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Traps, Cores, Extensions and Subdirect Decompositions of Unary Algebras*

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Abstract. Subdirect decompositions of unary algebras are studied in connection with one-element subalgebras, cores, Rees extensions of congruences of subalgebras, dense extensions and disjunctive elements. In particular, subdirectly irreducible unary algebras are described in terms of these notions.

1. Introduction

There are two well-known natural ways of viewing finite automata as algebraic structures. If one regards the input sequences of an automaton as elements of the free monoid generated by a finite alphabet of input symbols, then it is natural to treat the automaton as a finite monoid

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of transformations. This approach has been very successful indeed. In particular, it has been the starting point for the elegant classification theory of regular languages based on syntactic monoids and varieties of finite monoids, and it is the basis of the renowned Krohn-Rhodes decomposition theory of finite automata. These theories not only rely on semigroup theory, but they have also greatly stimulated the study of finite semigroups. However, it is equally natural to regard an automaton as a finite algebra in which each input symbol is realized as a unary operation. This interpretation, advocated by J.R. Büchi and J.B. Wright already in the late fifties, links automata with universal algebra, and the theory of tree automata in which arbitrary finite algebras are viewed as automata arose almost spontaneously from it ([1] offers an interesting historical perspective of these ideas).

Many basic notions of universal algebra have natural interpretations in the theory of automata. For example, an automaton with no proper subalgebras is strongly connected, morphisms are used for defining ways of representing an automaton by another, and direct products and subdirect representations correspond to parallel connections and parallel decompositions, respectively. In particular, the subdirect irreducibility of an automaton-algebra means that it cannot be realized by a parallel connection of simpler automata.

The subdirectly irreducible (finite) automata with one input symbol were characterized by Yoeli [15], and Wenzel [14] determined all subdirectly irreducible mono-unary algebras. The subdirectly irreducible automata with arbitrary input alphabets of some special types, such as nilpotent, commutative or definite automata, have been described in [3], [4], [7], [8] and [11].

In this paper we use both universal algebra and semigroup theory as we discuss subdirect decompositions and the subdirect irreducibility of unary algebras in terms of traps, cores, extensions and disjunctive elements, that is to say, in terms of notions originating mostly from semigroup theory. Since no finiteness assumptions are needed, we consider general unary algebras rather than just automata. In fact, the cardinality of the set of operation symbols may also be infinite.

In Section 2 we recall some basic concepts and fix our general notation. Traps, trap-connected (unary) algebras, cores and kernels are introduced in the following section. A trap is an element which forms a one-element subalgebra, and an algebra is said to be trap-connected if it has a trap and this trap appears in every subalgebra. If an algebra $\mathcal{A}$ has a least subalgebra, it is called the kernel of $\mathcal{A}$, and if $\mathcal{A}$ has a least nontrivial subalgebra, it is the core of $\mathcal{A}$. That nontrivial subdirectly irreducible unary algebras have cores was shown by Setoyanagi [11] and we study the properties of these cores. In Section 4 we consider Rees extensions of congruences of a subalgebra, a notion studied already in [14] and [11], and extensions of algebras. In particular, we show that if a unary algebra $\mathcal{A}$ is an extension of a subalgebra $\mathcal{B}$ by an algebra $\mathcal{C}$, then $\mathcal{A}$ is a subdirect product of $\mathcal{C}$ and a dense extension of $\mathcal{B}$; an algebra $\mathcal{A}$ is a dense extension of a subalgebra $\mathcal{B}$ if the diagonal relation on A is the only congruence on $\mathcal{A}$ which restricted to $\mathcal{B}$ is the diagonal on B.

In Section 5 we study traps, cores and disjunctive elements in subdirectly irreducible unary algebras, and we characterize subdirectly irreducible unary algebras in terms of these. An element a of an algebra $\mathcal{A}$ is said to be disjunctive if the principal congruence generated by it is the diagonal relation, that is to say, if a alone does not form a congruence class of any nontrivial congruence. That any nontrivial subdirectly irreducible algebra has disjunctive elements is known from before, but here we show that a unary algebra with two traps is subdirectly irreducible if and only if the traps are disjunctive.

2. Preliminaries

In what follows, X is always an alphabet, but not necessarily finite. As usual, X^* denotes the set of all (finite) words over X. The empty word is denoted by e. With the catenation of words as the operation and e as the unit element, X^* is the free monoid generated by X. However, we shall also treat X as a set of unary operation symbols and words over X are then to be regarded as X-terms over a one-element set of variables $\{\varepsilon\}$ written in reverse Polish notation: the empty word represents the term ε and any nonempty word $x_1x_2\dots x_n$ $(n \geq 1)$ the term $\varepsilon x_1x_2\dots x_n$. An X-*algebra* $\mathcal{A} = (A, X)$ is a system where A is a nonempty set and each symbol $x \in X$ is realized as a unary operation $x^{\mathcal{A}} : A \to A$. For any $a \in A$ and $x \in X$, we write $ax^{\mathcal{A}}$ for $x^{\mathcal{A}}(a)$. For any word $w = x_1x_2\dots x_n$ $(\in X^*)$, $w^{\mathcal{A}} : A \to A$ is defined as the composition of the mappings $x_1^{\mathcal{A}}, x_2^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}$, that is to say, $aw^{\mathcal{A}} = ax_1^{\mathcal{A}}x_2^{\mathcal{A}}\dots x_n^{\mathcal{A}}$ for every $a \in A$. In particular, $e^{\mathcal{A}}$ is the identity mapping 1_A of A. If $\mathcal{A}$ is known from the context, we write simply aw instead of $aw^{\mathcal{A}}$. An X-algebra $\mathcal{A} = (A, X)$ is *finite* if A is a finite set, and $\mathcal{A}$ is *trivial* if A has only one element. In what follows, we often assume, without saying so, that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are the X-algebras (A, X), (B, X) and (C, X), respectively.

Subalgebras, morphisms, congruences, quotiens and direct products of X-algebras are defined as for algebras in general (*cf.* [2] or [5], for example). Hence $\mathcal{B} = (B, X)$ is a *subalgebra* of an X-algebra $\mathcal{A} = (A, X)$ if $B \subseteq A$ and $bx^{\mathcal{B}} = bx^{\mathcal{A}}$ for all $x \in X$ and $b \in B$, and then B is a *closed subset* of $\mathcal{A}$, *i.e.*, $bx^{\mathcal{A}} \in B$ for all $x \in X$ and $b \in B$. For any $H \subseteq A$, we denote the least closed subset containing H as a subset by $\langle H\rangle$. If $H \neq \emptyset$, this is the *subalgebra generated* by H. It is obvious that $\langle H\rangle = \{aw : a \in H, w \in X^*\}$ for every $H \subseteq A$. For a singleton set $H = \{a\}$, we use the notation $\langle a\rangle$.

A *morphism* from $\mathcal{A}$ to $\mathcal{B}$ is mapping $\varphi : A \to B$ such that $ax^{\mathcal{A}}\varphi = a\varphi x^{\mathcal{B}}$ for all $a \in A$ and $x \in X$, and we write then $\varphi : \mathcal{A} \to \mathcal{B}$. A morphism is called an *isomorphism*, a *monomorphism* or an *epimorphism* if it is, respectively, bijective, injective or surjective. Two X-algebras $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if there is an isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ and this is expressed by writing $\mathcal{A} \cong \mathcal{B}$.

The set of all equivalences on a set A is denoted by $\mathrm{Eq}(A)$. If $\theta \in \mathrm{Eq}(A)$, the θ-class $\{b \in A : a\,\theta\, b\}$ of any $a \in A$ is denoted by $a\theta$. An equivalence $\theta \in \mathrm{Eq}(A)$ is a *congruence* on an algebra $\mathcal{A} = (A, X)$ if $ax\,\theta\, bx$ for all $x \in X$ whenever $a\,\theta\, b$. The set of all congruences on $\mathcal{A}$ is denoted by $\mathrm{Con}(\mathcal{A})$. Obviously, $\mathrm{Con}(\mathcal{A})$ always includes the *diagonal relation* $\Delta_A = \{(a, a) : a \in A\}$ and the *universal relation* $\nabla_A = A \times A$. Note that if $\theta \in \mathrm{Con}(\mathcal{A})$ and $a\,\theta\, b$, then $aw\,\theta\, bw$ for all $w \in X^*$. For any $\theta \in \mathrm{Con}(\mathcal{A})$, the *quotient algebra* $\mathcal{A}/\theta = (A/\theta, X)$ is defined so that $(a\theta)x^{\mathcal{A}/\theta} = (ax^{\mathcal{A}})\theta$ for all $a \in A$ and $x \in X$. The corresponding *canonical epimorphism* $\theta^\sharp : \mathcal{A} \to \mathcal{A}/\theta$ is defined by the condition $a\theta^\sharp = a\theta$ ($a \in A$).

Direct products and *subdirect decompositions* of X-algebras are also defined as usual. However, here it suffices to know

(1) that every proper subdirect decomposition of an algebra $\mathcal{A}$ is obtained from a family of nontrivial congruences $\mathcal{F}$ on $\mathcal{A}$ such that $\bigcap \mathcal{F} = \Delta_A$, and

(2) that a nontrivial X-algebra is *subdirectly irreducible* if and only if it has a least nontrivial congruence.

These facts are due to G. Birkhoff (*cf.* [2] or [5]).

An X-algebra $\mathcal{A} = (A, X)$ is called also an *automaton* if X and A are finite. In fact, our work is motivated to a great extent by the theory of automata and we shall also use some terminology from that theory. An X-algebra $\mathcal{A}$ is *connected* if for all pairs of elements $a, b \in A$, there are words $u, v \in X^*$ such that $au = bv$, and it is *strongly connected* if for all pairs of elements $a, b \in A$, there exists a word $w \in X^*$ such that $aw = b$. Obviously, $\mathcal{A} = (A, X)$ is strongly connected if and only if $\langle a \rangle = A$ for every element $a \in A$.

3. Traps, Cores and Kernels

An element $a \in A$ of an X-algebra $\mathcal{A} = (A, X)$ is called a *trap* if $ax = a$ for every $x \in X$, and $\mathcal{A}$ is a *trap algebra* if it has a trap. Obviously, a connected X-algebra can have at most one trap, and if it has a trap, it is said to be *trap-connected*. Furthermore, a nontrivial X-algebra $\mathcal{A} = (A, X)$ is *strongly trap-connected* if it has a trap $a_0 \in A$ and $\langle a \rangle = A$ for every $a \in A \setminus \{a_0\}$. Any strongly trap-connected algebra is trap-connected, but the converse does not hold. An X-algebra is called *discrete* if all of its elements are traps.

The least ideal of a semigroup S, if it exists, is called the kernel of S (*cf.* [6], for example). We call the least subalgebra of an X-algebra $\mathcal{A}$, if it exists, the *kernel* of $\mathcal{A}$. If $\mathcal{A}$ has a least non-trivial subalgebra, it is called the *core* of $\mathcal{A}$. Cores of automata were considered also by Setoyanagi [11]. It is clear that if an X-algebra $\mathcal{A}$ has a core, it is the intersection of all non-trivial subalgebras of $\mathcal{A}$. Similarly, the kernel, when it exists, is the intersection of all subalgebras of $\mathcal{A}$.

The following observations are immediate consequences of the fact that any set of traps is a closed subset.

Lemma 3.1. *If an X-algebra has a kernel, then it has at most one trap, and an X-algebra with a core has at most two traps.*

Theorem 3.1. *Let $\mathcal{A}$ be any X-algebra.*

(a) The kernel of $\mathcal{A}$, if it exists, is strongly connected.

(b) The core of $\mathcal{A}$, if it exists, is strongly connected, strongly trap-connected or discrete.

Proof:
If $\mathcal{K} = (K, X)$ is the kernel of $\mathcal{A}$, then $K \subseteq \langle a \rangle \subseteq K$ for every $a \in K$, and hence $\mathcal{K}$ is strongly connected.

Assume now that $\mathcal{A}$ has a core $\mathcal{C}$. By Lemma 3.1, $\mathcal{C}$ contains at most two traps. If there is no trap, then $\langle a \rangle$ is a nontrivial subalgebra for every $a \in C$, and hence $\mathcal{C}$ must be strongly connected. If $\mathcal{C}$ contains one trap a_0, then $\langle a \rangle \subseteq C \subseteq \langle a \rangle$ for every $a \in C \setminus \{a_0\}$, and hence $\mathcal{C}$ is strongly trap-connected. Finally, if $\mathcal{C}$ contains two traps a and b, then $C = \{a, b\}$ must hold and $\mathcal{C}$ is discrete. □

4. Rees Congruences. Extensions of Algebras

As noted in [13], for example, one may associate with any subalgebra of an X-algebra $\mathcal{A}$ a congruence on $\mathcal{A}$ akin to the Rees congruence defined by an ideal on a semigroup. The following extension of this construction was introduced for mono-unary algebras in [14] and for general unary algebras in [11].

Let $\mathcal{B} = (B, X)$ be a subalgebra of an X-algebra $\mathcal{A} = (A, X)$. The *Rees extension* to $\mathcal{A}$ of a congruence θ of $\mathcal{B}$ is defined as the relation $R(\theta) = \theta \cup \Delta_A$. It is clear that $R(\theta) \in \mathrm{Con}(\mathcal{A})$. In particular, the *Rees congruence* $\rho_{\mathcal{B}}$ modulo $\mathcal{B}$ is the Rees extension $R(\nabla_B)$ of the universal relation ∇_B. The quotient algebra $\mathcal{A}/\rho_{\mathcal{B}}$ is denoted also by $\mathcal{A}/\mathcal{B}$.

We say that an X-algebra $\mathcal{A}$ is an *extension* of $\mathcal{B}$ by $\mathcal{C}$ if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\mathcal{A}/\mathcal{B} \cong \mathcal{C}$. If this is the case, $\mathcal{C}$ evidently has a trap which corresponds to the image of $\mathcal{B}$ under the canonical epimorphism $\mathcal{A} \to \mathcal{A}/\mathcal{B}$. In other words, we may regard $\mathcal{C}$ as the result of contracting the subautomaton $\mathcal{B}$ of $\mathcal{A}$ into one element, a trap of $\mathcal{C}$. A *trap-extension* of an algebra is obtained by adjoining to it a trap, that is to say, $\mathcal{A}$ is a trap-extension of an algebra $\mathcal{B}$ if it is an extension of $\mathcal{B}$ by a two-element discrete algebra.

Theorem 4.1. *If $\mathcal{B}$ is a subalgebra of an X-algebra $\mathcal{A} = (A, X)$, then $\theta \mapsto R(\theta)$ defines an isomorphism from the congruence lattice of $\mathcal{B}$ onto the principal ideal $(\rho_{\mathcal{B}}] = \{\rho \in \mathrm{Con}(\mathcal{A}) : \Delta_A \subseteq \rho \subseteq \rho_{\mathcal{B}}\}$ of the lattice $\mathrm{Con}(\mathcal{A})$.*

Proof:
It is clear that $R(\theta_1) \neq R(\theta_2)$ for any two distinct congruences θ_1 and θ_2 of $\mathcal{B}$. It is also obvious that if $\rho \in (\rho_{\mathcal{B}}]$, then $R(\rho \cap \nabla_B) = (\rho \cap \nabla_B) \cup \Delta_A = \rho$. Hence, R defines a bijection between $\mathrm{Con}(\mathcal{B})$ and $(\rho_{\mathcal{B}}]$. Furthermore, for any congruences θ_1 and θ_2 on $\mathcal{B}$,

$$\theta_1 \subseteq \theta_2 \Longleftrightarrow \theta_1 \cup \Delta_A \subseteq \theta_2 \cup \Delta_A,$$

and hence $R : \mathrm{Con}(\mathcal{B}) \to (\rho_{\mathcal{B}}]$, $\rho \mapsto R(\rho)$, is an order-isomorphism. □

Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. A congruence θ on $\mathcal{A}$ is called a *$\mathcal{B}$-congruence* if $\theta \cap \nabla_B = \Delta_B$. If Δ_A is the only $\mathcal{B}$-congruence on $\mathcal{A}$, we say that $\mathcal{A}$ is a *dense extension* of $\mathcal{B}$. In particular, every algebra is a dense extension of itself.

Theorem 4.2. *If an X-algebra $\mathcal{A}$ is an extension of an X-algebra $\mathcal{B}$ by an X-algebra $\mathcal{C}$, then $\mathcal{A}$ is a subdirect product of $\mathcal{C}$ and some dense extension of $\mathcal{B}$.*

Proof:
Since the union of any chain of $\mathcal{B}$-congruences on $\mathcal{A}$ is obviously also a $\mathcal{B}$-congruence, it follows from Zorn's Lemma that $\mathcal{A}$ has maximal $\mathcal{B}$-congruences. Let μ be any maximal $\mathcal{B}$-congruence and let $\mathcal{D} = \mathcal{A}/\mu$. Since μ is a $\mathcal{B}$-congruence, $\mathcal{B}$ is isomorphic to the subalgebra $\mathcal{B}' = (B', X)$ of $\mathcal{D}$, where $B' = \{a\mu : a \in B\}$ and thus $\mathcal{D}$ may be regarded as an extension of $\mathcal{B}$. Let us now show that this extension is dense.

By the Correspondence Theorem (*cf.* p.49 in [2]), the mapping

$$\theta \mapsto \theta/\mu = \{(a\mu, b\mu) : (a, b) \in \theta\}$$

is an isomorphism from the interval $[\mu, \nabla_A]$ of the lattice Con($\mathcal{A}$) onto Con($\mathcal{D}$). Let $\theta \in [\mu, \nabla_A]$ be such that θ/μ is a $\mathcal{B}'$-congruence on $\mathcal{D}$. If $a\,\theta\, b$ for some $a, b \in B$, then $a\mu, b\mu \in B'$ and $(a\mu, b\mu) \in \theta/\mu$, and hence $a\mu = b\mu$. Since μ is a $\mathcal{B}$-congruence, this implies $a = b$. Thus θ is a $\mathcal{B}$-congruence such that $\mu \subseteq \theta$, and hence $\theta = \mu$ by the maximality of μ. This shows that $\mathcal{D}$ has no nontrivial $\mathcal{B}'$-congruence, and therefore $\mathcal{D}$ is a dense extension of $\mathcal{B}$.

Finally, $\mu \cap \rho_{\mathcal{B}} = \Delta_A$ since μ is a $\mathcal{B}$-congruence, and hence $\mathcal{A}$ is a subdirect product of $\mathcal{C} \cong \mathcal{A}/\mathcal{B}$ and $\mathcal{D} = \mathcal{A}/\mu$. □

Remark 4.1. The subdirect decomposition described above is proper exactly in case the subalgebra $\mathcal{B}$ is nontrivial and $\mathcal{A}$ has nontrivial $\mathcal{B}$-congruences. Indeed, $\Delta_A \subset \rho_{\mathcal{B}}$ if and only if $|B| > 1$, and $\Delta_A \subset \mu$ if and only if $\mathcal{A}$ has nontrivial $\mathcal{B}$-congruences.

An epimorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of an X-algebra $\mathcal{A}$ onto a subalgebra $\mathcal{B}$ is called a *retraction* if $b\varphi = b$ for all $b \in B$. If such a retraction exists, $\mathcal{B}$ is a *retract* of $\mathcal{A}$ and $\mathcal{A}$ is a *retractive extension* of $\mathcal{B}$.

Lemma 4.1. *If $\varphi : \mathcal{A} \to \mathcal{B}$ is a retraction, then* ker φ *is a maximal $\mathcal{B}$-congruence on $\mathcal{A}$.*

Proof:
That ker φ is a $\mathcal{B}$-congruence follows from the fact that $b\varphi = b$ for every $b \in B$. Let θ be any $\mathcal{B}$-congruence on $\mathcal{A}$ such that ker $\varphi \subseteq \theta$. If $a\,\theta\, b$ for some $a, b \in A$, then $a\varphi\,\theta\, b\varphi$ by transitivity since $(a, a\varphi), (b, b\varphi) \in$ ker $\varphi \subseteq \theta$. Since $a\varphi, b\varphi \in B$, this implies $a\varphi = b\varphi$, and hence ker $\varphi = \theta$ must hold. □

Lemma 4.1 yields immediately the following sharper form of Theorem 4.2 for retractive extensions.

Corollary 4.1. *If $\mathcal{A}$ is a retractive extension of $\mathcal{B}$ by $\mathcal{C}$, then $\mathcal{A}$ is a subdirect product of $\mathcal{C}$ and $\mathcal{B}$.*

5. Subdirectly Irreducible X-algebras

In this section we examine the subdirect irreducibility of X-algebras in terms of traps, cores, extensions and disjunctive elements. The following fact was observed both in [14] and [11], but we restate it and give a short direct proof.

Lemma 5.1. *An X-algebra is subdirectly irreducible if and only if all of its subalgebras are subdirectly irreducible.*

Proof:
The condition is clearly sufficient. Suppose now that a subalgebra $\mathcal{B}$ of an X-algebra $\mathcal{A}$ has a proper subdirect decomposition. This means that $\mathcal{B}$ has a family of nontrivial congruences $\{\theta_i : i \in I\}$ such that $\bigcap\{\theta_i : i \in I\} = \Delta_B$. But then also $\mathcal{A}$ has a proper subdirect decomposition since $\bigcap\{R(\theta_i) : i \in I\} = \Delta_A$ while $R(\theta_i) = \theta_i \cup \Delta_A \supset \Delta_A$ for every $i \in I$. □

Theorem 5.1. *If an X-algebra $\mathcal{A}$ has a nontrivial subalgebra $\mathcal{B}$, then $\mathcal{A}$ is subdirectly irreducible if and only if $\mathcal{B}$ is subdirectly irreducible and $\mathcal{A}$ is a dense extension of $\mathcal{B}$.*

Proof:
Assume that $\mathcal{A}$ is subdirectly irreducible. Then $\mathcal{B}$ is subdirectly irreducible by Lemma 5.1, and by Theorem 4.2 (and its proof) $\mathcal{A}$ is a subdirect product of $\mathcal{A}/\mathcal{B}$ and a dense extension $\mathcal{A}/\mu$ of $\mathcal{B}$. Since $\rho_{\mathcal{B}} \supset \Delta_A$ and $\mathcal{A}$ is subdirectly irreducible, $\mu = \Delta_A$ must hold, and hence $\mathcal{A}$ itself is a dense extension of $\mathcal{B}$.

Suppose now that $\mathcal{B}$ is subdirectly irreducible and that $\mathcal{A}$ is a dense extension of $\mathcal{B}$. If $\{\theta_i : i \in I\}$ is a family of congruences on $\mathcal{A}$ such that $\bigcap\{\theta_i : i \in I\} = \Delta_A$, then $\theta_i \cap \nabla_B \in \mathrm{Con}(\mathcal{B})$ for every $i \in I$ and $\bigcap\{\theta_i \cap \nabla_B : i \in I\} = \Delta_B$. Since $\mathcal{B}$ is subdirectly irreducible, this means that $\theta_i \cap \nabla_B = \Delta_B$ for some $i \in I$, and hence $\theta_i = \Delta_A$ as $\mathcal{A}$ is a dense extension of $\mathcal{B}$. This shows that $\mathcal{A}$ has no proper subdirect decomposition. □

A subset H of a set A is *saturated* by an equivalence θ on A if H is the union of some θ-classes. For any subset H of an X-algebra $\mathcal{A} = (A, \mathrm{X})$, the relation σ_H on A defined so that for any $a, b \in A$,

$$a \,\sigma_H\, b \iff (\forall u \in \mathrm{X}^*)(au \in H \iff bu \in H),$$

is the greatest congruence on $\mathcal{A}$ which saturates H. This fact has been proved in different forms in various contexts. For example, in semigroup theory the counterparts of the congruences σ_H are called *principal congruences*, and for a language H regarded as a subset of a finitely generated free monoid, σ_H is the *syntactic congruence* of H. More generally, the syntactic congruence of a subset of any algebra can be obtained in a similar manner (*cf.* [12], for example). A subset H of an X-algebra $\mathcal{A} = (A, X)$ is called *disjunctive* if $\sigma_H = \Delta_A$, and an element $a \in A$ is *disjunctive* if $\{a\}$ is disjunctive.

The following fact was first proved for semigroups by Schein [10] and that it holds for algebras in general was noted in [12].

Lemma 5.2. *Any nontrivial subdirectly irreducible X-algebra has at least two disjunctive elements.*

Theorem 5.2. *Any X-algebra $\mathcal{A}$ with at least two disjunctive elements has a core and all disjunctive elements of $\mathcal{A}$ are in the core.*

Proof:
Let $\mathcal{B}$ be a nontrivial subalgebra of $\mathcal{A}$. If $a \in A \setminus B$, then $\{a\}$ is saturated by $\rho_{\mathcal{B}}(\supset \Delta_A)$, and hence a is not disjunctive. This means that the intersection of the nontrivial subalgebras of $\mathcal{A}$ contains all disjunctive elements. Since $\mathcal{A}$ has at least two disjunctive elements, this intersection is the core of $\mathcal{A}$. □

Lemma 5.2, Theorem 5.2 and Lemma 3.1 yield the following result which combines Lemma 2.3 and Theorem 2.2 of [11].

Corollary 5.1. *Every nontrivial subdirectly irreducible X-algebra has a core and at most two traps.*

Theorem 5.3. *A nontrivial X-algebra $\mathcal{A}$ is subdirectly irreducible if and only if it is a dense extension of a nontrivial subdirectly irreducible subalgebra $\mathcal{B}$ by a trap-connected algebra and this $\mathcal{B}$ satisfies one of the following conditions:*

($C0$) *$\mathcal{B}$ is the core of $\mathcal{A}$ and strongly connected;*
($C1$) *$\mathcal{B}$ is the core of $\mathcal{A}$ and strongly trap-connected, or $\mathcal{B}$ is a trap-extension of the core of $\mathcal{A}$ and the core is strongly connected;*
($C2$) *$\mathcal{B}$ is the core of $\mathcal{A}$ and a two-element discrete X-algebra.*

Moreover, for each $k = 0, 1, 2$, $\mathcal{B}$ satisfies condition (Ck) if and only if $\mathcal{A}$ has exactly k traps.

Proof:
Assume first that $\mathcal{A}$ is subdirectly irreducible. Then $\mathcal{A}$ has by Corollary 5.1 a core $\mathcal{C} = (C, X)$ and at most two traps. ¿From Theorem 5.1 it follows now that $\mathcal{C}$ is subdirectly irreducible and that $\mathcal{A}$ is a dense extension of $\mathcal{C}$. We distinguish three cases according to the number of traps of $\mathcal{A}$.

1. If $\mathcal{A}$ has no traps, then the core $\mathcal{C}$ is strongly connected. Moreover, $\mathcal{A}/\mathcal{C}$ is trap-connected since $C \subseteq \langle a \rangle$ for every $a \in A$. Hence ($C0$) is satisfied for $\mathcal{B} = \mathcal{C}$.

2. Assume that $\mathcal{A}$ has just one trap a_0. If $a_0 \in C$, then $\mathcal{C}$ is strongly trap-connected as shown in the proof of Theorem 3.1. Furthermore, if $a \in A \setminus \{a_0\}$, then $\langle a \rangle$ is a nontrivial subalgebra and hence $C \subseteq \langle a \rangle$. This means that $\mathcal{A}/\mathcal{C}$ is trap-connected. Hence, the first alternative of ($C1$) is satisfied for $\mathcal{B} = \mathcal{C}$. If $a_0 \notin C$, then $\mathcal{C}$ is strongly connected and $\mathcal{B} = (B, X)$, where $B = C \cup \{a_0\}$, is a trap-extension of $\mathcal{C}$. Again $C \subseteq \langle a \rangle$ for every $a \in A \setminus \{a_0\}$, and therefore $\mathcal{A}/\mathcal{B}$ is trap-connected. Hence, the second alternative of ($C1$) holds.

3. If $\mathcal{A}$ has two traps a_1 and a_2, then obviously $C = \{a_1, a_2\}$. Moreover, for every $a \in A \setminus C$, $\langle a \rangle$ is a nontrivial subalgebra and therefore $C \subseteq \langle a \rangle$. This means that $\mathcal{A}/\mathcal{C}$ is trap-connected, and hence ($C2$) is satisfied for $\mathcal{B} = \mathcal{C}$.

Conversely, let $\mathcal{A}$ be a dense extension of a nontrivial subdirectly irreducible subalgebra $\mathcal{B}$ by a trap-connected algebra, where $\mathcal{B}$ satisfies one of the conditions $(C0)$-$(C2)$. Then $\mathcal{A}$ is subdirectly irreducible by Theorem 5.1. Moreover, since $\mathcal{A}/\mathcal{B}$ is trap-connected, all traps of $\mathcal{A}$, if there are any, are in B. It is clear that if $\mathcal{B}$ satisfies condition (Ck) for some $k = 0, 1, 2$, then there are exactly k traps. □

The following corollary is an immediate consequence of Lemma 5.2, Theorem 5.2 and Lemma 5.1.

Corollary 5.2. *Any nontrivial subdirectly irreducible X-algebra has a subdirectly irreducible core and at least two disjunctive elements.*

Theorem 5.4. *An X-algebra with two traps is subdirectly irreducible if and only if its traps are disjunctive.*

Proof:
Let $\mathcal{A} = (A, X)$ have two traps a_1 and a_2. If $\mathcal{A}$ is subdirectly irreducible, it follows from Theorem 5.3 that $\mathcal{A}$ has a core $\mathcal{C} = (C, X)$ and that $C = \{a_1, a_2\}$. By Lemma 5.2, $\mathcal{A}$ has two disjunctive elements and by Theorem 5.2 these are in C. Hence a_1 and a_2 must be disjunctive.

Assume now that the traps a_1 and a_2 are disjunctive. By Theorem 5.2, $\mathcal{A}$ has a core which contains a_1 and a_2, and it is then clear that $\mathcal{C} = (\{a_1, a_2\}, X)$ is the core. Consider any $\mathcal{C}$-congruence θ on $\mathcal{A}$. Suppose first that $a\,\theta\,b$ for some $a, b \in A \setminus \{a_1, a_2\}$, $a \neq b$. Since $\{a_1, a_2\} \subseteq \langle a\rangle, \langle b\rangle$ and a_1 is disjunctive, there exists a word $w \in X^*$ such that either $aw = a_1 \neq bw$ or $aw \neq a_1 = bw$. Furthermore, in the former case $bw \neq a_2$ and in the second case $aw \neq a_2$, because otherwise $a_1\,\theta\,a_2$ would hold. In both cases $aw\,\theta\,bw$, and hence we may conclude that if $\theta \neq \Delta_A$, then either $a_1\,\theta\,b$ or $a_2\,\theta\,b$ for some $b \in A \setminus \{a_1, a_2\}$. Let us suppose that $a_1\,\theta\,b$. Then $a_1\,\theta\,bw$ for every $w \in X^*$. Since $(a_1, a_2) \notin \theta$, this means that $a_1w, bw \notin \{a_2\}$ for all $w \in X^*$, and hence $a_1\,\sigma_{\{a_2\}}\,b$. But this is impossible since a_2 is disjunctive, and we can conclude that $\mathcal{A}$ is a dense extension of $\mathcal{C}$. On the other hand, it is clear that $\mathcal{C}$ is subdirectly irreducible, and it follows then from Theorem 5.1 that $\mathcal{A}$ is subdirectly irreducible. □

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