Fuzzy social network analysis[*]

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Abstract

The purpose of this paper is twofold. First, we intend to encourage researchers in social sciences to use some contemporary mathematical methods different than the traditional methods of statistics and data analysis, in particular, to encourage them to use methods of the social network analysis. Second, we want to propagate the fuzzy approach as a way to overcome the vagueness that is always present in social sciences. We give a brief overview of the main ideas and recent results in the positional analysis of fuzzy social networks, and we point to relationships between the social network analysis and certain up-to-date areas of computer science and mathematics.

1. Introduction

Network analysis has originated as a branch of sociology and mathematics which provides formal models and methods for the systematic study of social structures, and it has an especially long tradition in sociology, social psychology and anthropology. But, concepts of network analysis capture the common properties of all networks and its methods are applicable to the analysis of networks in general. For that reason, methods of network analysis are nowadays increasingly applied to many networks which are not social networks but share a number of commonalities with social networks, such as the hyperlink structure on the Web, the electric grid, computer networks, information networks or various large-scale networks appearing in nature. Network analysis is carried out in areas such as project planning, complex systems, electrical circuits, social networks, transportation systems, communication networks, epidemiology, bioinformatics, hypertext systems, text analysis, organization theory, event analysis, bibliometrics, genealogical research, and others. In most of these applications network analysis rely on a formal basis that is fairly coherent. In this paper, however, we have in mind primarily the application in the study of social structures.

The key difference between network analysis and other approaches is the focus on relationships among actors (social entities) rather than the attributes of individual actors. Network analysis takes a global view on network structures, based on the belief that types and patterns of relationships emerge from individual connectivity and that the presence (or absence) of such types and patterns have substantial effects on the network and its

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constituents. In particular, the network structure provides opportunities and imposes constraints on the individual actors by determining the transfer or flow of resources (material or immaterial) across the network. Such an approach requires a set of methods and analytic concepts that are distinct from the methods of traditional statistics and data analysis.

The natural means to model networks mathematically is provided by the mathematical notion of a *relation*, and because we work only with finite sets of actors, these relations are usually represented by *matrices* and visualized as *graphs*. Consequently, methods of network analysis primarily originate from semigroup theory (semigroups of relations), linear algebra and graph theory. This formality served network analysis to reduce the vagueness in formulating and testing its theories, and contributed to more coherence in the field by allowing researchers to carry out more precise discussions in the literature and to compare results across studies. More information on various aspects of the network analysis and its applications can be found in the books [6, 8, 16, 22, 26, 32].

Nevertheless, the above mentioned vagueness in social networks (as well as in many other kinds of networks) can not be completely avoided, since relations between actors are in essence vague. This vagueness can be overcame applying fuzzy approach to the network analysis, but just few authors dealt with this topic so far (cf. [12, 14, 15, 23, 24]).

Our interest in social networks originates from our research in the theory of fuzzy automata. It turned out that some very important concepts that we used in the state reduction of fuzzy automata are closely related to the concept of a regular equivalence, which is fundamental in the positional analysis of social networks. The methodology that was developed in [11,31], has been since applied in [17] to the general study of certain systems of fuzzy relation equations and inequalities, what led to results that are directly applicable to fuzzy social networks. Here we give a brief overview of the main ideas and recent results in the positional analysis of fuzzy social networks, and we point to relationships between the social network analysis and certain up-to-date areas of computer science and mathematics.

2. Fuzzy sets and fuzzy relations

Fuzzy sets were introduced in 1965 by L. A. Zadeh [34], as a method for representing some imprecise aspects of human knowledge that would be used in dealing with problems when the source of imprecision is the absence of sharply defined criteria of class membership. Such problems are very often when one deals with classes of objects encountered in the real physical world, and for that reason fuzzy sets have significant applications in many scientific fields.

Unlike ordinary set theory and classical logic, where membership degrees and truth values have only values 1 or 0 (*true* or *false*; *yes* or *no*), fuzzy set theory and fuzzy logic allow intermediate membership degrees and truth values, i.e., they are taken from some larger set L. In other words, a *fuzzy subset* of a set U is defined as any function f which maps U in L, i.e., $f: U \to L$, where L is a given set of truth values. For every $x \in U$, the value $f(x)$ is the *membership degree* of the element x to the fuzzy subset f. To be able to say that something is more or less true than something else, the set L of truth values has to be ordered, and also has to have 1 (*absolutely true*) as the greatest element and 0 (*absolutely false*) as the smallest element. Moreover, to define operations on fuzzy subsets analogous to operations on ordinary crisp sets, certain lattice-theoretical operations on the set L of truth values are

also needed. The most studied and applied structures of truth values are those defined on the real unit interval [0,1] by means of certain operations on it, but in our research we use complete residuated lattices, which include as special cases all structures of truth values traditionally used in fuzzy logic.

A *residuated lattice* is an algebra $\mathbb{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,

(L3) $\otimes$ and $\rightarrow$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leq z \Leftrightarrow x \leq y \rightarrow z.$$

If, moreover, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathbb{L}$ is called a *complete residuated lattice*.

The operations $\otimes$ (called *multiplication*) and $\rightarrow$ (called *residuum*) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum ($\vee$) and infimum ($\wedge$) are intended for modeling of the existential and general quantifier, respectively. An operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$$

called *biresiduum* (or *biimplication*), is used for modeling the equivalence of truth values.

The most studied and applied structures of truth values, defined on the real unit interval [0,1] with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are the *Łukasiewicz structure* (where $x \otimes y = \max(x + y - 1, 0)$, $x \rightarrow y = \min(1 - x + y, 1)$), the *Goguen structure* (or *product structure*) (with $x \otimes y = x \cdot y$, $x \rightarrow y = 1$ if $x \leq y$, and $= y/x$ otherwise), and the *Gödel structure* (with $x \otimes y = \min(x, y)$, $x \rightarrow y = 1$ if $x \leq y$, and $= y$ otherwise). More generally, an algebra $([0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and the residuum is defined by $x \rightarrow y = \vee \{u \in [0,1] | u \otimes x \leq y\}$ (cf. [1,2]). Another important set of truth values is the set $\{a_0, a_1, \dots, a_n\}$, $0 = a_0 < \cdots < a_n = 1$, with $a_k \otimes a_l = a_{\max(k+l-n, 0)}$ and $a_k \rightarrow a_l = a_{\min(n-k+l, n)}$. A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0,1\}$. The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations. This structure of truth values we call the *Boolean structure*. If any finitelly generated subalgebra of a residuated lattice $\mathbb{L}$ is finite, then $\mathbb{L}$ is called *locally finite*. For example, every Gödel structure is locally finite, whereas the product structure is not locally finite.

In the rest of the paper, if not noted otherwise, $\mathbb{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ will be a complete residuated lattice, and all fuzzy sets and fuzzy relations that will be considered will take their membership values in $\mathbb{L}$ (i.e., in L).

The concept of a fuzzy relation naturally arose from that of fuzzy set in Zadeh's very first paper on fuzzy sets [34], and it was further developed in his paper [35], where the notions of a fuzzy equivalence relation and fuzzy ordering were introduced. After that, a number of papers dealing with various aspects related to these relations have appeared, and today, the theory of fuzzy binary relations is probably one of the most important and influential branches of the fuzzy set theory. By allowing intermediate degrees of relationship, fuzzy relations provide much more freedom to express the subtle nuances of human preferences so they found natural applications in modeling various concepts inherent to so-called "soft"

sciences like psychology, sociology, linguistics, etc. The concept of a fuzzy relation will be also fundamental in our fuzzy approach to social networks.

A *fuzzy relation* on a set U is any fuzzy subset of the Cartesian product $U \times U$, that is, any function $A : U \times U \to L$. Let A and B be fuzzy relations on the set U. The *equality* of A and B is defined as the usual equality of functions, i.e., $A = B$ if and only if $A(x, y) = B(x, y)$, for all $x, y \in U$. The *inclusion* of A in B is also defined pointwise: we put $A \leq B$ if and only if $A(x, y) \leq B(x, y)$, for all $x, y \in U$.

Endowed with this partial order the set $\mathcal{R}(U)$ of all fuzzy relations on U forms a complete lattice, in which the meet (intersection) $\bigwedge_{i \in I} A_i$ and the join (union) $\bigvee_{i \in I} A_i$ of a family $\{A_i\}_{i \in I}$ of fuzzy relations on U are functions from $U \times U$ into L defined by

$$\left(\bigwedge_{i \in I} A_i \right)(x, y) = \bigwedge_{i \in I} A_i(x, y), \quad \left(\bigvee_{i \in I} A_i \right)(x, y) = \bigvee_{i \in I} A_i(x, y).$$

A *crisp relation* on U is a fuzzy relation on U which takes values only in the set $\{0,1\}$. If A is a crisp relation on U, then expressions "$A(x) = 1$" and "$x \in A$" will have the same meaning, i.e., A is considered as an ordinary relation on U. The *crisp part* of a fuzzy relation A on U is a crisp relation $A^c : U \times U \to L$ defined by $A^c(x) = 1$, if $A(x) = 1$, and $A^c(x) = 0$, if $A(x) < 1$, i.e., $A^c = \{x \in A \mid A(x) = 1\}$.

For two fuzzy relations A and B on the set U, the *composition* or *product* of A and B is a fuzzy relation $A \circ B$ on U defined by

$$(A \circ B)(x, y) = \bigvee_{z \in U} A(x, z) \otimes B(z, y),$$

for all $x, y \in U$. If U is a finite set with n elements, then A and B can be regarded as $n \times n$ fuzzy matrices over $\mathbb{L}$ and $A \circ B$ can be regarded as the matrix product.

A *fuzzy equivalence* on a set U is a fuzzy relation E on U which satisfies $E(x, x) = 1$ (*reflexivity*), $E(x, y) = E(y, x)$ (*symmetry*), and $E(x, y) \otimes E(y, x) \leq E(x, z)$ (*transitivity*), for all $x, y, z \in U$.

For undefined notion and notation the reader is referred to [1,2,9-11,17,31].

3. Positional analysis of fuzzy social networks and regular fuzzy equivalences

A *fuzzy social network* is defined as a fuzzy relational structure $\mathbb{U} = (U, \{A_i\}_{i \in I})$, where U is a non-empty set of *actors* or *nodes*, and $\{A_i\}_{i \in I}$ is a given family of fuzzy relations on U, i.e., a family of fuzzy relations between actors. As usual, the set U is assumed to be finite. We also call $\mathbb{U}$ shortly a *fuzzy network*. If $\{A_i\}_{i \in I}$ are crisp relations (or $\mathbb{L}$ is assumed to be the Boolean structure), then $\mathbb{U}$ is an ordinary crisp social network, or just a social network.

Because U is finite, fuzzy relations $\{A_i\}_{i \in I}$ can be regarded as fuzzy matrices, which was often done in the literature dealing with social networks. Also, social networks were often regarded as multigraphs or labeled graphs (with labels taken from the index set I), and in the fuzzy context, we can regard fuzzy social networks as directed labeled fuzzy graphs. In many situations not only basic relationships given by fuzzy relations A_i are interesting, but also composite relationships between actors given as compositions of basic relations. Seeing that fuzzy and crisp relations form monoids under the composition, the semigroup-

theoretical approach to the study of social networks was often applied. It is worth noting that fuzzy and crisp social networks can be also regarded as fuzzy and non-deterministic automata, but the automata-theoretical approach has not been used until now. However, it will be applied in our further research.

In large and complex systems it is impossible to understand the relationship between each pair of individuals, but to a certain extent, it may be possible to understand the system, by classifying individuals and describing relationships on the class level. In networks, for instance, nodes in the same class can be considered to occupy the same *position*, or play the same *role* in the network. The main aim of the positional analysis of networks is to find similarities between nodes which have to reflect their position in a network. These similarities have been formalized first by Lorrain and White [20] by the concept of a *structural equivalence*. Informally speaking, two nodes are considered to be structurally equivalent if they have identical neighborhoods. However, in many situations this concept has shown oneself to be too strong, and weakening it sufficiently to make it more appropriate for modeling social positions, White and Reitz [33] have introduced the concept of a *regular equivalence*. Here, two nodes are considered to be regularly equivalent if they are equally related to equivalent others. Afterwards, regular equivalences have been studied in numerous papers, e.g., in [3, 4, 5, 6, 7, 13, 25, 26]. One of the main problems discussed in these papers was to compute the greatest regular equivalence on a given social network, or in some cases, the greatest one contained in a given equivalence. The greatest regular equivalence is extremely important because it has a minimal number of equivalence classes, and therefore, it provides a minimal number of positions in a social network.

As we have already mentioned, fuzzy equivalences were introduced by Zadeh [35] as a generalization of ordinary crisp equivalences and equality to the fuzzy framework. They have been since widely studied as a way to measure the degree of indistinguishability or similarity between the objects of a given universe of discourse, and they have shown to be useful in different contexts such as fuzzy control, approximate reasoning, cluster analysis, etc. Depending on the authors and the context in which they appears, they have received other names such as *similarity relations* (original Zadeh's name) or *indistinguishability operators*.

Using the concept of a fuzzy equivalence, Fan et al. [14, 15] have extended the notion of a regular equivalence to the fuzzy framework. They defined a *regular fuzzy equivalence* on a fuzzy network $\mathbb{U} = (U, \{A_i\}_{i \in I})$ as any fuzzy equivalence E on U such that $A_i \circ E = E \circ A_i$, for each $i \in I$. In other words, regular fuzzy equivalences are just solutions to the system of fuzzy relation equations

$$A_i \circ X = X \circ A_i \ (i \in I),$$

where X is an unknown fuzzy relation and solutions are sought in the set $\mathcal{E}(U)$ of all fuzzy equivalences on U. In a similar way, the notion of a structural equivalence can be extended to the fuzzy framework. Namely, a *structural fuzzy equivalence* on a fuzzy network $\mathbb{U}$ is any solution to the system of fuzzy relation equations

$$A_i \circ X = A_i \ , \ X \circ A_i = A_i \ (i \in I),$$

where X is an unknown fuzzy relation and solutions are also sought in $\mathcal{E}(U)$. In particular, if $\mathbb{L}$ is the two-element Boolean algebra (i.e., if we deal with crisp relations), then solutions to this system are exactly the structural equivalences in the sense of Lorrain and White [20]

(cf., e.g., [19]). Therefore, our concept of a structural fuzzy equivalence generalizes the concept of a structural equivalence.

Fan et al. [14, 15] proved the existence of the greatest regular fuzzy equivalence on a fuzzy network contained in a given fuzzy equivalence, and provided procedures for computing the greatest regular fuzzy equivalence and the greatest regular crisp equivalence contained in a given fuzzy (resp. crisp) equivalence. However, they have considered only fuzzy networks over the Gödel structure. Here we discuss a more general case of fuzzy networks over a complete residuated lattice, where some difficulties appear which are not present when we work with fuzzy networks over the Gödel structure.

4. Computing the greatest regular fuzzy equivalence on a fuzzy network

As we have noted in the previous section, the greatest structural fuzzy equivalence on the fuzzy network $\mathbb{U}$ is the greatest solution in $\mathcal{E}(U)$ to the system

(4.1) $A_i \circ X = A_i \;,\; X \circ A_i = A_i \;\; (i \in I)$.

Clearly, the system (4.1) is the conjunction of two systems of fuzzy relation equations:

(4.2) $A_i \circ X = A_i \;\; (i \in I)$,

(4.3) $X \circ A_i = A_i \;\; (i \in I)$,

Using the well-known results by Sanchez [30] (see also [27-29]), it has been shown in [17] that the greatest fuzzy equivalences E^{ls} and E^{rs} which are solutions to (4.2) and (4.3), respectively, are given by

$$E^{ls}(x,y) = \bigwedge_{i \in I} \bigwedge_{z \in U} A_i(z,x) \leftrightarrow A_i(z,y),$$

$$E^{rs}(x,y) = \bigwedge_{i \in I} \bigwedge_{z \in U} A_i(x,z) \leftrightarrow A_i(y,z),$$

for all $x, y \in U$.

Consequently, the greatest structural fuzzy equivalence E^s on the fuzzy network $\mathbb{U}$ is the intersection of E^{ls} and E^{rs}, i.e., $E^s = E^{ls} \wedge E^{rs}$. Moreover, for any given fuzzy equivalence F on U, the greatest structural fuzzy equivalence on $\mathbb{U}$ contained in F is just the intersection $E^s \wedge F$ of E^s and F.

As well as the greatest regular fuzzy equivalence is concerned, the situation is much complicated. It has been found in [17] that it is very convenient to represent the system

(4.4) $A_i \circ X = X \circ A_i \;\; (i \in I)$

as the conjunction of two systems of fuzzy relation inequalities:

(4.5) $A_i \circ X \leq X \circ A_i \;\; (i \in I)$,

(4.6) $X \circ A_i \leq A_i \circ X \;\; (i \in I)$.

The splitting of the system (4.4) is extremely important for computing its greatest solution. Namely, it has been shown in [17] that the system (4.5) is equivalent to the system of fuzzy relation equations

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(4.7) $X \circ A_i \circ X = X \circ A_i \ (i \in I)$,

and (4.6) is equivalent to the system

(4.8) $X \circ A_i \circ X = A_i \circ X \ (i \in I)$,

and it turned out the systems (4.7) and (4.8) are more convenient for computing the greatest solutions. Indeed, let us define functions ϕ^l, ϕ^r, and ϕ of $\mathcal{E}(U)$ into itself by

(4.9) $(\phi^l(E))(x, y) = \bigwedge_{i \in I} \bigwedge_{z \in U} (E \circ A_i)(z, x) \leftrightarrow (E \circ A_i)(z, y)$,

(4.10) $(\phi^r(E))(x, y) = \bigwedge_{i \in I} \bigwedge_{z \in U} (A_i \circ E)(x, z) \leftrightarrow (A_i \circ E)(y, z)$,

(4.11) $\phi(E) = \phi^l(E) \wedge \phi^r(E)$,

for all $x, y \in U$ and $E \in \mathcal{E}(U)$. Then the systems (4.7) and (4.8), i.e., the systems (4.5) and (4.6), are respectively equivalent to the following fuzzy relation inequalities

$$X \leq \phi^l(X), \quad X \leq \phi^r(X),$$

and consequently, the system (4.4) is equivalent to the fuzzy relation inequality

$$X \leq \phi(X).$$

Therefore, the greatest regular fuzzy equivalence on the fuzzy network $\mathbb{U}$ contained in a given fuzzy equivalence F on U can be computed as the greatest solution in $\mathcal{E}(U)$ to the following system of fuzzy relation inequalities

(4.12) $X \leq \phi(X), \quad X \leq F$.

In the sequel we present an algorithm for computing the greatest solution to (4.12).
    Let us define a sequence $\{E_k\}_{k \in \mathbb{N}}$ of fuzzy equivalences on U by

(4.13) $E_1 = F, \quad E_{k+1} = E_k \wedge \phi(E_k), \ \text{for every } k \in \mathbb{N}$.

The sequence $\{E_k\}_{k \in \mathbb{N}}$ is descending, and if it is finite, i.e., if $E_k = E_{k+l}$, for some $k, l \in \mathbb{N}$, then $E_k = E_{k+m}$, for every $m \in \mathbb{N}$, and E_k equals the greatest regular fuzzy equivalence contained in the given fuzzy equivalence F (cf. [17]).
    Therefore, we have the following algorithm:

Algorithm. (*Construction of the greatest regular fuzzy equivalences*). The input of this algorithm are a fuzzy network $\mathbb{U} = (U, \{A_i\}_{i \in I})$ and a given fuzzy eqivalence F on U, and the output is the greatest regular fuzzy equivalence E_F^r contained in F.
    The procedure is to construct the descending sequence $\{E_k\}_{k \in \mathbb{N}}$ which correspond to $\mathbb{U}$ and F. It is constructed inductively in the following way:

 (A1) The first member of the sequence is F, i.e., we put $E_1 = F$.
 (A2) After the k-th step let the k-th member E_k of the sequence have been constructed.
 (A3) In the next step we construct the $(k + 1)$-st member E_{k+1} of the sequence by means of the formulas (4.9)–(4.11) and (4.13).
 (A4) Then we check whether $E_k = E_{k+1}$. If it is not true, we go to the next step. Otherwise, if it is true, then the procedure terminates and outputs $E_F^r = E_k$.

If the underlying structure of truth values $\mathbb{L}$ is *locally finite*, i.e., if every finitely generated subalgebra of $\mathbb{L}$ is finite, then the algorithm terminates in a finite number of steps, for every fuzzy network over $\mathbb{L}$. In particular, this is true in the traditional fuzzy logic based on the Gödel structure. But, if $\mathbb{L}$ is not locally finite, then the sequence $\{E_k\}_{k \in \mathbb{N}}$ may not be finite, and then the algorithm does not terminate in a finite number of steps, although the greatest regular fuzzy equivalence E_F^r contained in F always exist. In fact, we have that

$$(4.14) \qquad E_F^r \leq \bigwedge_{k \in \mathbb{N}} E_k,$$

but the equality in (4.14) does not necessarily hold. Certain necessary conditions under which the sequence $\{E_k\}_{k \in \mathbb{N}}$ must be finite, or the equality in (4.14) must hold, have been described in [17].

It is clear that every solution to the system (4.1) must be a solution to the system (4.4), and hence, every structural fuzzy equivalence is a regular fuzzy equivalence. As we have seen, the greatest structural fuzzy equivalences are easier to compute than the greatest regular ones, and they can be computed for fuzzy networks over an arbitrary complete residuated lattice, even in cases when the greatest regular fuzzy equivalence can not be computed by means of the above given procedure. Therefore, in cases when we are not able to compute the greatest regular fuzzy equivalence, instead we can use the greatest structural one. However, the greatest structural fuzzy equivalences give worse results in the positional analysis of social networks than the greatest regular ones, because they may give strictly greater number of positions.

Another option that can be used when we are unable to compute the greatest regular fuzzy equivalence is to work with the greatest regular crisp equivalence. Namely, the function ϕ can be modified so that it "computes" the greatest regular crisp equivalence contained in a given fuzzy or crisp equivalence. Let $\mathcal{E}^c(U)$ denote the set of all crisp equivalences on U, and let us define a function $\phi^c : \mathcal{E}^c(U) \to \mathcal{E}^c(U)$ by $\phi^c(E) = (\phi(E))^c$, for every $E \in \mathcal{E}^c(U)$. Now, let a sequence $\{C_k\}_{k \in \mathbb{N}}$ of crisp equivalences on U be defined by

$$(4.15) \qquad C_1 = F^c, \quad C_{k+1} = C_k \wedge \phi^c(C_k), \text{ for every } k \in \mathbb{N}.$$

The sequence $\{C_k\}_{k \in \mathbb{N}}$ is also descending, and it is always finite, independently of the local finiteness of $\mathbb{L}$, since U and $\mathcal{E}^c(U)$ are finite sets, and $\{C_k\}_{k \in \mathbb{N}}$ is contained in $\mathcal{E}^c(U)$. Consequently, we have that the sequence $\{C_k\}_{k \in \mathbb{N}}$ stabilizes at some C_k ($k \in \mathbb{N}$), i.e., C_k is the least member of this sequence, and this C_k is the greatest regular crisp equivalence contained in the given fuzzy equivalence F (cf. [17]).

It is worth noting that the greatest regular crisp equivalences also give worse results in the positional analysis of social networks than the greatest regular fuzzy equivalences, i.e., they may also give strictly greater numbers of positions. However, the greatest regular crisp equivalences are better than the greatest structural fuzzy equivalences (cf. [17]).

5. Related work

As we have already mentioned, fuzzy social networks have been recently studied by Fan et al. [14, 15], who have discussed fuzzy social networks over the Gödel structure. Unlike them, we have studied fuzzy social networks over a far more general structure, a complete

residuated lattice, where some difficulties have appeared which are not present when we work with fuzzy social networks over the Gödel structure. Algorithms given in [14, 15] and here, for computing the greatest regular fuzzy equivalence and the greatest regular crisp equivalence contained in a given fuzzy or crisp equivalence, are similar, but quite different. Actually, they are obtained using completely different methodologies. The results that are presented here, concerning regular and structural fuzzy equivalences, have been obtained in [17], where not only solutions to the systems (4.1) and (4.4) have been studied, but also solutions to the systems (4.2) and (4.3), and (4.5) and (4.6). Solutions to these systems have been sought not only in the set $\mathcal{E}(U)$ of all fuzzy equivalences on U, but also in the set $\mathcal{Q}(U)$ of all fuzzy quasi-orders (reflexive and transitive fuzzy relations) on U.

Solutions to the systems (4.5) and (4.6) in $\mathcal{Q}(U)$ and $\mathcal{E}(U)$ play a very important role in the fuzzy automata theory, where they are respectively called *left invariant* and *right invariant* fuzzy quasi-orders and equivalences (cf. [11, 31]; in [17] they are called *left regular* and *right regular*). They have been used in [11, 31] in the state reduction of fuzzy automata and fuzzy recognizers. Equivalences which are solutions to a system that correspond to (4.6), known as *bisimulation equivalences*, play an outstanding role in the concurrency theory. Bisimulations and bisimulation equivalences emerged from the concurrency theory, but they are employed today in a number of areas of computer science and mathematics, such as functional languages, object-oriented languages, types, data types, domains, databases, compiler optimizations, program analysis, verification tools, modal logic, set theory, etc. For more information about bisimulations and bisimulation equivalences the reader is referred to [11,17,31].

Резиме

Овај рад има више циљева. Прво, намера нам је да истраживаче у друштвеним наукама охрабримо да користе неке савремене математичке методе другачије од традиционалних метода статистике и анализе података, а посебно, да користе методе анализе социјалних мрежа. Друго, желимо да пропагирамо фази приступ као начин за превазилажење неодређености које су увек присутне у истраживањима у друштвеним наукама. У раду је дат сажет преглед главних идеја и најновијих резултата у области позиционе анализе социјалних мрежа и указано је на везе између анализе социјалних мрежа и неких актуелних области рачунарских и математичких наука.

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