. Then $E(S)$ is a Rédei's band. □

In any semigroup $S$, define a relation $\mathbf{K}$ by:

$$a\mathbf{K}b \Leftrightarrow (\exists m,\ n \in Z^+)\ a^m = b^n.$$

It is immediate that $\mathbf{K}$ is an equivalence relation. The $\mathbf{K}$-class containing an element $a$ is denoted by $K_a$. In particular, if $e$ is an idempotent then $K_e = \{a \in S \mid (\exists n \in Z^+)\ a^n = e\}$. If $S$ is periodic then $S$ is the union of its $\mathbf{K}$-classes $K_e, e \in E(S)$.

LEMMA 2.6. Let $S$ be a $U$-semigroup and let $x \in K_e$, $y \in K_f$, $e \neq f$. Then

$$ef = e \Rightarrow xy \in \langle x \rangle \ \text{ and } \ ef = f \Rightarrow xy \in \langle y \rangle. \tag{2.1}$$

**Proof.** Assume that $ef = e$ and $xy = y^p$ for some $p \in Z^+$. By assumption we have $x^m = e$, $y^k = f$ for some $m,n \in Z^+$. Consequently $e = x^m y^k = x^{m-1}xyy^{k-1} = x^{m-1}y^{p+k-1} = \ldots = y^{mp+k-m}$ so $e \in K_f$, which is a contradiction. A symmetric proof shows that the second implication holds. □

LEMMA 2.7. $S$ is a completely simple $U_{n+1}$-semigroup if and only if $S$ is a $U$-group or a singular band.

**Proof.** Let $S$ be a completely simple $U_{n+1}$-semigroup. By Lemma 2.5. $E(S)$ is a subsemigroup of $S$ and a Rédei's band, so $E(S)$ is a rectangular Rédei's band. Therefore, by Lemma 2.4. it follows that $E(S)$ is a singular band. If $|E(S)| = 1$, then it is clear that $S$ is a $U$-group. Let $E(S)$ be a left zero band and let $|E(S)| \geq 2$. Let $x \in G_e$, $e \in E(S)$. Then there exists $f \in E(S)$ such that $f \neq e$. Now $fx = f...fx \in \langle f \rangle \cup \langle x \rangle$ and for $fx = x^k$ for some $k \in \mathbf{Z}^+$ we obtain that $f = fe = e$, which is not possible. Thus $fx = f$ so $x = ex = = efx = ef = e$. Therefore, $|G_e| = 1$ for every $e \in E(S)$ so $S$ is a left zero band. The similar proof we have if $|E(S)| \geq 2$ and it is a right zero band.

The converse follows immediately. □

THEOREM 2.2. The following conditions are equivalent on a semigroup $S$:

(i) $S$ is a regular $U_{n+1}$-semigroup;
(ii) $S$ is a regular $U$-semigroup;
(iii) $S$ is an ordinal sum of $U$-groups and singular bands.

**Proof.** (i) ⇒ (ii). Let $S$ be a regular $U_{n+1}$-semigroup. For $a \in S$ there exists $x \in S$ such that $a = axa$ and $x = xax$. By Lemma 2.5. it follows that $axxa = ax$ or $axxa = xa$. Let $ax^2a = ax$. Then $a = axa = ax^2a^2 \in aSa^2$. Let $ax^2a = xa$. If $n + 1 = 2k$, $k \in Z^+$, then $xa = (xa)^k \in \langle x \rangle \cup \langle a \rangle$. If $xa = x^p$ for some $p \in Z^+$, then $x = xax = x^{p+1}$ and $x^2a = x^{p+1} = x$ so $ax = ax^2a = xa$ and $a = axa = ax^2a^2 \in aSa^2$. If $xa = a^p$ for some $p \in Z^+$, then $a = axa = a^{p+1} \in aSa^2$. Let $n + 1 = 2k + 1$, $k \in Z^+$. Then $xa = (ax)(xa)^k \in \langle ax \rangle \cup \langle a \rangle \cup \langle x \rangle$. The case $xa \in \langle a \rangle \cup \langle x \rangle$ we prove as in the previous. If $xa = ax$, then $a = axaxa = axxaa = ax^2a^2 \in aSa^2$. Hence $S$ is completely regular. Let $x, y \in S$. Then $x \in G_e$, $y \in G_f$ for some $e, f \in E(S)$ so $xy = xyf = xyf...f \in \langle x \rangle \cup \langle y \rangle \cup \langle f \rangle \subseteq \langle x \rangle \cup \langle y \rangle$. Thus $S$ is a $U$-semigroup.

(ii) ⇒ (iii). As above it can be prove that $S$ is completely regular, so $S$ is a semilattice $Y$ of completely simple $U$-semigroups $S_\alpha$, $\alpha \in Y$. By Lemma 2.7. $S_\alpha$, $\alpha \in Y$, is a $U$-group or a singular band. Let $\alpha, \beta \in Y$, $\alpha \neq \beta$, and let $e \in E(S_\alpha)$, $f \in E(S_\beta)$. Then $\alpha < \beta$ if $ef = e$ and $\beta < \alpha$ if $ef = f$ so $Y$ is a chain. Let $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$, $\alpha \neq \beta$. Assume that $\alpha < \beta$. Let $x \in G_e$, $e \in E(S_\alpha)$. Then by Lemma 2.6. it follows that $ey = e$ whence $xy = xey = xe = x$, and similarly $yx = x$. Thus $S$ is an ordinal sum of semigroups $S_\alpha$, $\alpha \in Y$..

(iii) ⇒ (i). This follows immediately. □

LEMMA 2.8. $S$ is a $U_{n+1}$-semigroup with only one idempotent if and only if $S$ is an ideal extension of a $U$-group by a $U_{n+1}$-nil-semigroup.

**Proof.** Let $S$ be an ideal extension of a $U$-group $G$ by a $U_{n+1}$-nil-semigroup $Q$. Then the mapping $\varphi: S \to G$ defined by $\varphi(x) = xe$, $x \in S$, where $e$ is the identity of $G$, is a retraction. Let $x_1, x_2, \ldots, x_{n+1} \in S$. If

$x_1x_2...x_{n+1} \in S - G$ then $x_1, x_2,...,x_{n+1} \in Q - \{0\}$ and $x_1x_2...x_{n+1} \neq 0$ in $Q$. Hence $x_1x_2...x_{n+1} = x_i^k \neq 0$ in $Q$, $i \in \{1, 2,..., n+1\}$, $k \in \mathbf{Z}^+$, whence $x_1x_2... x_{n+1} = x_i^k \notin G$ in $S$. Assume that $x_1 x_2...x_{n+1} \in G$. Then $x_1 x_2...x_{n+1} = \varphi(x_1 x_2...x_{n+1}) = \varphi(x_1)\ \varphi(x_2) ...\varphi(x_{n+1}) = \varphi(x_i^k)$ for some $i \in \{1, 2,..., n+1\}$, $k \in \mathbf{Z}^+$. Moreover, we have that $x_i^m = e$ for some $m \in \mathbf{Z}^+$ so $\varphi(x_i^k) = x_i^k e = x_i^k x_i^m = x_i^{k+m} \in \langle x_i \rangle$. Therefore $S$ is a $U_{n+1}$-semigroup.

The converse follows by Corollary 2.1, Theorem VI 3.2.2. [3] and by Lemma 2.1. □

DEFINITION 2.1. A band $Y$ of semigroups $S_\alpha$, $\alpha \in Y$, is a $U_{n+1}$-band of semigroups if

$$x_1x_2...x_{n+1} \in \langle x_1 \rangle \cup \langle x_2 \rangle \cup ... \cup \langle x_{n+1} \rangle$$

for every $x_1 \in S_{\alpha_1}, x_2 \in S_{\alpha_2}, ..., x_{n+1} \in S_{\alpha_{n+1}}$ such that there exist $i,j \in \{1, 2,..., n+1\}$ such that $S_{\alpha_i} \neq S_{\alpha_j}$. One defines analogously $U_{n+1}$-semilattice and $U_{n+1}$-chain of semigroups.

THEOREM 2.3. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a $U_{n+1}$-semigroup;

(ii) $S$ is a $U_{n+1}$-chain of retract extension of a $U$-group or a singular band by a $U_{n+1}$-nil-semigroup;

(iii) $S$ is a $U_{n+1}$-band of ideal extensions of a $U$-group by a $U_{n+1}$-nil-semigroup.

**Proof.** (i) ⇒ (ii). Let $S$ be a $U_{n+1}$-semigroup. By Corollary 2.1. $S$ is periodic, so $S$ is $\pi$-regular and $E(S) \neq \emptyset$. By Lemma 2.5. we have that $E(S)$ is a subsemigroup of $S$ so by Proposition X 2.1. [3] it follows that Reg($S$) is a subsemigroup of $S$. By Lemma 2.1. and Theorem 2.2. we have that Reg($S$) = Gr($S$), so by Theorem X 1.1. [3] $S$ is a semilattice $Y$ of semigroups $S_\alpha$, $\alpha \in Y$, and $S_\alpha$ is a nil-extension of a completely simple semigroup $T_\alpha$, $\alpha \in Y$. By Lemma 2.7. $T_\alpha$ is a $U$-group or a singular band. Let $\alpha \in Y$ and let $T_\alpha$ be a left zero band (the similar proof we have if $T_\alpha$ is a right zero band). Then $S_\alpha = \bigcup_{e \in T_\alpha} K_e$ and by Theorem X 1.1. [3] $K_e$ are $\mathbf{H}^*$-classes and $K_e$ are nil-semigroups. Let $x \in K_e$, $y \in K_f$, $e,f \in T_\alpha$, $e \neq f$. Assume that $xy \in K_g$ for some $g \in T$. We have that $xf = xf...f \in \langle x \rangle \cup \langle f \rangle$ and if $xf = f$ then $x^k f = f$ for every $k \in \mathbf{Z}^+$, whence $e = ef = f$, which is not possible. Thus $xf = x^k$ for some $k \in \mathbf{Z}^+$. Since $T_\alpha$ is an ideal of $S_\alpha$, then $xf \in T_\alpha$, i.e. $xf = e$. In a similar way we prove that $yg = f$, whence $g = xyg = xf = e$ so $xy \in K_e = K_{ef}$. Therefore, the mapping $\varphi : S_\alpha \to T_\alpha$ defined by $\varphi(x) = e$ if $x \in K_e$ is a retraction of $S_\alpha$ onto $T_\alpha$, so $S_\alpha$ is a retract extension of a left zero band $T_\alpha$. If $T_\alpha$ is a group, then every ideal extension of $T_\alpha$ is a retract extension.

Since $E(S)$ is an ordinal sum of singular bands, then $Y$ is a chain, so $S$ is a $U_{n+1}$-chain of semigroups $S_\alpha$, $\alpha \in Y$.

(ii) $\Rightarrow$ (i). Let $S$ be a $U_{n+1}$-chain of semigroups $S_\alpha$, $\alpha\in Y$ and let $S_\alpha$ be a retract extension of $T_\alpha$ by a $U_{n+1}$-nil-semigroup, where $T_\alpha$ is a $U$-group or a singular band. Let $\alpha\in Y$ and let $T_\alpha$ be a singular band. Let $\varphi : S_\alpha \to T_\alpha$ be the retraction. Let $x_1, x_2,\dots,x_{n+1}\in S_\alpha$. If $x_1, x_2\dots x_{n+1}\in S_\alpha - T_\alpha$, then we have that $x_1x_2\dots x_{n+1} \neq 0$ in $Q = S_\alpha/T_\alpha$ so $x_1x_2\dots x_{n+1} = x_{i_k}^k \neq 0$ in $Q$, $i\in\{1, 2,\dots,n + 1\}$, $k\in\mathbf{Z}^+$, whence $x_1 x_2\dots x_{n+1} = x_i^k$ in $S$. Let $x_1 x_2\dots x_{n+1}\in T_\alpha$. Then $x_1 x_2\dots x_{n+1} = \varphi(x_1 x_2\dots x_{n+1}) = \varphi(x_1)\,\varphi(x_2)\dots\varphi(x_{n+1}) = \varphi(x_i)$, where $i = 1$ if $T_\alpha$ is a left zero band and $i = n + 1$ if $T_\alpha$ is a right zero band. Moreover, $x_i^m\in T_\alpha$ for some $m\in\mathbf{Z}^+$ so $x_i^m = \varphi(x_i^m) = (\varphi(x_i))^m = \varphi(x_i)$. Therefore $x_1x_2\dots x_{n+1} = \varphi(x_i) = x_i^m$, so $S_\alpha$ is a $U_{n+1}$-semigroup. If $T_\alpha$ is a group, then we use Lemma 2.8. Thus $S$ is a $U_{n+1}$-semigroup.

(ii) $\Rightarrow$ (iii). Let $S$ be a $U_{n+1}$-chain of semigroups $S_\alpha$, $\alpha\in Y$ and let $S_\alpha$ be a retracted extension of a semigroup $T_\alpha$, where $T_\alpha$ is a $U_2$-group or a singular band. Then $S$ is a $U_{n+1}$-semigroup, so by Corollary 2.1. it follows that $S$ is periodic, whence

$$S = \bigcup_{e\in E(S)} K_e,$$

and, also,

$$S_\alpha = \bigcup_{e\in T_\alpha} K_e, \qquad\qquad \alpha\in Y.$$

By Theorem X 1.1. [3] we have that $K_e$ are $\mathbf{H}^*$-classes and $K_e$ are nil-extensions of groups. By Lemma 2.1. we have that for any $e\in E(S)$ $K_e$ is an ideal extension of a $U_2$-group by a $U_{n+1}$-nil-semigroup.

Let $x\in K_e$, $y\in K_f$, $e \neq f$. If $e,f\in E(S_\alpha)$ for some $\alpha\in Y$, then $T_\alpha$ is a singular band, and by the proof of the part (i) $\Rightarrow$ (ii) we have that

$$xy\in K_e = K_{ef},$$

if $T_\alpha$ is a left zero band and

$$xy\in K_f = K_{ef},$$

if $T_\alpha$ is a right zero band.

Thus, in this case $xy\in K_{ef}$. Assume that $e\in E(S_\alpha)$, $f\in E(S_\beta)$, $\alpha \neq \beta$, let $\alpha\beta = \beta\alpha = \alpha$, then $xy\in S_\alpha$. If $T_\alpha$ is a group, then $xy\in S_\alpha = K_e = K_{ef}$. Let $T_\alpha$ be a left zero band, and assume that $xy\in K_g$ for some $g\in T_\alpha$, then $ye\in S_\alpha$ and

$$ye = ye\dots e\in\langle y\rangle\cup\{e\},$$

so $ye = e$. Now we have that $g = (xy)^k$ for some $k\in\mathbf{Z}^+$ and

$$g = ge = (xy)^k e = e,$$

so $xy \in K_e = K_{ef}$. Let $T_\alpha$ be a right zero band and let $xy \in K_g$ for some $g \in T_\alpha$, then $ey \in S_\alpha$ and

$$ey = e \dots ey \in \{e\} \cup \langle y \rangle,$$

so $ey = e$. Now we have that $g = (xy)^k$ for some $k \in \mathbf{Z}^+$, so

$$g = eg = e(xy)^k = e,$$

whence $xy \in K_e = K_{ef}$. The similar proof we have if $\alpha\beta = \beta\alpha = \beta$. Therefore, $S$ is a band $E(S)$ of semigroups $K_e$, $e \in E(S)$ (clearly, it is a $U_{n+1}$-band).

(iii) $\Rightarrow$ (i). This follows by Lemma 2.8. □

A subsemigroup $T$ of a semigroup $S$ is a *retract* of $S$ if there exists a retraction of $S$ onto $T$.

COROLLARY 2.5. Let $S$ be a $U_{n+1}$-semigroup. Then Reg$(S)$ is a retract of $S$.

**Proof.** By Lemma 2.4. and Proposition X 2.1. [3] we have that Reg $(S)$ is a subsemigroup of $S$. Define a mapping $\varphi : S \rightarrow \text{Reg}(S)$ by

$$\varphi(x) = ex \ \text{ if } \ x \in K_e,\ e \in E(S).$$

Let $x \in K_e$, $y \in K_f$, $e,f \in E(S)$. Then by Theorem 2.3. it follows that $xy \in K_{ef}$. If $ef = e$, then $xy \in K_e$, and by Theorem I 4.3. [3] we have that

$$\varphi(x)\ \varphi(y) = (ex)(fy) = xefy = xey = exy = \varphi(xy).$$

If $ef = f$, then $xy \in K_f$, and by Theorem I 4.3. [3] we have

$$\varphi(x)\ \varphi(y) = (ex)(fy) = xefy = xfy = xyf = fxy = \varphi(xy).$$

Therefore, $\varphi$ is a retraction. □

THEOREM 2.4. $S$ is a $U_{n+1}$-semigroup and Reg$(S)$ is an ideal of $S$ if and only if

$$x_1 x_2 \dots x_{n+1} \in \bigcup_{i=1}^{n+1} \left\{ x_i^k \mid k \geq 2 \right\} \tag{2.2}$$

for every $x_1, x_2, \dots, x_{n+1} \in S$.

**Proof.** Let $S$ be a $U_{n+1}$-semigroup and let Reg$(S)$ is an ideal of $S$. By Lemma 2.1. Reg$(S)$ is a $U_{n+1}$-semigroup. By Theorem 2.2. we have that Reg(S) = Gr$(S)$. For $x_1, x_2, \dots, x_{n+1} \in S$ from $x_1 x_2 \dots x_{n+1} = x_i$, $1 \leq i \leq n+1$ it follows that $x_i = (x_1 \dots x_{i-1})^r x_i (x_{i+1} \dots x_{n+1})^r$ for every $r \in \mathbf{Z}^+$. Since there exists $r \in \mathbf{Z}^+$ such that $(x_1 \dots x_{i-1})^r \in E(S)$, then we have that $x_i \in \text{Reg}(S) = \text{Gr}(S)$, whence $x_i = x_i^k$ for some $k \geq 2$ (since $S$ is periodic). So $x_i \in \left\{ x_i^k \mid k \geq 2 \right\}$. Therefore $S$ satisfies (2.2).

Conversely, if (2.2) holds, then for $e \in E(S)$ and $x \in S$ we have that $ex = e \dots eex \in \{e, (ex)^2, (ex)^3, \dots\}$. Thus $ex \in \text{Reg}(S)$. In a similar way it can be prove that $xe \in \mathrm{Reg}(S)$. Therefore $E(S)S \cup SE(S) \subseteq \mathrm{Reg}(S)$. Since $E(S)$ is a subsemigroup of $S$ then we have that $\mathrm{Reg}(S)$ is a subsemigroup of $S$. Now, for any $a \in \mathrm{Reg}(S)$ and $x \in S$ there exists $b \in S$ such that $ax = abax \in E(S)S \subseteq \mathrm{Reg}(S)$. Similarly, $xa \in \mathrm{Reg}(S)$. Hence, $\mathrm{Reg}(S)$ is an ideal of $S$. □

COROLLARY 2.6. A semigroup $S$ is a retract extension of a regular $U$-semigroup by a $U_{n+1}$-nil–semigroup if and only if $S$ satisfies (2.2).

**Proof.** Let $S$ be a retract extension of a regular $U$-semigroup $T$ by a $U_{n+1}$-nil-semigroup $Q$, with the retraction $\varphi$. Then $T$ is completely regular and $S$ is periodic. Let $x \in S$ and let $x^m = e \in E(S) \subseteq T$ for some $m \in \mathbf{Z}^+$. Then $e = x^m = \varphi(x^m) = (\varphi(x))^m$, whence $\varphi(x) \in G_e$ (Theorem I 4.1. [3]). Now we have that $\varphi(x) = e\varphi(x) = \varphi(e)\varphi(x) = \varphi(ex) = ex = x^{m+1}$. Therefore, for every $x \in S$ it is $\varphi(x) \in \langle x \rangle$. Let $x_1, x_2, \ldots, x_{n+1} \in S$. If $x_1x_2 \ldots x_{n+1} \in S - T$, then $x_1x_2 \ldots x_{n+1} \neq 0$ in $Q$, so $x_1, x_2, \ldots, x_{n+1} \in Q - \{0\} = S - T$ and $x_1x_2 \ldots x_{n+1} = x_i^k \neq 0$ in $Q$, $2 \leq i \leq n+1$, $k \in \mathbf{Z}^+$, whence $x_1x_2 \ldots x_{n+1} = x_i^k$ in $S$. Let $x_1x_2 \ldots x_{n+1} \in T$. Then $x_1x_2 \ldots x_{n+1} = \varphi(x_1x_2 \ldots x_{n+1}) = \varphi(x_1)\varphi(x_2) \ldots \varphi(x_{n+1}) \in \langle \varphi(x_1) \rangle \cup \langle \varphi(x_2) \rangle \cup \ldots \cup \langle \varphi(x_{n+1}) \rangle \subseteq \langle x_1 \rangle \cup \langle x_2 \rangle \cup \ldots \cup \langle x_{n+1} \rangle$. Thus, $S$ is a $U_{n+1}$-semigroup and, clearly, $\mathrm{Reg}(S) = T$ is an ideal of $S$, so by Theorem 2.4. it follows that $S$ satisfies (2.2). The converse follows by Theorem 2.4. and by Corollary 2.5. □

EXAMPLE. The semigroup given by the following table

| | $x$ | $e$ | $f$ |
|---|---|---|---|
| $x$ | $e$ | $e$ | $x$ |
| $e$ | $e$ | $e$ | $e$ |
| $f$ | $x$ | $e$ | $f$ |

is a $U$-semigroup, but $\mathrm{Reg}(S) = \{e, f\}$ is not an ideal of $S$ since $xf = x \notin \mathrm{Reg}(S)$.

LEMMA 2.9. [13]. If $x$ is a nonzero element of a nil-semigroup $S$, then $x \notin xS \cup Sx \cup SxS$. □

LEMMA 2.10. Let $S$ be a $U$-semigroup. Then $S$ is a nil-semigroup if and only if $S^5 = 0$.

**Proof.** Let $S$ be a $U$-nil-semigroup and let $a_i \in S$, $i = 1, 2, 3, 4, 5$. Let $a = a_1 a_2 a_3 a_4 a_5$. Since $a_1a_2 \in \langle a_1 \rangle \cup \langle a_2 \rangle$, then we may take $a_1a_2 = a_i^k$, $i \in \{1, 2\}$, $k \geq 2$ (using Lemma 2.9.). Then $a_i^k a_3 = a_j^m$, where $m \geq 3$ and $j \in \{i, 3\}$, i.e. $j \in \{1, 2, 3\}$, and $a_j^m a_4 = a_s^n$, where $n \geq 4$ and $s \in \{1, 2, 3, 4\}$.

Finally, $a_s^n a_5 = a_r^p$, where $p \geq 5$ and $r \in \{1, 2, 3, 4, 5\}$. From this and by Theorem 2.1. we obtain that $a = 0$, i.e. $S^5 = 0$.

The converse follows immediately. □

## 3.

In this section we consider some special $U_{n+1}$-semigroups. Some construction of these also will be given.

DEFINITION 3.1. $S$ is a $U_{(m_1,\dots,m_{n+1})}^{(k_1,\dots,k_{n+1})}$ *-semigroup*, $1 \leq k_i \leq m_i, i = 1, 2,\dots,n+1$, $n \in \mathbf{Z}^+$ if $x_1 x_2 \dots x_{n+1} \in \left\{x_1^{k_1}, x_1^{k_1+1},\dots, x_1^{m_1},\dots, x_{n+1}^{k_{n+1}}, x_{n+1}^{k_{n+1}+1},\dots, x_{n+1}^{m_{n+1}}\right\}$ for all $x_1,x_2,\dots,x_{n+1} \in S$. If $k_i = 1$ for all $i = 1, 2,\dots, n+1$, then we put $U_{(m_1,\dots,m_{n+1})} = U_{(m_1,\dots,m_{n+1})}^{(1,\dots,1)}$. A group $G$ is a $U_{(m_1,\dots,m_{n+1})}^{(k_1,\dots,k_{n+1})}$*-group* if $G$ is a $U_{(m_1,\dots,m_{n+1})}^{(k_1,\dots,k_{n+1})}$ -semigroup.

REMARK. $U_{(1,1,\dots,1)}$-semigroups are described by J. Pelikán [11], and $U_{(1,1,1)}$-semigroups are treated by B. Pondělíček [15]. $U_{(2,2)}$-semigroups are studied by A. E. Evseev [8].

LEMMA 3.1. Every subsemigroup and every homomorphic image of a $U_{(m_1,\dots,m_{n+1})}^{(k_1,\dots,k_{n+1})}$-semigroup is a $U_{(m_1,\dots,m_{n+1})}^{(k_1,\dots,k_{n+1})}$-semigroup. □

LEMMA 3.2. Let $n \geq 5$. Then the following conditions are equivalent on a semigroups $S$:

(i) $S$ is a $U$-nil-semigroup;
(ii) $S$ is a $U_{(n,n)}$-nil-semigroup;
(iii) $S$ is a $U_{(5,5)}$-nil-semigroup.

**Proof.** Follows by Lemma 2.10. □

LEMMA 3.3. Let $2 \leq n \leq 4$. Then $S$ is a $U_{(n,n)}$-nil-semigroup if and only if $S$ is a $U$-semigroup, $S^{n+1} = 0$ and

$$(\forall x, y \in S)\ \ xy = 0 \Rightarrow x^n = 0 \vee y^n = 0. \tag{3.1}$$

**Proof.** Let $S$ be a $U_{(n,n)}$-nil-semigroup and let $x \in S$. Then $x^{n+1} = xx^n \in \{x, x^2,\dots, x^n, x^{2n},\dots x^{n^2}\}$ whence $x^{n+1} = 0$. Let $n = 2$. Let $a_1,a_2,a_3 \in S$ and $a = a_1a_2a_3$. By Lemma 2.9. we have that $a_1a_2 = a_i^2$, $i \in \{1,2\}$ and $a_i a_3 = a_i^2$ or $a_i a_3 = a_3^2$. Assume that $a_i a_3 = a_i^2$ (the case $a_i a_3 = a_3^2$ is similar to the previous). Then $a = a_1a_2a_3 = a_i^2a_3 = a_i a_i a_3 = a_i a_i^2 = 0$. Hence, $S^3 = 0$. Let $n = 3$. Let $a_1, a_2, a_3, a_4$ $S$ and $a = a_1 a_2 a_3 a_4$. Then

$a_1a_2 = a_i^k$, $k \geq 2$, $i \in \{1,2\}$ and $a_3 a_4 = a_j^m$, $m \geq 2$, $j \in \{3, 4\}$. Also, $a_i a_j = a_i^p$, $p \geq 2$ or $a_i a_j = a_j^q$, $q \geq 2$. Assume that $a_i a_j = a_i^p$, $p \geq 2$ (the case $a_ia_j = a_j^q$, $q \geq 2$ is similar to the previous). Then

$$a = a_i^k a_j^m = a_i^{k-1} a_i a_j a_j^{m-1} = a_i^{k-1+p} a_j^{m-1} = \ldots = a_i^{mp+k-m} = 0$$

since $mp + k - m = m(p-1) + k \geq m + k \geq 4$. Thus $S^4 = 0$. If $n = 4$, then by Lemma 2.10. we have that $S^5 = 0$. Let $x,y \in S$, $xy = 0$. Then $xy = x^k$ or $xy = y^k$ for some $k \in \mathbf{Z}^+$, $1 \leq k \leq n$. So $x^k = 0$ or $y^k = 0$ for some $k \in \mathbf{Z}^+$, $1 \leq k \leq n$, whence $x^n = 0$ or $y^n = 0$. So (3.1) holds.

Conversely, let $S$ be a $U$-semigroup such that $S^{n+1} = 0$ and let (3.1) holds. Then $S$ is a $U$-nil-semigroup and by Lemma 3.2. $xy \in \{x,\ldots,x^4, x^5 = 0 = y^5, y,\ldots,y^4\}$ for all $x, y \in S$. Since $x^{n+1} = 0$ for all $x \in S$, we have that $xy \in \{x,\ldots,x^n,x^{n+1} = 0 = y^{n+1}, y,\ldots,y^n\}$ for all $x, y \in S$. Now, if $xy = x^{n+1} = y^{n+1} = 0$, then by (3.1) we have that $x^n = 0$ or $y^n = 0$, so $xy = x^n$ or $xy = y^n$. Therefore, $S$ is a $U_{(n,n)}$-nil-semigroup. □

REMARK. If $n = 4$, then the assertion of Lemma 3.3. holds without the condition $S^{n+1} = 0$.

DEFINITION 3.2. A band $Y$ of semigroups $S_\alpha$, $\alpha \in Y$ is a $U^{(k_1,\ldots,k_{n+1})}_{(m_1,\ldots,m_{n+1})}$ *-band* of semigroups if

$$x_1x_2\ldots x_{n+1} \in \{x^{k_1}, x^{k_1+1},\ldots,x^{m_1},\ldots,x^{k_{n+1}}, x^{k_{n+1}+1},\ldots,x^{m_{n+1}}\}$$

for every $x_1 \in S_{\alpha_1},\ldots, x_{n+1} \in S_{\alpha_{n+1}}$ and there exist $i, j \in \{1,\ldots, n+1\}$ such that $S_{\alpha_i} \neq S_{\alpha_j}$. One defines analogously $U^{(k_1,\ldots,k_{n+1})}_{(m_1,\ldots,m_{n+1})}$*-chain* of semigroups.

THEOREM 3.1. Let $n \geq 5$. Then the following conditions are equivalent on a semigroup $S$:

(i) $S$ is a $U_{(n,n)}$-semigroup;
(ii) $S$ is a $U_{(n,n)}$-band of $U_{(n,n)}$-semigroups with only one idempotent;
(iii) $S$ is a $U_{(n,n)}$-chain of $U_{(n,n)}$-semigroups with only one idempotent or of retract extensions of a singular band by a $U$-nil-semigroup.

**Proof.** (i) ⇒ (ii). This follows by Theorem 2.3. and by Lemma 3.1.
(ii) ⇒ (i). This implication follows immediately.
(i) ⇒ (iii). This implication follows by Theorem 2.3. and by Lemmas 3.1. and 3.2.
(iii) ⇒ (i). We prove only that if $S_\alpha$ is a retract extension of a left zero band $E_\alpha$ by a $U$-nil-semigroup with the retraction $\varphi : S_\alpha \to E_\alpha$, then $S_\alpha$ is a $U_{(5,5)}$-semigroup. If $x \in E_\alpha$ and $y \in S_\alpha - E_\alpha$, then $xy = x\varphi(y) = x$. If $x \in S_\alpha - E_\alpha$ and $y \in E_\alpha$ then $xy = \varphi(x)y = \varphi(x) = x^5$, since $x^5 \in S_\alpha^5 = E_\alpha$ and $x^5 = \varphi(x^5) = (\varphi(x))^5 = \varphi(x)$. Thus $S_\alpha$ is a $U_{(5,5)}$-semigroup. By symmetry, we also have that a retract extension of a right zero band by a $U$-nil-semigroup is a $U_{(5,5)}$-semigroup. Now, using the fact that $S$ is a $U_{(n,n)}$-chain $Y$ of semigroups $S_\alpha$, $\alpha \in Y$ we have that $S$ is a $U_{(n,n)}$-semigroup. □

THEOREM 3.2. Let $2 \leq n \leq 4$. Then the following conditions are equivalent on a semigroup $S$:

(i) $S$ is a $U_{(n,n)}$-semigroup;

(ii) $S$ is a $U_{(n,n)}$-chain $Y$ of semigroups $S_\alpha$, $\alpha \in Y$ and one of the following conditions hold:

(1) $S_\alpha$ is a retract extension of a singular band $E_\alpha$ by a $U_{(n,n)}$-nil-semigroup $Q_\alpha$ and

$$(\forall x \in Q_\alpha) \qquad x^n = 0; \tag{3.2}$$

(2) $S_\alpha$ is a $U_{(n,n)}$-semigroup with only one idempotent;

(iii) $S$ is a $U_{(n,n)}$-band of $U_{(n,n)}$-semigroups with only one idempotent.

**Proof.** (i) $\Rightarrow$ (ii). Let $S$ be a $U_{(n,n)}$-semigroup. Then $S$ is a $U$-semigroup and by Theorem 2.3. $S$ is a $U$-chain of semigroups $S_\alpha$, $\alpha \in Y$ and $S_\alpha$ is a retract extension of a singular band by a $U$-nil-semigroup or $S_\alpha$ is an ideal extension of a $U$-group by a $U$-nil-semigroup. Now by Lemma 3.1. we have that $S$ is a retract extension of a singular band by a $U_{(n,n)}$-nil-semigroup or $S$ is a $U_{(n,n)}$-semigroup with only one idempotent. Let $S_\alpha$ be a retract extension of a singular band $E_\alpha$ by a $U_{(n,n)}$-nil-semigroup $Q_\alpha$ with the retraction $\varphi$. Let $|E_\alpha| \geq 2$, $x \in S_\alpha - E_\alpha = Q_\alpha - \{0\}$, and let $\varphi(x) = e$. Then there exists $f \in E$ such that $f \neq e$, so $xf = \varphi(x)f = ef = e$, if $E_\alpha$ is a left zero band and $fx = f\varphi(x) = fe = e$, if $E_\alpha$ is a right zero band. Since we have that $xf, fx \in \{x, x^2, \ldots, x^n, f\}$, then $e = x^k$ for some $k$, $1 \leq k \leq n$, whence $x^n = e$. Thus $x^n = 0$ in $Q$.

(ii) $\Rightarrow$ (i). We prove only that a retract extension $S_\alpha$ of a left zero band $E_\alpha$ by a $U_{(n,n)}$-nil-semigroup $Q_\alpha$ with (3.2) is a $U_{(n,n)}$-semigroup. Let $\varphi : S_\alpha \to E_\alpha$ be the retraction. If $x \in E_\alpha$, $y \in S_\alpha - E_\alpha$, then $xy = x\varphi(y) = x$. If $x \in S_\alpha - E_\alpha$, $y \in E_\alpha$, then $xy = \varphi(x)y = \varphi(x) = x^n$, since by (3.2) $x^n = 0$ in $Q$, i.e. $x^n \in E_\alpha$ so $x^n = \varphi(x^n) = (\varphi(x))^n = \varphi(x)$. If $x, y \in S_\alpha - E_\alpha$ and $xy \in E_\alpha$, then $xy = \varphi(x)\varphi(y) = \varphi(x) = x^n$. Thus $S_\alpha$ is a $U_{(n,n)}$-semigroup. The remainder of the proof now follows without difficulty. We have the similar proof if $E_\alpha$ is a right zero band.

(i) $\Rightarrow$ (iii). This follows by Theorem 2.3.

(iii) $\Rightarrow$ (i). This follows immediately. □

The following theorem describe $U_{(n,n)}$-groups.

THEOREM 3.3. $G$ is a $U_{(n,n)}$-group if and only if $G$ is a cyclic group of order $p^k \leq n$ for some prime $p$, $k \in Z^+$.

**Proof.** Let $G$ be a $U_{(n,n)}$-group. Then $G$ is a $U$-group and by Lemma 2.3. $G$ is a cyclic group of order $p^k$ or $G$ is a $Z_{p^\infty}$ for some prime $p$. Let $xG$ and let $r(x)$ be the order of $x$. Let $e$ be the identity of $G$. Then $e = xx^{-1} \in \{x, x^2, \ldots, x^n, (x^{-1}), \ldots, (x^{-1})^n\}$ so $x^k = e$ for some $k$, $1 \leq k \leq n$. Therefore, for every $x \in G$, $r(x) \leq n$ so $G$ is a cyclic group of order $p^k \leq n$ for some prime $p$. The converse follows by Lemmas 2.2. and 2.3. □

THEOREM 3.4. Let $p$ be a prime. Let $A$ be a $U_{(p,\,p)}$-nil-semigroup with the zero $e$ and let $G$ be a cyclic group of the order $p$ with the identity $e$, such that $A\cap G = \{e\}$. Define a multiplication $*$ on $S = A\cup G$ by:

$$x * y = \begin{cases} xy & \text{if} \quad (x,\, y)\in A\times A \cup G\times G \\ x & \text{if} \quad (x,\, y)\in G\times A \\ y & \text{if} \quad (x,\, y)\in A\times G \end{cases}$$

Then $(S,\, *)$ is a $U_{(p,\,p)}$-semigroup with only one idempotent.

Conversely, every $U_{(p,\,p)}$-semigroup with only one idempotent which is an ideal extension of a group of order $p$ can be so constructed.

**Proof.** It is simple to verify that $(S,\, *)$ is a $U_{(p,p)}$-semigroup with only one idempotent. Conversely, let $S$ be a $U_{(p,\,p)}$-semigroup with only one idempotent $e$ and let $S$ be an ideal extension of a group $G = \{b, b^2,\dots,b^{p-1}, b^p = e\}$ of order $p$. Then this extension is retractive. Assume that $\varphi : S\to G$ is the retraction of $S$ onto $G$. If we put $Y_k = \varphi^{-1}(b^k)$, $k = 1,2,\dots,p$, then $b^k\in Y_k$, $Y_i\cap Y_j = \emptyset$ if $i\neq j$, $S = \bigcup_{k=1}^{p} Y_k$ and $Y_iY_j\subseteq Y_{i+j}$ if $i+j\le p$, and $Y_iY_j\subseteq Y_{i+j-p}$ if $i+j>p$, $1\le i,\ j\le p$. Let $x\in Y_k$ for some $k$, $1\le k\le p-1$. Then $ex\in G\cap Y_k = \{b^k\}$ so $ex = b^k$. On the other hand, $b^k = ex\in\{e,x,x^2,\dots,x^p\}$, and since $x^i\in Y_j$ where $j\equiv ki(\mathrm{mod}p)$, $i = 2, 3,\dots,p$ and $b^{ki}\neq b^k$, then we have that $b^k = x$. Indeed, if $b^k = b^{ki}$ for some $i\in\{2,\dots,p\}$ then $b^{k(i-1)} = e$, which is not possible, since the order of $b^k$ is $p$. Thus $Y_k = \{b^k\}$ for every $k\in\{1, 2,\dots,p-1\}$. Assume that $A = Y_p$. Then it is clear that $A = S/G$. By Lemma 3.1. $A$ is a $U_{(p,\,p)}$-nil-semigroup. We have, also, that $A\cap G = \{e\}$, $S = A\cup G$ and for all $x\in A$, $y\in G$ $xy = yx = y$. □

THEOREM 3.5. Let $A$ be a $U_{(3,\,3)}$-nil-semigroup with the zero element $e$.

(A) Let $B$ be a nonempty set such that $A\cap B = \emptyset$ and let $b$ be a fixed element from $B$. Let $\varphi : B\to A$ be a mapping such that

$$(\forall x\in A)(\forall y\in B)\ x\varphi(y) = \varphi(y)x = \varphi(b). \tag{3.3}$$

Assume that $B\times B = M\cup N$ and let

$$(\forall y\in B)\ (b,\, y)\in M \ \wedge\ (y,\, b)\in N. \tag{3.4}$$

Define a multiplication $*$ on $S = A\cup B$ by

$$x * y = \begin{cases} xy & \text{if} \quad (x,\, y)\in A\times A \\ b & \text{if} \quad (x,\, y)\in A\times B \cup B\times A \\ \varphi(x) & \text{if} \quad (x,\, y)\in M \\ \varphi(y) & \text{if} \quad (x,\, y)\in N \end{cases}$$

(B) Let $G$ be a cyclic group of the order 3 with the identity $e$ such that $A \cap G = \{e\}$. Define a multiplication on $S = A \cup G$ by:

$$x * y = \begin{cases} xy & \text{if} \quad (x, y) \in A \times A \cup G \times G \\ x & \text{if} \quad (x, y) \in G \times A \\ y & \text{if} \quad (x, y) \in A \times G \end{cases}$$

Then $(S, *)$ from (A) or (B) is a $U_{(3,3)}$-semigroup with only one idempotent.

Conversely, every $U_{(3,3)}$-semigroup with only one idempotent is a $U_{(3,3)}$-nil-semigroup or it can be constructed by (A) or by (B).

**Proof.** (A) Let $x,y,z \in S$. If $(x,y,z) \in A \times A \times B$ then $x * (y * z) = x * b = b = (xy) * z = (x * y) * z$. The associativity in the case $(x,y,z) \in A \times B \times B$ can be proved in a similar way as above. Let $(x,y,z) \in B \times A \times B$. Then $(x * y) * z = b * z = \varphi(b)$ and $x * (y * z) = x * b = \varphi(b)$. So $(x * y) * z = x * (y * z)$. Finally, let $(x,y,z) \in B \times B \times B$. Then

$$(x * y) * z = \begin{cases} \varphi(x) * z = b & \text{if} \quad (x, y) \in M \\ \varphi(y) * z = b & \text{if} \quad (x, y) \in N \end{cases}$$

$$x * (y * z) = \begin{cases} x * \varphi(y) = b & \text{if} \quad (y, z,) \in M \\ x * \varphi(z) = b & \text{if} \quad (y, z) \in N \end{cases}$$

so $(x * y) * z = x * (y * z)$. Therefore, $(S, *)$ is a semigroup. It is clear that $S$ contains only one idempotent. Let $(x, y) \in A \times B$. Then $x * y = b$ and $y^2 = y * y = \varphi(y)$, $y^3 = y^2 * y = \varphi(y) * y = b$. Thus, $x * y = y^3$. Analogously, if $(x,y) \in B \times A$, then $x * y = b = x^3$. If $x,y \in B$, then $x * y \in \{\varphi(x), \varphi(y)\} = \{x^2, y^2\}$. Therefore, $S$ is a $U_{(3,3)}$-semigroup.

(B) That $(S, *)$ from the construction is a $U_{(3,3)}$-semigroup with only one idempotent follows by Theorem 3.4.

Conversely, let $S$ be a $U_{(3,3)}$-semigroup with only one idempotent $e$. Then by Lemma 2.8. $S$ is an ideal extension of a $U$-group $G$ by a $U$-nil-semigroup. By Lemma 3.3. $S$ is an ideal extension of a $U_{(3,3)}$-group by a $U_{(3,3)}$-nil-semigroup. By Theorem 3.3. $G$ is a cyclic group of order $r \geq 3$. If $r = 1$, then $S$ is a $U_{(3,3)}$-nil-semigroup. If $r = 3$, then the assertion follows by Theorem 3.4. Let $G = \{e, b\}$. In this case $S$ is a retract extension of $G$ with the retraction $f : S \to G$. Let $A = f^{-1}(e)$ and $B = f^{-1}(b)$. It is easily to see that $A^2 \subseteq A$, $B^2 \subseteq A$, $AB \subseteq B$ and $BA \subseteq B$. Define a mapping $\varphi : B \to A$ with $\varphi(y) = y^2$, $y \in B$. Since $xy \in \{x, x^2, x^3, y, y^2, y^3\}$ for $x \in A$, $y \in B$ and $\{x, x^2, x^3, y^2\} \subseteq A$ we then have that $xy \in \{y, y^3\}$. If $xy = y$, then $x^4 y = y$. By Lemma 3.3. $x^4 = e$. So $y = x^4 y = ey \in G \cap B$, whence $y = b = b^3$. Therefore, if $(x, y) \in A \times B$, then $xy = y^3$. Similarly, $yx = y^3$. If we put $x = e$, then we obtain that $y^3 = ey = b$. Thus, $(x, y) \in A \times B$ implies that

$xy = yx = b$. Furthermore, $x\varphi(y) = xy^2 = xyy = by = e = yb = yyx = y^2x = \varphi(y)x$, i.e. (3.3) holds, since $\varphi(b) = b^2 = e$. Let $M = \{(x, y) \in B\times B \mid xy = x^2\}$, $N = \{(x, y) \in B\times B \mid xy = y^2\}$. Then from $xy \in \{x, x^2, x^3, y, y^2, y^3\}$ and from the fact that $B^2 \subseteq A$ and $\{x, x^3, y, y^3\} \subseteq B$ for all $x,y \in B$ we have that $xy \in \{x^2, y^2\}$ for all $x,y \in B$, so $B\times B = M\cup N$. If $(x, y) \in M$ then $xy = \varphi(x)$ and if $(x,y) \in N$, then $xy = \varphi(y)$. Also, for $y \in B$, $by$, $yb \in G\cap A$, i.e. $by = yb = e = b^2$. Thus, (3.4) holds. Therefore, if $|G| = 2$, then $S$ can be obtained by $(A)$. □

THEOREM 3.6. A semigroup $S$ is an $n$-inflation of a Rédei's band $T$ if and only if

$$x_1x_2\dots x_{n+1} \in \left\{x_1^{n+2}, x_2^{n+2}\dots x_{n+1}^{n+2}\right\} \tag{3.5}$$

for every $x_1, x_2, \dots, x_{n+1} \in S$.

**Proof.** If (3.5) holds, then $x^{n+1} = x^{n+2}$ for all $x \in S$, so $x^{n+1} \in E(S)$ for every $x \in S$. Thus, $S^{n+1} = E(S)$ and by Lemma 2.5. $E(S)$ is a Rédei's band. Let $x^{n+1} = e$, $y^{n+1} = f$, $e,f \in E(S)$. Then by Theorem 2.3. we have that $(xy)^{n+1} = ef = x^{n+1}y^{n+1}$. Therefore, by Theorem 3.4. [5] we have that $S$ is an $n$-inflation of a Rédei's band.

Conversely, let $S$ be an $n$-inflation of a Rédei's band $T$. Then $S^{n+1} = T$ and there exists a retraction $\varphi$ of $S$ onto $T$. Then for every $x_1, x_2, \dots, x_{n+1} \in S$ we have that

$$x_1x_2\dots x_{n+1} = \varphi(x_1x_2\dots x_{n+1}) = \varphi(x_1)\,\varphi(x_2)\dots\varphi(x_{n+1}).$$

Since

$$\varphi(x_i) = (\varphi(x_i))^{n+2} = \varphi(x_i^{n+2}) = x_i^{n+2}$$

for all $i \in \{1, 2, \dots n+1\}$, then

$$x_1x_2\dots x_{n+1} = x_1^{n+2}, x_2^{n+2}\dots x_{n+1}^{n+2} \in \{x_1^{n+2}, x_2^{n+2}, \dots, x_{n+1}^{n+2}\}. \ \square$$

COROLLARY 3.1. (A. E. Evseev [8]). A semigroup is an inflation (1-inflation) of a Rédei's band ($U_{(1,1)}$-semigroup) if and only if $xy \in \{x^3, y^3\}$ for every $x,y \in S$. □

COROLLARY 3.2. A semigroup $S$ is a 2-inflation of a Rédei's band $T$ and $x^2 \in T$ for all $x \in S$ if and only if $xyz \in \{x^2, y^2, z^2\}$ for all $x, y, z \in S$.

**Proof.** Let $xyz \in x^2, y^2, z^2$ for all $x, y, z \in S$. Then $x^3 = x^2 \in E(S)$ for every $x \in S$. So $x^2 = x^4$ for every $x \in S$, whence $xyz \in \{x^4, y^4, z^4\}$ for every $x, y, z \in S$, and by Theorem 3.6. we have the assertion.

The converse follows immediately by Theorem 3.6. □

COROLLARY 3.3. A semigroup $S$ contains only one idempotent and $xyz \in \{x^4, y^4, z^4\}$ for all $x, y, z \in S$ if and only if $S^3 = 0$. □

COROLLARY 3.4. A semigroup $S$ contains only one idempotent and $xyz \in \{x^2, y^2, z^2\}$ for all $x, y, z \in S$ if and only if $S^3 = 0$ and $x^2 = 0$ for every $x \in S$. □

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