FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 11 (1996), 143–156

THE LATTICE OF POSITIVE QUASI-ORDERS ON AN AUTOMATON*

Miroslav Ćirić, Stojan Bogdanović and Tatjana Petković

Abstract. Making a specialization of the general definition of positive quasi-orders on universal algebras to automata, treated as unary algebras, we define positive quasi-orders on automata, and we study them from the aspects of their relationships with lattices of subautomata of automata and direct sum decompositions of automata.

1. Introduction and Preliminaries

Positive quasi-orders have been first defined and studied by B. M. Schein [18], 1965, in Theory of semigroups, where they shown oneself to be very useful, especially in investigations of semilattice decompositions of semigroups and related decompositions, carried out by T. Tamura in [21], M. S. Putcha in [15], [16], and S. Bogdanović and M. Ćirić in [4]–[6] and [8]–[10]. By M. Ćirić, S. Bogdanović and T. Petković in [12], the definition of positive quasi-orders was extended to an arbitrary universal algebra, and making a specialization of this general notion to automata, treated as unary algebras, one obtains the positive quasi-orders on an automaton, which are the topic of the present investigation.

Positive quasi-orders on an automaton will be here studied from the aspect of their relationships with the lattice of subautomata of this automaton, and from the aspect of their usage in the general theory of direct sum decompositions of automata, developed by M. Ćirić and S. Bogdanović in [11]. Many of the results obtained here generalize certain results obtained in [11].

Throughout this paper, $\mathbb{N}$ will denote the set of all positive integers.

Let ξ be a binary relation on a set H. For $n \in \mathbb{N}$, ξ^n will denote the n-th power of ξ in the semigroup of binary relations on H, ξT will denote the

---

Received July 17, 1996.
1991 *Mathematics Subject Classification*. Primary Primary 68Q70, 06A15.
*Supported by Grant 04M03B of RFNS through Math. Inst. SANU

transitive closure of ξ and ξ^{-1} will denote the relation defined by: $a\,\xi^{-1}\,b \Leftrightarrow b\,\xi\,a$. For $a \in H$, the set $a\xi = \{x \in H \mid a\,\xi\,x\}$ will be called the *left coset* of H determined by a, and the set $\xi a = \{x \in H \mid x\,\xi\,a\}$ will be called the *right coset* of H determined by a. Similarly, for $G \subseteq H$, the sets

$$G\xi = \bigcup_{a\in G} a\xi \qquad \text{and} \qquad \xi G = \bigcup_{a\in G} \xi a$$

will be called the *left coset* and the *right coset* of H determined by G, respectively. By Δ_H we denote the equality relation on H. If it is clear on which set this relation is considered, then we write simply Δ.

Let L be a complete lattice. If K is a subset of L containing the meet of any its nonempty subset, then K is called a *complete meet-subsemilattice* of L. A *complete join-subsemilattice* is defined dually. If K is both complete meet-subsemilattice and complete join-subsemilattice of L, then it is called a *complete sublattice* of L. For a complete lattice L, $\mathcal{C}(L)$ will denote the lattice of all complete 0,1-sublattices of L, and for a complete Boolean algebra B, $\mathcal{B}(B)$ will denote the lattice of all complete Boolean subalgebras of B. These lattices are also complete. A sublattice K of L containing the zero and the unity of L is called a *0,1-sublattice* of L.

A complete lattice L is *infinitely distributive for meets* if $a \wedge (\bigvee_{\alpha\in Y} x_\alpha) = \bigvee_{\alpha\in Y}(a \wedge x_\alpha)$, for every $a \in L$ and every nonempty subset $\{x_\alpha \mid \alpha \in Y\}$ of L. Such lattices are also called *complete Brouwerian lattices*. An element a of a lattice L with the zero 0 is an *atom* of L, if $0 < a$ and there exists no $x \in L$ such that $0 < x < a$. A complete Boolean algebra B is *atomic* if every element of B is the join of some set of atoms of B.

It is well-known that the set of all elements of a distributive lattice L with zero 0 and unity 1, having a complement with respect to 0 and 1, is a Boolean algebra, which we call the *Boolean part* of L and which we denote by $B(L)$. If L is a complete Brouwerian lattice, then the mapping B of $\mathcal{C}(L)$ into $\mathcal{B}(B(L))$ is defined by $B : K \mapsto B(K)$, where $B(K)$ is the Boolean part of K.

A mapping φ of a poset P into a poset Q is *isotone* (*antitone*) if for $x, y \in P$, $x \le y$ implies $x\varphi \le y\varphi$ ($x \le y$ implies $y\varphi \le x\varphi$), and φ is an *order isomorphism* (*dual order isomorphism*) if it is an isotone (antitone) bijection with isotone (antitone) inverse. Note that a poset isomorphic or dually isomorphic to a (complete) lattice is also a (complete) lattice, and by Lemma II 3.2 [1] and its dual, any (dual) order isomorphism between lattices is a (dual) lattice isomorphism.

For a nonempty set H, $\mathcal{P}(H)$ will denote the *lattice of subsets* of H. Let H be a nonempty set and let L be a sublattice of $\mathcal{P}(H)$ containing its unity and having the property that any nonempty intersection of elements of L is also in L. Then for any $a \in H$ there exists the smallest element of L containing a (it is the intersection of all elements of L containing a), which will be called the *principal element* of L generated by a. The set of all principal elements of L is called the *principal part* of L.

Given a nonempty set H. By $\mathfrak{B}(H)$ we denote the Boolean algebra of all binary relations on H. By a *quasi-order* on H we mean a reflexive and transitive binary relation on H. The set $\mathcal{Q}(H)$ of quasi-orders on an nonempty set H, partially ordered by inclusion of relations, is a complete lattice, in which the meet and the join of a subset Q of $\mathcal{Q}(H)$ are defined as follows. The meet of Q equals the set-theoretical intersection of all elements of Q, and the join of Q equals the set-theoretical union of all elements of the subsemigroup generated by Q of the semigroup of all binary relations on H. The subset $\mathcal{E}(H)$ of $\mathcal{Q}(S)$ consisting of all equivalence relations on H is a complete sublattice of $\mathcal{Q}(H)$.

Given a complete lattice L. By an *operator* on L we mean any mapping of L into itself. The set $\mathcal{O}(L)$ of all operators on L is partially ordered by a relation $\leq$ defined by: for $M, N \in \mathcal{O}(L)$, $M \leq N \quad \Leftrightarrow \quad (\forall a \in L)\ aM \leq aN$. With such a partial ordering, $\mathcal{O}(L)$ is a complete lattice in which the meet and the join of a subset $\{M_i \,|\, i \in I\}$ of $\mathcal{O}(L)$ are the operators on L defined respectively by

$$a\Big(\bigwedge_{i\in I} M_i\Big) = \bigwedge_{i\in I}(aM_i) \qquad \text{and} \qquad a\Big(\bigvee_{i\in I} M_i\Big) = \bigvee_{i\in I}(aM_i),$$

for $a \in L$. On the other hand, with respect to the usual multiplication of mappings, i.e. the multiplication defined for $M, N \in \mathcal{O}(L)$ by $a(MN) = (aM)N$, for $a \in L$, $\mathcal{O}(L)$ is a semigroup.

An operator M on L is called *extensive*, if $a \leq aM$, for any $a \in L$, and *idempotent*, if $M^2 = M$. An extensive, isotone operator is called a *semi-closure operator* [20], and an idempotent semi-closure operator is called a *closure operator*. If M is an operator on L, we say that an element $a \in L$ is *closed with respect to* M, or shortly M*-closed*, if $aM = a$.

As was proved by T. Tamura in [20], the set $\mathcal{S}(L)$ of all semi-closure operators on L is a complete sublattice and a subsemigroup of $\mathcal{O}(L)$, and the partial order $\leq$ is compatible with the multiplication on $\mathcal{S}(L)$. Furthermore, the partial order $\leq$ is positive, i.e. $\mathcal{S}(L)$ is a positively ordered semigroup. Closure operators on L are exactly the idempotents of the semigroup $\mathcal{S}(L)$.

A semi-closure operator M on a complete lattice L will be called *algebraic* (*join-conservative*, in Tamura's terms) if $\big(\bigvee_{i\in I} a_i\big)M = \bigvee_{i\in I}(a_iM)$, for any directed subset $\{a_i \,|\, i \in I\}$ of L, where a non-empty subset K of L is defined to be *directed* if any finite subset of K has a upper bound in K. Further, if $\big(\bigvee_{i\in I} a_i\big)M = \bigvee_{i\in I}(a_iM)$, for each subset $\{a_i \,|\, i \in I\}$ of L, then we say that M is a *complete semi-closure operator*.

The following two useful propositions were proved by T. Tamura in [20].

Proposition 1. *Let $\{M_i \,|\, i \in I\}$ be a family of semi-closure operators on L and let an operator M on L be defined by:*

$$aM = \bigwedge\{b \in L \,|\, a \le b \text{ and } b \text{ is } M_i\text{-closed, for all } i \in I\}, \tag{1}$$

for $a \in L$. Then M is the smallest closure operator on L which is a upper bound of $\{M_i \,|\, i \in I\}$.

The above defined closure operator M is called a *closure operator* on L *generated by* $\{M_i \,|\, i \in I\}$, and is denoted by $M = \{M_i \,|\, i \in I\}^{\#}$.

Proposition 2. *Let $\{M_i \,|\, i \in\}$ be a family of algebraic semi-closure operators on L. Then*

$$\{M_i \,|\, i \in I\}^{\#} = \bigvee \langle M_i \,|\, i \in I\rangle\,, \tag{2}$$

where $\langle M_i \,|\, i \in I\rangle$ denotes the subsemigroup generated by $\{M_i \,|\, i \in I\}$ of the semigroup $\mathcal{S}(L)$.

Moreover, for a finite set $\{M_1, M_2, \dots, M_k\}$ of algebraic semi-closure operators on L we have

$$\{M_1, M_2, \dots, M_k\}^{\#} = \bigvee_{n\in\mathbb{N}} (M_{1\sigma}M_{2\sigma}\cdots M_{k\sigma})^n, \tag{3}$$

for an arbitrary permutation σ of the set $\{1, 2, \dots, k\}$.

All automata which will be considered throughout this paper will be automata without outputs, in the sense of definition from the book of F. Gécseg and I. Peák [13]. Since there will be no danger of confusion if the set of states of an automaton A we denote by the same letter A, this will be done in order to simplify the notations. For any considered automaton A, its input alphabet will be denoted by X, and the free monoid over X, i.e. the input monoid of A, will be denoted by X^*. By au we will denote the state of an automaton A in which A goes from the state a of A under the input word $u \in X^*$. Since only automata without outputs will be considered, and it is well-known that these automata can be treated as unary algebras, and vice versa, the notions such as *congruence*, *subautomaton*, *generating set* etc., will have their usual algebraic meanings. The set $\mathrm{Sub}(A)$ of all subautomata of an automaton A, where we also include the empty set – the *empty subautomaton* of A, is a complete sublattice of the lattice $\mathcal{P}(A)$ of all subsets of A, and it is a complete Brouwerian lattice.

An automaton A is a *direct sum* of its subautomata A_α, $\alpha \in Y$, in notation $A = \sum_{\alpha \in Y} A_\alpha$, if

$$A = \bigcup_{\alpha \in Y} A_\alpha \qquad \text{and} \qquad A_\alpha \cap A_\beta = \varnothing, \quad \text{for } \alpha \neq \beta.$$

As was proved by M. Ćirić and S. Bogdanović in [11], the equivalence relation on A whose classes are different A_α, $\alpha \in Y$, is a congruence on A, and it is called a *direct sum congruence* on A. The related partition of A is called a *direct sum decomposition* of A, and the automata A_α, $\alpha \in Y$ are called *direct summands* of A.

A subset H of an automaton A will be called a *consistent subset* of A, if for all $a \in A$, $u \in X^*$, $au \in H$ implies $a \in H$. Clearly, the free monoid X^* can be replaced above by the input alphabet X. A consistent subautomaton of A will be called a *filter* of A. The empty subautomaton of A is also defined to be a filter of A. A non-empty filter of A different than the whole automaton A is called a *proper filter* of A.

For undefined notions and notations we refer to the books [1], [7], [13], [14], [17] and [19].

2. Direct Sum Congruence Generating

In this section we give a new representation for the equivalence closure operator and we show how we can generate the smallest direct sum congruence on an automaton containing a given relation.

For a non-empty set H let $\mathfrak{B}(H)$ denote the lattice of all binary relations on H. Define operators U and L on $\mathfrak{B}(H)$ in the following way: for $\xi \in \mathfrak{B}(H)$ let

$$\xi U = \xi\xi^{-1} \qquad \text{and} \qquad \xi L = \xi^{-1}\xi.$$

Equivalently, U and L can be defined by: for $\xi \in \mathfrak{B}(H)$,

$$\begin{aligned} (a,b) \in \xi U \quad &\Leftrightarrow \quad (\exists c \in H)\ a\,\xi\,c \ \&\ b\,\xi\,c; \\ (a,b) \in \xi L \quad &\Leftrightarrow \quad (\exists c \in H)\ c\,\xi\,a \ \&\ c\,\xi\,b. \end{aligned}$$

By the following two lemmas we describe some properties of the operators U and L:

Lemma 1. *For an arbitrary $\xi \in \mathfrak{B}(H)$, ξU and ξL are symmetric relations.*

If $\xi \in \mathfrak{B}(H)$ is a reflexive relation, then ξU and ξL are also reflexive relations.

Proof. We have $(\xi U)^{-1} = (\xi\xi^{-1})^{-1} = \xi\xi^{-1} = \xi U$, so ξU is symmetric.

If ξ is reflexive, i.e. if $\Delta \subseteq \xi$, then $\Delta = \Delta^{-1} \subseteq \xi^{-1}$ and $\Delta = \Delta^2 \subseteq \xi\xi^{-1} = \xi U$, which means that ξU is reflexive.

The assertions concerning the operator L we prove similarly. □

Lemma 2. *The operators U and L are isotone operators on $\mathfrak{B}(H)$.*

On the lattice of reflexive binary relations on H they are extensive operators.

Proof. Assume $\xi, \eta \in \mathfrak{B}(H)$ such that $\xi \subseteq \eta$. Then $\xi^{-1} \subseteq \eta^{-1}$, so $\xi U = \xi\xi^{-1} \subseteq \eta\eta^{-1} = \eta U$, whence it follows that U is an isotone operator. If $\xi \in \mathfrak{B}(H)$ is a reflexive relation, i.e. if $\Delta \subseteq \xi$, then $\Delta = \Delta^{-1} \subseteq \xi^{-1}$ and $\xi = \xi\Delta \subseteq \xi\xi^{-1} = \xi U$, which proves that U is an extensive operator on the lattice of reflexive binary relations on H.

Similarly we prove the assertions concerning the operator L. □

Let R, S, T and E denote the reflexive, symmetric, transitive and equivalence closure operator on $\mathfrak{B}(H)$, respectively. It is well-known that the operator E has the representations $E = RST = SRT = STR$ (see T. Tamura [20] and T. Tamura and R. Dickinson [22]). Here we give some other representations of the operator E:

Theorem 1. *On the lattice $\mathfrak{B}(H)$ of binary relations on an arbitrary non-empty set H the following equalities hold:*

$$E = RUT = RLT.$$

Proof. Assume an arbitrary $\xi \in \mathfrak{B}(H)$ and set $\eta = \xi E$. Then η is an equivalence relation, so $\eta^{-1} = \eta = \eta^2$, whence $\eta U = \eta\eta^{-1} = \eta^2 = \eta$, and hence $\xi EU = \xi E$. Therefore, $EU = E$. By this, by Lemma 2 and by $R \leq E$ we obtain $RU \leq EU = E$ and so $RUT \leq ET = E$. Hence, $RUT \leq E$.

On the other hand, by Lemma 1 we have that $RS \leq RU$, whence $E = RST \leq RUT$, which completes the proof of the assertion $E = RUT$.

Similarly we prove $E = RLT$. □

The general definition of a positive relation on an algebra, given by the authors in [12], applied to an automaton considered as a unary algebra, yields the following definition: A relation ξ on an automaton A is defined to be *positive* if $a\,\xi\,au$, for any $a \in A$ and $u \in X^+$. The smallest positive relation on an automaton A is $\{(a, au) \mid a \in A,\, u \in X^+\}$, and the set of all positive relations on A is the principal dual ideal of $\mathfrak{B}(A)$ generated by this relation. Moreover, the operator P on $\mathfrak{B}(A)$ defined by

$$\xi P = \xi \cup \{(a, au) \,|\, a \in A,\, u \in X^+\},$$

for $\xi \in \mathfrak{B}(A)$, is a closure operator on $\mathfrak{B}(A)$, and the set of all P-closed elements of $\mathfrak{B}(A)$ is exactly the set of all positive relations on A. The operator P will be called the *positive closure operator* on $\mathfrak{B}(A)$.

A natural generalization of the *division relation on a semigroup* is the *division relation on an algebra*, defined by the authors in [12]. Applying this definition to automata we obtain the *division relation on an automaton* A, denoted by $|$, which is defined by M. Ćirić and S. Bogdanović in [11] as follows: for $a, b \in A$, $a \mid b \quad \Leftrightarrow \quad (\exists u \in X^*)\ b = au$. The division relation on A is positive and reflexive, and furthermore, positive reflexive relations on A form a principal dual ideal of $\mathfrak{B}(A)$, which is generated by the division relation, and these are precisely the closed elements of $\mathfrak{B}(A)$ corresponding to the closure operator $RP = PR$.

M. Ćirić and S. Bogdanović proved in [11] that the set $\mathcal{D}(A)$ of all direct sum congruences on an automaton A is a principal dual ideal of the lattice $\mathcal{E}(A)$ of equivalence relations on A, and hence, it is a complete meet-subsemilattice of $\mathfrak{B}(A)$ containing its unity. Therefore, for any $\xi \in \mathfrak{B}(A)$, the intersection of all direct sum congruences on A containing ξ, denoted by ξD, is the smallest direct sum congruence on A containing ξ. In other words, the operator $D : \xi \mapsto \xi D$ is a closure operator on $\mathfrak{B}(A)$ and the set of all D-closed elements of $\mathfrak{B}(A)$ equals $\mathcal{D}(A)$. The operator D is also characterized in the following way:

Theorem 2. *On the lattice $\mathfrak{B}(A)$ of binary relations on an arbitrary automaton A, the following equalities hold:*

$$D = \{P, E\}^{\#} = PE.$$

Proof. As we noted above, semi-closure operators on $\mathfrak{B}(A)$ form a subsemigroup of the semigroup of operators on $\mathfrak{B}(A)$, so PE is a semi-closure operator. On the other hand, the operator E preserves the positivity, since positive relations form a principal dual ideal of $\mathfrak{B}(A)$, whence $PEP = PE$. Therefore, $(PE)^2 = (PEP)E = PE^2 = PE$, so PE is a closure operator on $\mathfrak{B}(A)$. Clearly, $P \leq PE$ and $E \leq PE$. If we assume an arbitrary closure operator on $\mathfrak{B}(A)$ such that $P \leq M$ and $E \leq M$, then $PE \leq M^2 = M$. Hence, we have proved $\{P, E\}^{\#} = PE$.

Note that by Lemma 3.1 of [11], a relation $\xi \in \mathfrak{B}(A)$ is a positive equivalence relation on A if and only if it is a direct sum congruence on A. Then for any $\xi \in \mathfrak{B}(A)$, ξD is a positive equivalence relation, whence $P \leq D$ and $E \leq D$, and hence, $PE \leq D$. On the other hand, for any $\xi \in \mathfrak{B}(A)$, ξPE is a direct sum congruence on A, whence $D \leq PE$. Therefore, we have $D = PE$. □

Applying the above theorem to positive quasi-orders we obtain the following:

Corollary 1. *On the lattice $\mathcal{Q}^p(A)$ of positive quasi-orders on an arbitrary automaton A, the following equalities hold:*

$$D = ST = UT = LT.$$

Note that $\mathcal{Q}^p(A)$ is the principal dual ideal of the lattice $\mathcal{Q}(A)$ of quasi-orders on A generated by the division relation on A.

3. The Lattice of Positive Quasi-Orders

Further we consider positive quasi-orders on automata. The next lemma gives some useful properties of such quasi-orders, concerning the structure of their left and right cosets.

Lemma 3. *The following conditions for a quasi-order ξ on an automaton A are equivalent:*

(i) *ξ is positive;*
(ii) $(\forall a \in A)(\forall u \in X^*)\ (au)\xi \subseteq a\xi$;
(iii) $(\forall a \in A)(\forall u \in X^*)\ \xi a \subseteq \xi(au)$;
(iv) *$a\xi$ is a subautomaton of A, for each $a \in A$;*
(v) *ξa is a consistent subset of A, for each $a \in A$.*

Proof. (i) ⇒ (ii). Assume arbitrary $a \in A$, $u \in X^*$. If $b \in (au)\xi$, i.e. $au\,\xi\, b$, then by the positivity and the transitivity of ξ we have $a\,\xi\, au\,\xi\, b$, which yields $a\,\xi\, b$, i.e. $b \in a\xi$, which was to be proved.

(ii) ⇒ (iv). Assume an arbitrary $a \in A$, and assume $b \in a\xi$, $u \in X^*$. By the transitivity of ξ we have $b\xi \subseteq a\xi$, so by the reflexivity of ξ and the hypothesis we obtain $bu \in (bu)\xi \subseteq b\xi \subseteq a\xi$, which was to be proved.

The implications (i) ⇒ (iii) and (iii) ⇒ (v) we prove similarly as (i) ⇒ (ii) and (ii) ⇒ (iv). Finally, the implications (iv) ⇒ (i) and (v) ⇒ (i) are obvious. □

Using the previous lemma we give the following characterization of the lattice of positive quasi-orders on an automaton:

Theorem 3. *The lattice $\mathcal{Q}^p(A)$ of positive quasi-orders on an automaton A is dually isomorphic to the lattice of complete 0,1-sublattices of* $\mathrm{Sub}(A)$.

Proof. As was noted by the first two authors in [4], the lattice $\mathcal{Q}(A)$ of quasi-orders on A is dually isomorphic to the lattice $\mathcal{C}(\mathcal{P}(A))$ of complete 0,1-sublattices of the lattice $\mathcal{P}(A)$ of all subsets of A. There are two dual isomorphisms of $\mathcal{Q}(A)$ onto $\mathcal{C}(\mathcal{P}(A))$. The first one, denoted by $\Lambda : \xi \mapsto \xi\Lambda$, is given in terms of the left cosets of ξ as follows:

$$\xi\Lambda = \{H \in \mathcal{P}(A) \,|\, H\xi = H\},$$

and the second one, denoted by $\Pi : \xi \mapsto \xi\Pi$, is given in terms of the right cosets of ξ as follows:

$$\xi\Pi = \{H \in \mathcal{P}(A) \,|\, \xi H = H\}.$$

In the proof of this theorem we will use the dual isomorphism Λ.

To prove the assertion of the theorem, it is enough to prove that a quasi-order ξ is positive if and only if $\xi\Lambda$ is contained in $\mathrm{Sub}(A)$.

Suppose that ξ is a positive quasi-order on A and assume an arbitrary $H \in \xi\Lambda$. Then by Lemma 3 we have

$$H = H\xi = \bigcup_{a\in H} a\xi \in \mathrm{Sub}(A),$$

since $\mathrm{Sub}(A)$ is a complete sublattice of $\mathcal{P}(A)$.

Conversely, let $\xi\Lambda$ is contained in $\mathrm{Sub}(A)$. Then for any $a \in A$ we have $a\xi \in \xi\Lambda \subseteq \mathrm{Sub}(A)$, so by Lemma 3 we have that ξ is positive. This completes the proof of the theorem. □

If we consider the dual isomorphism Π instead of Λ, we obtain the following theorem:

Theorem 4. *The lattice $\mathcal{Q}^p(A)$ of positive quasi-orders on an automaton A is dually isomorphic to the lattice of complete 0,1-sublattices of the lattice of consistent subsets of A.*

Note that the principal elements of the lattices $\xi\Lambda$ and $\xi\Pi$ are exactly the left cosets $a\xi$, $a \in A$, and the right cosets ξa, $a \in A$, respectively.

As a consequence of Theorems 3 and 4 we obtain the following theorem proved by M. Ćirić and S. Bogdanović in [11].

Theorem 5. *The lattice $\mathcal{D}(A)$ of direct sum congruences on an automaton A is dually isomorphic to the lattice of complete Boolean subalgebras of the Boolean algebra $\mathrm{F}(A)$ of filters of A.*

Proof. This theorem was proved by the first two authors in [11], but here we give another proof, using Theorems 3 and 4. Note that by Lemma 3.1 [11], direct sum congruences on A are exactly positive equivalence relations on A, so in view of Theorems 3 and 4, to prove this theorem it is enough to prove that for an arbitrary positive quasi-order ξ on A, ξ is an equivalence relation if and only if $\xi\Lambda$ is a Boolean subalgebra of $\mathrm{F}(A)$.

Suppose first that ξ is an equivalence relation. Then for an arbitrary $H \in \mathcal{P}(A)$ we have $H\xi = \xi H$, and by this and Theorems 3 and 4 it follows that $\xi\Lambda = \xi\Pi \subseteq \mathrm{F}(A)$. Assume an arbitrary $H \in \xi\Lambda$, Let G be the set-theoretic complement of H in A. Assume that $b \in G\xi$, i.e. $a\,\xi\,b$, for some $b \in G$. If $a \notin G$, then $a \in H$ and then $b \in \xi a \subseteq \xi H = H\xi = H$, which contradicts our assumption $b \in G$. Therefore, $a \in G$ and so $G\xi = G$, which yields $G \in \xi\Lambda$. Hence $\xi\Lambda$ is a Boolean subalgebra of $\mathrm{F}(A)$.

Conversely, suppose that $\xi\Lambda$ is a Boolean subalgebra of $\mathrm{F}(A)$. Then by Theorem 56 of [19], $\xi\Lambda$ is atomic, and the atoms of $\xi\Lambda$ are precisely its principal elements. Therefore, for arbitrary $a, b \in A$, either $a\xi \cap b\xi = \varnothing$, or $a\xi = b\xi$. Assume now $a, b \in A$ such that $a\,\xi\,b$. Then $a\xi \cap b\xi \neq \varnothing$, whence $a\xi = b\xi$, so $b\,\xi\,a$. Therefore, ξ is an equivalence relation. This completes the proof of the theorem. $\square$

We also prove the following interesting result:

Theorem 6. *For any automaton A, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{Q}^p(A) & \xrightarrow{\;D\;} & \mathcal{D}(A) \\ {\scriptstyle\Lambda}\big\downarrow & & \big\downarrow{\scriptstyle\Lambda} \\ \mathcal{C}(\mathrm{Sub}(A)) & \xrightarrow{\;B\;} & \mathcal{B}(\mathrm{F}(A)) \end{array}$$

In other words, for an any $\xi \in \mathcal{Q}_p(A)$, $(\xi D)\Lambda$ is the Boolean part of $\xi\Lambda$.

Proof. Assume an arbitrary $H \in (\xi D)\Lambda$. Then $H(\xi D) = H$, whence $H \subseteq H\xi \subseteq H(\xi D) = H$, so $H = H\xi$ and $H \in \xi\Lambda$. Therefore, $(\xi D)\Lambda$ is contained in $\xi\Lambda$, and, by the proof of Theorem 5, it is a complete Boolean subalgebra of $B(\xi\Lambda)$.

Conversely, assume an arbitrary $H \in B(\xi\Lambda)$. Then $H, H' \in \xi\Lambda$, i.e. $H\xi = H$ amd $H'\xi = H'$, where H' denotes the set-theoretical complement of H in A. To prove that $H(\xi D) = H$, it is enough to prove $H(\xi D) \subseteq H$, so assume an arbitrary $a \in H(\xi D)$. Then $(b, a) \in \xi D$, for some $b \in H$, and by Corollary 1 we have $(b, a) \in \xi D = \xi UT$. Therefore, there exist $c_1, \dots, c_n \in A$, $n \in \mathbb{N}$, $n \geq 2$, such that

$$c_1 = b, \; c_n = a \quad \text{and} \quad (c_{i-1}, c_i) \in \xi U, \text{ for all } i \in \{2, \dots, n\}.$$

Suppose that $c_i \notin H$, for some $i \in \{2, \dots, n\}$. Set

$$k = \min\{i \,|\, 2 \leq i \leq n \;\&\; c_i \notin H\}.$$

Then $c_k \notin H$, i.e. $c_k \in H'$, and also $c_{k-1} \in H$ and $(c_{k-1}, c_k) \in \xi U$, i.e. $c_{k-1}\xi \cap c_k\xi \neq \varnothing$. But we have that $c_{k-1}\xi \subseteq H\xi = H$ and $c_k\xi \subseteq H'\xi = H'$, so $c_{k-1}\xi \cap c_k\xi = \varnothing$, which gets a contradiction. Thus, we conclude that $c_i \in H$, for any $i \in \{1, \dots, n\}$, so $a = c_n \in H$, which was to be proved. Hence, $H(\xi D) = H$, i.e. $H \in (\xi D)\Lambda$, so $B(\xi\Lambda) \subseteq (\xi D)\Lambda$, which completes the proof of the theorem. □

Note that the mapping Λ in the previous theorem can be replaced by Π (in this case we replace $\mathrm{Sub}(A)$ by the lattice of consistent subset of A). This is in fact a consequence of the following theorem:

Theorem 7. *Let ξ be a positive quasi-order on an automaton A. Then*

$$B(\xi\Lambda) = B(\xi\Pi) = (\xi D)\Lambda = (\xi D)\Pi = \xi\Lambda \cap \xi\Pi.$$

Proof. It is enough to prove that

$$H \in B(\xi\Lambda) \quad \Leftrightarrow \quad H\xi = \xi H = H,$$

for an arbitrary $H \subseteq A$.

Suppose that $H \in B(\xi\Lambda)$. This means that $H, H' \in \xi\Lambda$, i.e. $H\xi = H$ and $H'\xi = H'$, where H' denotes the set-theoretical complement of H in A. In order to prove $\xi H \subseteq H$, assume $a \in \xi H$. Then $a \,\xi\, b$, for some $b \in H$, and if $a \notin H$, i.e. $a \in H'$, then $b \in a\xi \subseteq H'\xi = H'$, which gets a contradiction. By this we conclude that $a \in H$ and hence $\xi H \subseteq H$, so we proved $H\xi = \xi H = H$.

Conversely, assume that $H\xi = \xi H = H$. By the definition of the mapping Λ we have $H \in \xi\Lambda$, so it remains to prove that $H' \in \xi\Lambda$, i.e. that $H'\xi = H'$. Assume $a \in H'\xi$. Then $b \,\xi\, a$, for some $b \in H'$, and if $a \in H$, then $b \in \xi a \subseteq \xi H = H$, which gets a contradiction, so we conclude that $a \in H'$ and $H'\xi = H'$. This completes the proof of the theorem. □

For a positive quasi-order ξ on an automaton A define operators S^ξ, C^ξ and F^ξ on $\mathcal{P}(A)$ by

$$HS^\xi = H\xi,\ HC^\xi = \xi H \quad \text{and} \quad HF^\xi = H(\xi D),$$

for $H \in \mathcal{P}(A)$. Clearly, $H(\xi D) = (\xi D)H$, since ξD is an equivalence relation. By the above proved theorems, S^ξ, C^ξ and F^ξ are closure operators on $\mathcal{P}(A)$ and the sets of S^ξ-closed, C^ξ-closed and F^ξ-closed elements of $\mathcal{P}(A)$ are exactly $\xi\Lambda$, $\xi\Pi$ and $\xi\Lambda \cap \xi\Pi$, respectively. Moreover, we have the following:

Theorem 8. *Let ξ be a positive quasi-order on an automaton A. Then*

$$F^\xi = \{S^\xi, C^\xi\}^\# = \bigvee_{n\in\mathbb{N}} (S^\xi C^\xi)^n = \bigvee_{n\in\mathbb{N}} (C^\xi S^\xi)^n.$$

Proof. Set $M^\xi = \{S^\xi, C^\xi\}^\#$. For an arbitrary $H \in \mathcal{P}(A)$, by (1) and the definition of F^ξ we have

$$\begin{aligned} HM^\xi &= \bigwedge\{G \in \mathcal{P}(A) \mid H \subseteq G \text{ and } HS^\xi = HC^\xi = H\} \\ &= \bigwedge\{G \in \xi\Lambda \cap \xi\Pi \mid H \subseteq G\} = HF^\xi. \end{aligned}$$

Therefore, $M^\xi = F^\xi$.

On the other hand, it is easy to verify that S^ξ and C^ξ are complete closure operators on $\mathcal{P}(A)$, so using Proposition 2 we complete the proof of the theorem. □

For an arbitrary positive quasi-order ξ on an automaton A we have that the Boolean algebra $(\xi D)\Lambda = (\xi D)\Pi = \xi\Lambda \cap \xi\Pi$ is atomic, and for an arbitrary $a \in A$, the atom of $(\xi D)\Lambda$ containing a, i.e. the summand in the direct sum decomposition of A corresponding to the direct sum congruence ξD, is $a(\xi D) = aF^\xi$. The following consequence of the previous theorem give an algorithm for finding the atoms in $(\xi D)\Lambda$.

Corollary 2. *Let A be an automaton, $\xi \in \mathcal{Q}^p(A)$, $a \in A$ and let a sequence $\{U_n^\xi(a)\}_{n\in\mathbb{N}}$ of subsets of A be defined by:*

$$U_1^\xi(a) = \big(aS^\xi\big)C^\xi, \quad U_{n+1}^\xi(a) = \big(\big(U_n^\xi(a)\big)S^\xi\big)C^\xi, \ \textit{for } n \in \mathbb{N}.$$

Then $\{U_n^\xi(a)\}_{n\in\mathbb{N}}$ is an increasing sequence of sets and

$$aF^\xi = \bigcup_{n\in\mathbb{N}} U_n^\xi(a).$$

In view of Theorem 8, the final result of the above procedure will not be changed when we permute the operators S^{ξ} and C^{ξ}.

By the above corollary we obtain Theorems 2.2. and 2.3 of [11], which give algorithms for finding the summands in the greatest direct sum decomposition of an automaton.

REFERENCES

1. G. Birkhoff: *Lattice Theory.* Amer. Math. Soc, Coll. Publ. Vol. 25, (3rd. edition, 3rd. printing), Providence, 1979.
2. S. Bogdanović and M. Ćirić: *Orthogonal sums of semigroups.* Israel J. Math. **90** (1995), 423–428.
3. S. Bogdanović and M. Ćirić: *Decompostitons of semigroups with zero.* Publ. Inst. Math. Belgrade **57 (71)** (1995), 111–123.
4. S. Bogdanović and M. Ćirić: *Positive quasi-orders with the common multiple property on a semigroup.* in: Proc. Math. Conf. in Priština 1994, Lj. Kočinac, Ed., Priština, 1995, 1–6.
5. S. Bogdanović and M. Ćirić: *Quasi-orders and semilattice decompositions of semigroups (A survey).* (to apear).
6. S. Bogdanović and M. Ćirić: *Generation of lower-potent positive half-congruences.* SEA Bull. Math. (to appear).
7. S. Burris and H. P. Sankappanavar: *A Course in Universal Algebra.* Springer-Verlag, 1983.
8. M. Ćirić and S. Bogdanović: *The lattice of positive quasi-orders on a semigroup.* Israel J. Math. (to appear).
9. M. Ćirić and S. Bogdanović: *The lattice of positive quasi-orders on a semigroup* II. Facta Univ. (Niš), Ser. Math. Inform. **9**, (1994), 7–17.
10. M. Ćirić and S. Bogdanović: *Quasi-semilattice decompositions of semigroups with zero.* (to appear).
11. M. Ćirić and S. Bogdanović: *Lattices of subautomata and direct sum decompositions of automata.* (to appear).
12. M. Ćirić, S. Bogdanović and T. Petković: *Sums and limits of generalized direct families of algebras.* (to appear).
13. F. Gécseg and I. Peák: *Algebraic Theory of Automata.* Akadémiai Kiadó, Budapest, 1972.
14. G. Grätzer: *General Lattice Theory.* Akademie-Verlag, Berlin, 1978.
15. M. S. Putcha: *Positive quasi-orders on semigroups.* Duke Math. J. **40**, (1973), 857–869.
16. M. S. Putcha: *Paths in graphs and minimal π-sequences in semigroups.* Discrete Math. **11**, (1975), 173–185.
17. V. N. Saliĭ: *Universal Algebra and Automata.* Izd. Saratovskogo Univ., Saratov, 1988 (in Russian).

18. B. M. Schein: *On certain classes of semigroups of binary relations.* Sibirsk. Mat. Zhurn. **6**, (1965), 616–635 (in Russian).

19. G. Szász: *Théorie des Treillis.* Akadémiai Kiadó, Budapest, et Dunod, Paris, 1971.

20. T. Tamura: *The theory of operations on binary relations.* Trans. Amer. Math. Soc. **120**, (1965), 343–358.

21. T. Tamura: *Quasi-orders, generalized Archimedeaness, semilattice decompositions.* Math. Nachr. **68**, (1975), 201–220.

22. T. Tamura and R. Dickinson: *Semigroups connected with equivalence and congruence relations.* Proc. Japan Acad. **42**, (1966), 688–692.

University of Niš,
Faculty of Philosophy,
Department of Mathematics,
Ćirila i Metodija 2, P. O. Box 91,
18000 Niš, Yugoslavia
e-mail: `mciric@archimed.filfak.ni.ac.yu`

University of Niš,
Faculty of Economics,
Trg JNA 11, P. O. Box 121,
18000 Niš, Yugoslavia
e-mail: `sbogdan@archimed.filfak.ni.ac.yu`

University of Niš,
Faculty of Philosophy,
Department of Mathematics,
Ćirila i Metodija 2, P. O. Box 91,
18000 Niš, Yugoslavia
e-mail: `tanjapet@archimed.filfak.ni.ac.yu`