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MORE ON NORMAL BAND COMPOSITIONS OF SEMIGROUPS*

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. The authors in [1] gave a construction of a normal band of arbitrary semigroups, and in [2] they studied band compositions which are (punched) spined products of a band and a semilattice of semigroups. This paper is a continuation of these two papers. Using the constructions from [1], in this paper we give a construction of a semigroup which is a (punched) spined product of a normal band and a semilattice of semigroups, different than the one from [2]. The obtained results generalize the well-known characterizations of spined products of a normal band and a semilattice of groups through orthodox normal bands of groups and through strong semilattices of rectangular groups.

Let B be a band. By $\leq$ we will denote the natural partial order on B, i.e. the relation on B defined by: $j \leq i \;\Leftrightarrow\; ij = ji = j,\ (i, j \in B)$. By $\preccurlyeq$ we will denote a quasi-order on B defined by: $j \preccurlyeq i \quad\Leftrightarrow\quad j = jij,\ (i, j \in B)$. Let $i \mapsto [i],\ i \in B$, denote the natural homomorphism of the smallest semilattice congruence on B. It is easy to verify that $j \preccurlyeq i \;\Leftrightarrow\; [j] \leq [i]$, for all $i, j \in B$. If P and Q are two semigroups having a common homomorphic image Y, then the *spined product of P and Q with respect to Y* is $S = \{(a, b) \in P \times Q \mid a\varphi = b\psi\}$, where $\varphi : P \to Y$ and $\psi : Q \to Y$ are homomorphisms onto Y, [6], [8], [9]. If Y is a semilattice and P and Q are a semilattice Y of semigroups $P_\alpha,\ \alpha \in Y$, and $Q_\alpha,\ \alpha \in Y$, respectively, then the spined product of P and Q with respect to Y is $S = \cup_{\alpha\in Y} P_\alpha \times Q_\alpha$. A subsemigroup S of a spined product of semigroups P and Q with respect to Y, that is also a subdirect product of P and Q, is a *punched spined product of P and Q with respect to Y*, [5]. If ξ is a congruence on a semigroup S, $\xi^\natural$ will denote the natural homomorphism induced by ξ.

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Let B be a band. To each $i \in B$ we associate a semigroup S_i and an oversemigroup D_i of S_i such that $D_i \cap D_j = \emptyset$, if $i \neq j$. For $i, j \in B,\ i \succcurlyeq j$, let $\phi_{i,j}$ be a mapping of S_i into D_j and suppose that the family of $\phi_{i,j}$ satisfies the following conditions:

(1) $\phi_{i,i}$ is the identity mapping on S_i, for every $i \in B$;
(2) $(S_i\phi_{i,ij})(S_j\phi_{j,ij}) \subseteq S_{ij}$, for all $i, j \in B$;
(3) $[(a\phi_{i,ij})(b\phi_{j,ij})]\phi_{ij,k} = (a\phi_{i,k})(b\phi_{j,k})$, for $a \in S_i, b \in S_j, ij \succcurlyeq k, i, j, k \in B$.

Define a multiplication $*$ on $S = \cup_{i\in B} S_i$ by: $a * b = (a\phi_{i,ij})(b\phi_{j,ij})$, for $a \in S_i,\ b \in S_j$. Then S is a band B of semigroups $S_i,\ i \in B$, in notation $S = (B; S_i, \phi_{i,j}, D_i)$. If we assume $i = j$ in (3), then we obtain that $\phi_{i,k}$ is a homomorphism, for all $i, k \in B,\ i \succcurlyeq k$. If $D_i = S_i$, for each $i \in B$, then we write $S = (B; S_i, \phi_{i,j})$. Here the condition (2) can be omitted. If $S = (B; S_i, \phi_{i,j}, D_i)$ and if

(4) $S_i\phi_{i,j} \subseteq S_j$, for $[i] = [j],\ i, j \in B$;
(5) $\phi_{i,j}\phi_{j,k} = \phi_{i,k}$, for $[i] = [j] \geq [k],\ i, j, k \in B$;

then we will write $S = [\![B; S_i, \phi_{i,j}, D_i]\!]$. If $S = (B; S_i, \phi_{i,j})$ with (4) and (5), then we will write $S = [\![B; S_i, \phi_{i,j}]\!]$, [2]. If $S = (B; S_i, \phi_{i,j})$ and if $\{\phi_{i,j} \mid i, j \in B,\ i \succcurlyeq j\}$ is a *transitive system of homomorphisms*, i.e. if $\phi_{i,j}\phi_{j,k} = \phi_{i,k}$, for $i \succcurlyeq j \succcurlyeq k$, then we will write $S = [B; S_i, \phi_{i,j}]$, and we will say that S is a *strong band* B of semigroups S_i. If B is a semilattice, then we obtain a *strong semilattice* of semigroups. A strong semilattice of rectangular bands will be called a *normal band* and in the further considerations, the phrase "$B = [Y; B_\alpha, \theta_{\alpha,\beta}]$ is a normal band" will mean: a band B is a strong semilattice Y of rectangular bands $B_\alpha,\ \alpha \in Y$, with $\{\theta_{\alpha,\beta}\}$ as its transitive system of homomorphisms.

Let a band B be a semilattice Y of rectangular bands $B_\alpha,\ \alpha \in Y$, and let $\{S_i \mid i \in B\}$ be a family pairwise disjoint semigroups. The authors in [1] showed that a semigroup S is a band B of semigroups $S_i,\ i \in B$, if and only if the following conditions hold:

(6) $S = (Y; S_\alpha, \phi_{\alpha,\beta}, D_\alpha)$;
(7) for every $\alpha \in Y$, S_α is a matrix B_α of semigroups $S_i,\ i \in B_\alpha$;
(8) $(S_i\phi_{\alpha,\alpha\beta})(S_j\phi_{\beta,\alpha\beta}) \subseteq S_{ij}$, for all $i \in B_\alpha,\ j \in B_\beta,\ \alpha, \beta \in Y$. Furthermore, they proved that in a semigroup satisfying (6), (7) and (8), for each $\alpha \in Y$, D_α can be chosen to satisfy:
(9) D_α is a matrix B_α of semigroups $D_i,\ i \in B_\alpha$;
(10) $S_i \subseteq D_i$, for every $i \in B_\alpha$;

if and only if B is a normal band. This gives a construction of a normal band of arbitrary semigroups. Here we pose the following question: How to obtain a (punched) spined product of a normal band and a semilattice of semigroups from this construction? An answer for this question will be given by the next theorems.

For undefined notions and notations we refer to [1], [2] and [7].

In the further considerations of semigroups S satisfying the conditions (6), (7) and (8), $*$ will denote the multiplication in S, $\circ$ will denote the multiplications in D_α, $\alpha \in Y$, and the multiplications in S_i, $i \in B$, will be denoted in the usual way.

Theorem 1. *Let $B = [Y; B_\alpha, \theta_{\alpha,\beta}]$ be a normal band.*

If S is a semigroup which satisfies the conditions (6), (7) and (8), and if for each $\alpha \in Y$, a semigroup D_α can be chosen to satisfy:

(11) $D_\alpha = (B_\alpha; D_i, \chi_{i,j}, E_i)$;

(12) $S_i \subseteq D_i$, *for every* $i \in B_\alpha$.

Then

(A1) *a relation ρ on S defined by: $a\,\rho\,b$ if and only if $a \in S_i,\ b \in S_j,\ i,j \in B_\alpha,\ \alpha \in Y$, and*

$$a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},k} = b\phi_{\alpha,\beta}\chi_{j\theta_{\alpha,\beta},k},$$

for every $\beta \in Y,\ \alpha \geq \beta,\ k \in B_\beta$, is a congruence on S and the semigroup $T = S/\rho$ is a semilattice Y of semigroups $T_\alpha = S_\alpha\rho^\natural,\ \alpha \in Y$;

(A2) *S is a punched spined product of T and B with respect to Y.*

Conversely, if a semigroup S is a punched spined product of semigroups $T = (Y; T_\alpha,\ \omega_{\alpha,\beta}, U_\alpha)$ and B with respect to Y and if we assume that:

(B1) *$S_\alpha = (T_\alpha \times B_\alpha) \cap S,\ D_\alpha = U_\alpha \times B_\alpha$, for $\alpha \in Y$;*

(B2) *for $\alpha, \beta \in Y,\ \alpha \geq \beta$, a mapping $\phi_{\alpha,\beta} : S_\alpha \to D_\beta$ is defined by:*

$$(a,i)\phi_{\alpha,\beta} = (a\omega_{\alpha,\beta}, i\theta_{\alpha,\beta}), \qquad ((a,i) \in S_\alpha);$$

(B3) *$S_i = (T_\alpha \times \{i\}) \cap S,\ D_i = U_\alpha \times \{i\}$, for $\alpha \in Y,\ i \in B_\alpha$;*

then S satisfies the conditions (6), (7) i (8), for every $\alpha \in Y$, D_α is a strong matrix B_α of semigroups $D_i,\ i \in B_\alpha$, and the condition (12) holds.

Proof. Let S be a semigroup which satisfies (6), (7) and (8), and let for every $\alpha \in Y$, a semigroup D_α can be chosen to satisfy (11) and (12).

(A1) It is clear that ρ is reflexive and symmetric. Let $a\,\rho\,b$ and $b\,\rho\,c$, $a,b,c \in S$. Then $a \in S_i,\ b \in S_j,\ c \in S_k,\ i,j,k \in B_\alpha,\ \alpha \in Y$. Assume $\beta \in Y$ such that $\alpha \geq \beta$, and assume $l \in B_\beta$. Then

$$a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},l} = b\phi_{\alpha,\beta}\chi_{j\theta_{\alpha,\beta},l} = c\phi_{\alpha,\beta}\chi_{k\theta_{\alpha,\beta},l},$$

whence $a\,\rho\,c$. Thus, ρ is transitive.

Assume $a, b \in S$ such that $a\ \rho\ b$, and assume $x \in S$. Let $a \in S_i,\ b \in S_j,\ i, j \in B_\alpha,\ x \in S_k,\ k \in B_\beta,\ \alpha, \beta \in Y$. Then $a * x \in S_{ik}$, $b * x \in S_{jk}$, and $ik, jk \in B_{\alpha\beta}$. Assume $\gamma \in Y$ such that $\alpha\beta \geq \gamma$, and assume $l \in B_\gamma$. Then

$$\begin{aligned}(a * x)\phi_{\alpha\beta,\gamma}\chi_{(ik)\theta_{\alpha\beta,\gamma},l} &= [(a\phi_{\alpha,\alpha\beta}) \circ (x\phi_{\beta,\alpha\beta})]\phi_{\alpha\beta,\gamma}\chi_{(ik)\theta_{\alpha\beta,\gamma},l} \\ &= [(a\phi_{\alpha,\gamma}) \circ (x\phi_{\beta,\gamma})]\chi_{(ik)\theta_{\alpha\beta,\gamma},l} \\ &= [(a\phi_{\alpha,\gamma}\chi_{i\theta_{\alpha,\gamma},(i\theta_{\alpha,\gamma})(k\theta_{\beta,\gamma})})(x\phi_{\beta,\gamma}\chi_{k\theta_{\beta,\gamma},(i\theta_{\alpha,\gamma})(k\theta_{\beta,\gamma})}]\chi_{(i\theta_{\alpha,\gamma})(k\theta_{\beta,\gamma}),l} \\ &= (a\phi_{\alpha,\gamma}\chi_{i\theta_{\alpha,\gamma},l})(x\phi_{\beta,\gamma}\chi_{k\theta_{\beta,\gamma},l}) = (b\phi_{\alpha,\gamma}\chi_{j\theta_{\alpha,\gamma},l})(x\phi_{\beta,\gamma}\chi_{k\theta_{\beta,\gamma},l}) \\ &= \cdots = (b * x)\phi_{\alpha\beta,\gamma}\chi_{(jk)\theta_{\alpha\beta,\gamma},l},\end{aligned}$$

since $(ik)\theta_{\alpha\beta,\gamma} = [(i\theta_{\alpha,\alpha\beta})(k\theta_{\beta,\alpha\beta})]\theta_{\alpha\beta,\gamma} = (i\theta_{\alpha,\gamma})(k\theta_{\beta,\gamma})$. Thus, $a*x\ \rho\ b*x$. Similarly, $x * a\ \rho\ x * b$. Hence, ρ is a congruence. Let η be a semilattice congruence on S whose classes are the various $S_\alpha,\ \alpha \in Y$. Then $\rho \subseteq \eta$, so $T = S/\rho$ is a semilattice Y of semigroups $T_\alpha = S_\alpha\rho^\natural$.

(A2) Let ξ be the band congruence on S determined by the partition $\{S_i \mid i \in B\}$. Clearly, $\rho \cap \xi = \Delta$, where Δ is the equality relation on S. Thus S is a subdirect product of T and B, where an one-to-one homomorphism Φ of S into $T \times B$ is given by: $a\Phi = (a\rho^\natural, a\xi^\natural),\ a \in S$. Assume $a \in S$. Let $a \in S_i,\ i \in B_\alpha,\ \alpha \in Y$. Then by $a \in S_\alpha$ we obtain that $a\rho^\natural \in T_\alpha,\ a\xi^\natural = i \in B_\alpha$, i.e. $a\Phi \in T_\alpha \times B_\alpha$. Thus, $S\Phi \subseteq \cup_{\alpha \in Y} T_\alpha \times B_\alpha$. Hence, S is a punched spined product of semigroups T and B with respect to Y.

The converse follows immediately. □

Theorem 2. *Let $B = [Y; B_\alpha, \theta_{\alpha,\beta}]$ be a normal band.*

If S is a semigroup which satisfies the conditions (6), (7) *and* (8), *and if for every $\alpha \in Y$, a semigroup D_α can be chosen to satisfy:*

(13) $D_\alpha = [B_\alpha; D_i, \chi_{i,j}]$;
(14) $S_i \subseteq D_i$, *for each* $i \in B_\alpha$;
(15) $S_i\chi_{i,j} \subseteq S_j$, *for all* $i, j \in B_\alpha$;
(16) $a\chi_{i,j}\phi_{\alpha,\beta} = a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},j\theta_{\alpha,\beta}}$, *for* $\alpha, \beta \in Y,\ \alpha \geq \beta,\ i, j \in B_\alpha,\ a \in S_i$;

then the condition (A1) *of Theorem* 1 *holds and S is a spined product of semigroups T and B with respect to Y.*

Conversely, if a semigroup S is a spined product of semigroups $T = (Y; T_\alpha, \omega_{\alpha,\beta}, U_\alpha)$ and B with respect to Y, then in notations from (B1)–(B3) *of Theorem* 1, *S satisfies the conditions* (6)–(8) *and* (13)–(16).

Proof. Let a semigroup S satisfies the conditions (6)–(8) and let for every $\alpha \in Y$, a semigroup D_α can be chosen to satisfy the conditions (13)–(16). Note that for $\alpha, \beta \in Y,\ \alpha \geq \beta,\ i \in B_\alpha$, by Theorem 2 [1] $S_i\phi_{\alpha,\beta} \subseteq D_{i\theta_{\alpha,\beta}}$, so the right hand side of (16) is correct. Also, the conditions (11) and (12) hold, so by Theorem 1 we have that (A1) holds and that S is a punched spined product of semigroups T and B with respect to Y.

Assume an arbitrary $(u,i)\in\cup_{\alpha\in Y}T_\alpha\times B_\alpha$. Then $(u,i)\in T_\alpha\times B_\alpha$, for some $\alpha\in Y$, and $u=b\rho^\natural$, for some $b\in S_\alpha$. Let $b\in S_j$, $j\in B_\alpha$, and let $a=b\chi_{j,i}$. By (15), $a\in S_i$. Assume $\beta\in Y$ such that $\alpha\geq\beta$, and assume $k\in B_\beta$. Then by (13) and (16) we obtain

$$a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},k}=b\chi_{j,i}\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},k}=b\phi_{\alpha,\beta}\chi_{j\theta_{\alpha,\beta},i\theta_{\alpha,\beta}}\chi_{i\theta_{\alpha,\beta},k}=b\phi_{\alpha,\beta}\chi_{j\theta_{\alpha,\beta},k}.$$

Hence, $a\;\rho b$, so $a\rho^\natural=b\rho^\natural=u$, whence $a\Phi=(u,i)$. Thus, $S\Phi=\cup_{\alpha\in Y}T_\alpha\times B_\alpha$, so S is a spined product of semigroups T and B with respect to Y.

The converse follows immediately. □

Theorem 3. *Let $B=[Y;B_\alpha,\theta_{\alpha,\beta}]$ be a normal band and let $\{S_i\mid i\in B\}$ be a family of pairwise disjoint semigroups. Then*

(a) *$S=[\![B;S_i,\phi_{i,j}]\!]$ if and only if $S=(Y;S_\alpha,\phi_{\alpha,\beta})$, $S_\alpha=[B_\alpha;S_i,\chi_{i,j}]$, for every $\alpha\in Y$, and*

$$a\chi_{i,j}\phi_{\alpha,\beta}=a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},j\theta_{\alpha,\beta}},\tag{17}$$

for $\alpha,\beta\in Y,\ \alpha\geq\beta,\ i,j\in B_\alpha,\ a\in S_i$;

(b) *$S=[B;S_i,\phi_{i,j}]$ if and only if $S=[Y;S_\alpha,\phi_{\alpha,\beta}]$, $S_\alpha=[B_\alpha;S_i,\chi_{i,j}]$, for every $\alpha\in Y$, and (17) holds.*

Proof. (a) Let $S=[\![B;S_i,\phi_{i,j}]\!]$. By Theorem 3 [2] we obtain that S is a spined product of semigroups $T=(Y;T_\alpha,\omega_{\alpha,\beta})$ and B with respect to Y. Without loss of generality, we can assume that $S=\cup_{\alpha\in Y}T_\alpha\times B_\alpha$. Let $S_\alpha=T_\alpha\times B_\alpha$, $\alpha\in Y$, for $\alpha,\beta\in Y$, $\alpha\geq\beta$, let a mapping $\phi_{\alpha,\beta}$ of S_α into S_β be defined by

$$(a,i)\phi_{\alpha,\beta}=(a\omega_{\alpha,\beta},i\theta_{\alpha,\beta}),\qquad((a,i)\in S_\alpha),$$

$S_i=T_\alpha\times\{i\}$, $i\in B_\alpha$, $\alpha\in Y$, and for $\alpha\in Y$, $i,j\in B_\alpha$, let a mapping $\chi_{i,j}$ of S_i into S_j be defined by

$$(a,i)\chi_{i,j}=(a,j),\qquad((a,i)\in S_i).$$

Then it is easy to verify that $S=(Y;S_\alpha,\phi_{\alpha,\beta})$, $S_\alpha=[B_\alpha;S_i,\chi_{i,j}]$, for every $\alpha\in Y$, and that (17) holds.

Conversely, let $S=(Y;S_\alpha,\phi_{\alpha,\beta})$, $S_\alpha=[B_\alpha;S_i,\chi_{i,j}]$, for every $\alpha\in Y$, and let (17) holds. For $i,j\in B$, $i\succcurlyeq j$, $\alpha=[i]$, $\beta=[j]$, let $\phi_{i,j}$ be a mapping of S_i into S_j defined by

$$a\phi_{i,j}=a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},j},\qquad(a\in S_i).$$

Clearly, $a\phi_{i,i} = a\phi_{\alpha,\alpha}\chi_{i,i} = a$, for every $i \in B$. Assume $\alpha, \beta, \gamma \in Y,\ \alpha\beta \geq \gamma,\ i \in B_\alpha,\ j \in B_\beta,\ k \in B_\gamma,\ a \in S_i,\ b \in S_j$. Then

$$\begin{aligned}[(a\phi_{i,ij})(b\phi_{j,ij})]\phi_{ij,k} &= [(a\phi_{\alpha,\alpha\beta}\chi_{i\theta_{\alpha,\alpha\beta},ij})(b\phi_{\beta,\alpha\beta}\chi_{j\theta_{\beta,\alpha\beta},ij})]\phi_{ij,k} \\ &= [(a\phi_{\alpha,\alpha\beta}) \circ (b\phi_{\beta,\alpha\beta})]\phi_{\alpha\beta,\gamma}\chi_{(ij)\theta_{\alpha\beta,\gamma},k} = [(a\phi_{\alpha,\gamma}) \circ (b\phi_{\beta,\gamma})]\chi_{(ij)\theta_{\alpha\beta,\gamma},k} \\ &= [(a\phi_{\alpha,\gamma}\chi_{i\theta_{\alpha,\gamma},(i\theta_{\alpha,\gamma})(j\theta_{\beta,\gamma})})(b\phi_{\beta,\gamma}\chi_{j\theta_{\beta,\gamma},(i\theta_{\alpha,\gamma})(j\theta_{\beta,\gamma})})]\chi_{(i\theta_{\alpha,\gamma})(j\theta_{\beta,\gamma}),k} \\ &= (a\phi_{\alpha,\gamma}\chi_{i\theta_{\alpha,\gamma},k})(b\phi_{\beta,\gamma}\chi_{j\theta_{\beta,\gamma},k}) = (a\phi_{i,k})(b\phi_{j,k}),\end{aligned}$$

since $(ij)\theta_{\alpha\beta,\gamma} = [(i\theta_{\alpha,\alpha\beta})(j\theta_{\beta,\alpha\beta})]\theta_{\alpha\beta,\gamma} = (i\theta_{\alpha,\gamma})(j\theta_{\beta,\gamma})$, and

$$\begin{aligned}a * b &= (a\phi_{\alpha,\alpha\beta}) \circ (b\phi_{\beta,\alpha\beta}) = (a\phi_{\alpha,\alpha\beta}\chi_{i\theta_{\alpha,\alpha\beta},ij})(b\phi_{\beta,\alpha\beta}\chi_{j\theta_{\beta,\alpha\beta},ij}) \\ &= (a\phi_{i,ij})(b\phi_{j,ij}).\end{aligned}$$

Therefore, $S = (B; S_i, \phi_{i,j})$.

Assume $\alpha, \beta \in Y,\ \alpha \geq \beta,\ i, j \in B_\alpha,\ k \in B_\beta,\ a \in S_i$. Then by (17),

$$\begin{aligned}a\phi_{i,j}\phi_{j,k} &= a\phi_{\alpha,\alpha}\chi_{i,j}\phi_{\alpha,\beta}\chi_{j\theta_{\alpha,\beta},k} = a\chi_{i,j}\phi_{\alpha,\beta}\chi_{j\theta_{\alpha,\beta},k} \\ &= a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},j\theta_{\alpha,\beta}}\chi_{j\theta_{\alpha,\beta},k} = a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},k} = a\phi_{i,k},\end{aligned}$$

Hence, $S = [\![B; S_i, \phi_{i,j}]\!]$.

(b) Let $S = [B; S_i, \phi_{i,j}]$. Assume that $S_\alpha = \cup_{i \in B_\alpha} S_i$. Clearly, $S_\alpha = [B_\alpha; S_i, \chi_{i,j}]$, for every $\alpha \in Y$, where $\chi_{i,j} = \phi_{i,j},\ i, j \in B_\alpha$. For $\alpha, \beta \in Y,\ \alpha \geq \beta$, define a mapping $\phi_{\alpha,\beta}$ of S_α into S_β by

$$a\phi_{\alpha,\beta} = a\phi_{i,i\theta_{\alpha,\beta}}, \qquad (a \in S_i,\ i \in B_\alpha).$$

Clearly, $\{\phi_{\alpha,\beta}\}$ is a transitive system of homomorphisms. Assume $\alpha, \beta \in Y,\ i \in B_\alpha,\ j \in B_\beta,\ a \in S_i,\ b \in S_j$. Then

$$\begin{aligned}(a\phi_{\alpha,\alpha\beta}) \circ (b\phi_{\beta,\alpha\beta}) &= (a\phi_{i,i\theta_{\alpha,\alpha\beta}}) \circ (b\phi_{j,j\theta_{\beta,\alpha\beta}}) \\ &= (a\phi_{i,i\theta_{\alpha,\alpha\beta}}\phi_{i\theta_{\alpha,\alpha\beta},ij})(b\phi_{j,j\theta_{\beta,\alpha\beta}}\phi_{j\theta_{\beta,\alpha\beta},ij}) = (a\phi_{i,ij})(b\phi_{j,ij}) = a * b,\end{aligned}$$

since $ij = (i\theta_{\alpha,\alpha\beta})(j\theta_{\beta,\alpha\beta})$. Therefore, $S = [Y; S_\alpha, \phi_{\alpha,\beta}]$. Assume $\alpha, \beta \in Y,\ \alpha \geq \beta,\ i, j \in B_\alpha,\ a \in S_i$. Then

$$\begin{aligned}a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},j\theta_{\alpha,\beta}} &= a\phi_{i,i\theta_{\alpha,\beta}}\phi_{i\theta_{\alpha,\beta},j\theta_{\alpha,\beta}} = a\phi_{i,j\theta_{\alpha,\beta}} = a\phi_{i,j}\phi_{j,j\theta_{\alpha,\beta}} \\ &= a\chi_{i,j}\phi_{\alpha,\beta}.\end{aligned}$$

Therefore, (17) holds.

Conversely, let $S=[Y;S_\alpha,\phi_{\alpha,\beta}]$, $S_\alpha=[B_\alpha;S_i,\chi_{i,j}]$, for every $\alpha\in Y$, and let (17) holds. By (a) we have that $S=[\![B;S_i,\phi_{i,j}]\!]$. In notations from (a), assume that $i,j,k\in B$ such that $i\succcurlyeq j\succcurlyeq k$, i.e. $i\in B_\alpha,\ j\in B_\beta,\ k\in B_\gamma,\ \alpha,\beta\,\gamma\in Y,\ \alpha\ge\beta\ge\gamma$. Let $a\in S_i$. Then

$$\begin{aligned}a\phi_{i,j}\phi_{j,k}&=a\phi_{\alpha,\beta}\chi_{i\theta_{\alpha,\beta},j}\phi_{\beta,\gamma}\chi_{j\theta_{\beta,\gamma},k}=a\phi_{\alpha,\beta}\phi_{\beta,\gamma}\chi_{i\theta_{\alpha,\beta}\theta_{\beta,\gamma},j\theta_{\beta,\gamma}}\chi_{j\theta_{\beta,\gamma},k}\\&=a\phi_{\alpha,\gamma}\chi_{i\theta_{\alpha,\gamma},k}=a\phi_{i,k}.\end{aligned}$$

Therefore, $S=[B;S_i,\phi_{i,j}]$. □

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