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SEMIGROUPS OF GALBIATI–VERONESI III (SEMILATTICE OF NIL-EXTENSIONS OF LEFT AND RIGHT GROUPS)

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Abstract. This paper is the continuation of [1] and [2]. Here we consider semigroups which are semilattice of nil-extensions of left and right groups.

1. Introduction and preliminares

J. L. Galbiati and M. L. Veronesi [6] studied π-regular semigroups in which every regular element is completely regular (semigruppi quasi fortemente regolari). These semigroups are completely described by M. L. Veronesi in [11]. L. N. Ševrin [9],[10] has anounced several conditions equivalent to various decomposition of *completely π-regular* (called quasiperiodic) semigroups, but the details are not now available to the authors. Semigroups which are semilattice of nil-extensions of rectangular groups are described by the first author in [1]. Using the well known method of the semilattice decomposition of Tamura and Putcha, here we consider semigroups which are semilattice of nil-extensions of left and right groups and several subclasses of these semigroups.

Throughout this paper, $\mathbb{Z}^+$ will denote the set of all positive integers. A semigroup S is π-regular if for every $a \in S$ there exists $m \in \mathbb{Z}^+$ such that $a^m \in a^m S a^m$. Let us denote by $\mathrm{Reg}(S)$ $(\mathrm{Gr}(S), \mathrm{E}(S))$ the set of all regular (completely regular, idenpotent) elements of a semigroup S. S is a *GV-semigroup* (*semigroup of Galbiati-Veronesi*) if S is *π-regular* and $\mathrm{Reg}(S) = \mathrm{Gr}(S)$ (see [6]). A semigroup S is a *π-group* if for every $a \in S$ there exists $n \in \mathbb{Z}^+$ such that $a^n \in G$, where G is a subgroup of S.

For undefined notions and notations we refer to [5] and [8].

In our investigations the following result is fundamental (see [1, Theorem 2.1]).

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Theorem (Bogdanović). *S is a semilattice of nil-extensions of rectangular groups if and only if S is a GV-semigroup and for every $e, f \in \mathrm{E}(S)$ there exists $n \in \mathbb{Z}^+$ such that $(ef)^n = (ef)^{n+1}$.*

2. Weakened Rédei's band

A band S is a singular band either S is a left or a right zero band. In this paper semilattices and chains of singular bands will be described.

Definition 2.1. *A semigroup S is a weakened Rédei's band (or simply WR-band) if*

$$(\forall x, y \in S) \qquad xy \in \{x, y\} \quad \vee \quad yx \in \{x, y\}. \tag{2.1}$$

Lemma 2.1. *S is a rectangular WR-band if and only if S is a singular band.*

Proof. Let S be a rectangular WR-band. Let $a, b \in S$. If $ba = a$, then $ab = bab = b$. If $ba = b$, then $ab = aba = a$. Hence, by this and by (2.1) it follows that

$$ab = a \qquad \text{or} \qquad ab = b$$

Assume that $ab = a$. Then we will prove that $xy = x$ for all $x, y \in S$. Let $x, y \in S$. In a similar way we prove that

$$xa \in \{x, a\} \qquad and \qquad xb \in \{x, b\} \quad ,$$

$$ya \in \{y, a\} \qquad and \qquad yb \in \{y, b\} \quad .$$

By Proposition IV 3.2. [7] we have that

$$xa = xab = xb \in \{x, a\} \cap \{x, b\} = \{x\}$$

and

$$ya = yab = yb \in \{y, a\} \cap \{y, b\} = \{y\}$$

Thus, $xa = xb = x$ and $ya = yb = y$. Now by Proposition IV 3.2. [7] we have that

$$xy = xya = xa = x.$$

Therefore, S is a left zero band. Similarly, if $ab = b$ for some $a, b \in S$, then $xy = y$ for all $x, y \in S$, so S is a right zero band.

The converse follows immediately. $\square$

Now we have the following theorem:

Theorem 2.1. *S is a WR-band if and only if S is a chan of singular bands.*

Proof. Let S be a WR-band. By Theorem IV 3.1 [7] we have that S is a semilattice Y of rectangular bands $S_\alpha, \alpha \in Y$. By Lemma 2.1 it follows that S_α is a singular band for every $\alpha \in Y$, and by (2.1) it follows that Y is a chain.

Conversely, let S be a chain Y of singular bands $S_\alpha, \alpha \in Y$. Let $x \in S_\alpha,\quad y \in S_\beta,\quad \alpha, \beta \in Y$. Then $xy = xxy = x$ if $\alpha \le \beta$ and S_α is a left zero band, $xy = xyy = y$ if $\beta \le \alpha$ and S_β is a right zero band, $yx = yxx = x$ if $\alpha \le \beta$ and S_α is a right zero band and $yx = yyx = y$ if $\beta \le \alpha$ and S_β is a left zero band. Thus, S is a WR-band. □

A semigroup S is a *Rédei's band* (L. Rédei) if $xy \in \{x, y\}$ for every $x, y \in S$. It is clear that every Rédei's band is a WR-band. A chain Y of semigroups $S_\alpha, \alpha \in Y$ is an *ordinal sum of semigroups* S_α if $xy = yx = x$ for all $x \in S_\alpha$, $y \in S_\beta$, $\alpha \le \beta$, $\alpha, \beta \in Y$.

Corollary 2.1 (L. Rédei)**.** *S is a Rédei's band if and only if S is an ordinal sum of singular bands.*

Definition 2.2. *A semigroup S is a left-weakened Rédei's band* (*or simply LWR-band*) *if*

$$(\forall x, y \in S)\;\; xy = x \quad \vee \quad yx = y \tag{2.2}$$

Dually we define a *right weakened Rédei's band* (*or simply RWR-band*).

Corollary 2.2. *S is a chain of left zero bands if and only if S is a LWR-band.*

Example 2.1. A semigroup S given by the following table

	e	f	g	h
e	e	e	e	e
f	f	f	f	f
g	f	f	g	h
h	f	f	g	h

is a chain of singular bands i.e. it is a WR-band, but this is not a Rédei's band, since $ge = f \notin \{g, e\}$. Also, this is not an LWR-band, since $gh = h \ne g$ and $hg = g \ne h$, and this is not an RWR-band, since $ef = e \ne f$ and $fe = f \ne e$.

Example 2.2. A semigroup given by the following table

	e	f	g	h
e	e	e	e	e
f	f	f	f	f
g	f	f	g	g
h	f	f	h	h

is an LWR-band, so this is a WR-band, but this is not a Rédei's band and this is not an RWR-band.

Example 2.3. A semigroup given by the following table

	e	f	g
e	e	g	g
f	g	f	g
g	g	g	g

is a semilattice, but it is not a chain, so it is not a WR-band.

Definition 2.3. *A band S is an LR-band if*

$$(\forall x, y \in S) \quad xy = xyx \quad \vee \quad xy = yxy \tag{2.3}$$

Lemma 2.2. *S is a rectangular LR-band if and only if S is a singular band.*

Proof. This follows by (2.3), by Proposition IV 3.2 [7] and by Lemma 2.1. □

Theorem 2.2. *The following condition on a semigroup S are equivalent:*
(i) *S is an LR-band;*
(ii) *S is a semilattice of singular bands;*
(iii) *S is regular and*

$$(\forall x, y \in S) \quad xy = xyx \quad \vee \quad xy = yxy \tag{2.4}$$

(iv) *S is regular and*

$$(\forall x, y \in S) \quad xyx = xy \quad \vee \quad xyx = yx. \tag{2.5}$$

Proof. (i)⇒ (ii). This follows by Theorem IV 3.1 [7] and by Lemma 2.2.

(ii)⇒ (iii). Let S be a semilattice Y of singular bands S_α, $\alpha \in Y$ It is clear that S is regular. Let $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$. Then $xy, yx \in S_{\alpha\beta}$ so

$$xy = (xy)(yx) = xyx \ ,$$

if $S_{\alpha\beta}$ is a left zero band, and

$$xy = (yx)(xy) = yxy,$$

if $S_{\alpha\beta}$ is a right zero band. Thus, (2.4) holds.

(iii)⇒ (i). Let S be regular and (2.4) holds. Let $a \in S$. Then $a = axa$ for some $x \in S$, and by (2.4) we have that $ax = axa = a$ or $ax = xax$. If $ax = a$, then $a^2 = a$. If $ax = xax$, then $a = axa = (xa)^2 = xa$, whence $a^2 = a$. Therefore, S is a band and it is an LR-band.

(i)⇒ (iv). Let S be an LR-band. Then S is regular. Let $x, y \in S$. If $xy = yxy$, then $xyx = (yx) = yx$. By this and by (2.3) it follow that (2.5) holds.

(iv)⇒ (i). Let S be regular and let (2.5) holds. As in the proof of (iii)⇒ (i) we have that S is a band. Let $x, y \in S$ and $xyx = yx$. Then $yxy = (xy)^2 = xy$. Thus, (2.3) holds, i.e. S is an LR-band. □

Theorem 2.3. *A semigroup S is a chain of rectangular bands if and only if*

$$(\forall x, y \in S) \quad x = xyx \quad \vee \quad y = yxy. \tag{2.6}$$

Proof. Let S be a chain Y of rectangular bands S_α, $\alpha \in Y$. Let $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$. Assume that $\alpha \leq \beta$. Then $x, xy \in S_\alpha$, whence

$$xyx = x(xy)x = x$$

Similarly, if $\beta \leq \alpha$, then $yxy = y$. Thus, (2.6) holds.

Conversely, let (2.6) holds. Let $x \in S$. Then by (2.6) we have that $x = x^3$, and

$$x = xx^2x = x^4 = x^2 \quad \text{or} \quad x^2 = x^2xx^2 = x^5 = x$$

Thus, S is a band. By theorem IV 3.1 [7] we have that S is a semilattice Y of rectangular bands, and by (2.6) it follows that Y is a chain. □

3. Semilattice of nil-extensions of left and right groups

By $L(a)$ and $R(a)$ we denote the principal left ideal and the principal right ideal of a semigroup S generated by the element a of S.

Definition 3.1. *A semigroup S is an LR-semigroup if*

$$(\forall x, y \in S)(\exists n \in \mathbb{Z}^+)\ (xy)^n \in L(x) \cup R(y).$$

Lemma 3.1. *A semigroup S is an LR-semigroup if and only if*

$$(\forall x, y \in S)(\exists n \in \mathbb{Z}^+)\ (xy)^n \in Sx \cup yS. \tag{3.2}$$

Proof. Let S be an LR-semigroup and let $x, y \in S$. Then $(xy)^n \in L(x) \cup R(y) = x \cup Sx \cup y \cup yS = \{x, y\} \cup Sx \cup yS$, for some $n \in \mathbb{Z}^+$. If $(xy)^n \in \{x, y\}$, then $(xy)^{2n} \in \{x^2, y^2\} \subset Sx \cup yS$. Hence, (3.2) holds.

The converse follows immediately. □

Lemma 3.2. *If S is a semilattice of left and right archimedean semigroups, then S is an LR-semigroup.*

Proof. Let S be a semilattice Y of left and right arhimedian semigroups S_α, $\alpha \in Y$ Than for $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$ we have that $xy, yx \in S_{\alpha\beta}$, whence

$$(xy)^n \in S_{\alpha\beta} yx \subset Sx,$$

for some $n \in \mathbb{Z}^+$, if $S_{\alpha\beta}$ is a left archimedian semigroup, and

$$(xy)^n \in yxS_{\alpha\beta} yx \subset yS,$$

for some $n \in \mathbb{Z}^+$, if $S_{\alpha\beta}$ is a right archimedian semigroup. Now, by Lemma 3.1. it follows that S is an LR-semigroup. □

Lemma 3.3 [3]**.** *A semigroup S is a left (right) group if and only if $x \in xSa$ ($x \in aSx$) for all $a, x \in S$.*

Theorem 3.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left and right groups;*
(ii) $(\forall x, y \in S)(\exists n \in \mathbb{Z}^+)\ (xy)^n \in (xy)^n S(yx)^n \cup (yx)^n S(xy)^n$;
(iii) *S is a π-regular LR-semigroup;*
(iv) *S a GV-semigroup and for every $e, f \in \mathrm{E}(S)$ there exists $n \in \mathbb{Z}^+$ such that:*

$$(ef)^n = (efe)^n \quad \vee \quad (ef)^n = (fef)^n; \tag{3.3}$$

(v) *S is π-regular and*

$$a = axa \quad \Rightarrow \quad ax = ax^2a \quad \vee \quad ax = xa^2x. \tag{3.4}$$

Proof. (i)⇒(ii). Let S be a semilattice Y of semigroups S_α , $\alpha \in Y$ and let S_α be a nil-extension of T_α, where T_α , $\alpha \in Y$ is a left or a right group. For $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$ we have that $xy, yx \in S_{\alpha,\beta}$. Then there exists $n \in \mathbb{Z}^+$ such that $(xy)^n, (yx)^n \in T_{\alpha,\beta}$. By Lemma 3.3 it follows that

$$(xy)^n \in (xy)^n S (yx)^n ,$$

if $T_{\alpha,\beta}$ is a left group, and

$$(xy)^n \in (yx)^n S (xy)^n ,$$

if $T_{\alpha,\beta}$ is a right group. Thus, (ii) holds.

(ii) ⇒ (iii). Let (ii) holds. For any $x \in S$ there exists $n \in \mathbb{Z}^+$ such that $x^{2n} \in x^{2n} S x^{2n}$, so S is $\pi-$ regular. By (ii) it follows that S is an LR-semigroup.

(iii) ⇒ (iv). Let S be a $\pi-$regular LR-semigroup. Let $a \in \mathrm{Reg}(S)$. Then there exists $b \in S$ such that $a = aba$. By (3.2) we have that $ab \in Sa \cup bS$ and $ba \in Sb \cup aS$. Let $ab = ua$ for some $u \in S$. Then $a = aba = ua^2 \in Sa^2$. Let $ab = bv$ for some $v \in S$. Then $a = aba = bva$, whence $a^2 = abva$ and $a = bva = babva = ba^2 \in Sa^2$. Similarly, from $ba \in Sb \cup aS$ it follows that $a \in a^2 S$. Therefore, a is a completely regular element and so S is a GV-semigroup.

Let $e, f \in \mathrm{E}(S)$. Then there exists $n \in \mathbb{Z}^+$ such that $(ef)^n \in Se \cup fS$. Let $(ef)^n = ue$ for some $u \in S$. Then

$$(efe)^n = (ef)^n e = uee = ue = (ef)^n .$$

From $(ef)^n \in fS$ it follows that $(fef)^n = (ef)^n$ for some $n \in \mathbb{Z}^+$. Therefore, (3.3) holds.

(iv) ⇒ (i). By (3.3) we obtain that

$$(ef)^{n+1} = (ef)^n ef = (efe)^n f = (ef)^n f = (ef)^n$$

or

$$(ef)^{n+1} = ef(ef)^n = e(fef)^n = e(ef)^n = (ef)^n$$

for some $n \in \mathbb{Z}^+$. Now, by Theorem of Bogdanović we have that S is a semilattice Y of semigroups S_α, $\alpha \in Y$ where S_α is a nil-extension of a rectangular group T_α, $\alpha \in Y$. Since $E_\alpha = \mathrm{E}(S_\alpha) = \mathrm{E}(T_\alpha)$, $\alpha \in Y$ is a rectangular band, then we have that for every $e, f \in E_\alpha$ there exists $n \in \mathbb{Z}^+$ such that

$$ef = (ef)^n = (efe)^n = e^n = e$$

or

$$ef = (ef)^n = (fef)^n = f^n = f.$$

Thus, E_α is a rectangular Rédei's band, so by Lemma 2.1 we have that E_α is a singular band. Now, $T_\alpha, \alpha \in Y$ is a left or right group, i.e. S_α, $\alpha \in Y$ is a nil-extension of a left or a right group.

(i) $\Rightarrow$ (v). Let S be a semilattice Y of semigroups $S_\alpha,\ \alpha \in Y$, where S_α is a nil-extension of a left or a right group $T_\alpha,\ \alpha \in Y$. It is clear that S is π-regular. Assume that $a = axa$, wrere $a \in S_\alpha, \quad x \in S_\beta, \quad \alpha, \beta \in Y$. Then

$$ax, xa \in S_{\alpha\beta} \cup \mathrm{E}(S) = \mathrm{E}(S_{\alpha\beta}) = E_{\alpha\beta} ,$$

whence

$$ax = (ax)(xa) = ax^2a ,$$

if $E_{\alpha\beta}$ is a left zero band, and

$$ax = (xa)(ax) = xa^2x ,$$

if $E_{\alpha\beta}$ is a right zero band. Thus (3.4) holds.

(v) $\Rightarrow$ (i). Let S be $\pi-$regular and let (3.4) holds. Assume that $a = axa$, $a, x \in S$. By (3.4) we have that

$$a = (ax)a = (ax^2a)a = ax^2a^2$$

or

$$a = (ax)^2a = (ax)(xa^2x)a = ax^2a(axa) = ax^2a^2 .$$

Now, by Theorem 2.1 [1] we have that S is a semilattice Y of semigroups S_α , $\alpha \in Y$ where S_α is a nil-extension of a rectangular group T_α , $\alpha \in Y$. Let $e, f \in \mathrm{E}(S_\alpha) = \mathrm{E}(T_\alpha)$. Since $\mathrm{E}(T_\alpha)$ is a rectangular band, then by (3.4) we have that $e = efe$ implies that

$$ef = ef^2e = efe = e \text{ or } ef = fe^2f = fef = f .$$

Hence, $\mathrm{E}(T_\alpha)$ is a rectangular Rédei's band and by Lemma 2.1 it follows that $\mathrm{E}(T_\alpha)$ is a singular band. Then by Theorem IV 3.9 ([8]) T_α is a left or a right group. Therefore, S is a semilattice of nil-extensions of left and right groups. $\square$

Corollary 3.1. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of left groups ;*
(ii) $(\forall x, y \in S)(\exists n \in \mathbb{Z}^+)\ \ (xy)^n \in (xy)^n S(yx)^n$ *;*
(iii) *S is $\pi-$regular and*

(3.5) $$(\forall x, y \in S)(\exists n \in \mathbb{Z}^+)\ \ (xy)^n \in L(x) .$$

Theorem 3.2. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of singular bands ;*
(ii) $(\forall x, y \in S)(\exists n \in \mathbb{Z}^+)\ \ (xy)^n = (xy)^n x \ \vee\ (xy)^n = y(xy)^n$ *;*
(iii) *S is $\pi-$regular LR-semigroup and* $\mathrm{Reg}(S) = \mathrm{E}(S)$ *;*
(iv) *S is $\pi-$regular and*

(3.6) $$a = axa \ \Rightarrow\ a = axa \ \vee\ a = xa .$$

Proof. (i) $\Rightarrow$ (ii). Let S be a semilattice Y of semigroups S_α , $\alpha \in Y$ and let S_α is a nil-extension of a singular band E_α , $\alpha \in Y$. Let $x \in S_\alpha, y \in S_\alpha, \alpha, \beta \in Y$. Then $xy, yx \in S_{\alpha\beta}$ and there exists $n \in \mathbb{Z}^+$ such that $(xy)^n, (yx)^n \in E_{\alpha\beta}$. Since $E_{\alpha\beta}$ is an ideal of $S_{\alpha\beta}$, then we have that

$$(xy)^n x \;=\; (xy)^n (xy)^{2n} x \;=\; (xy)^n$$

if $E_{\alpha\beta}$ is a left zero band, and

$$y(xy)^n \;=\; y(xy)^{2n}(xy)^n \;=\; (xy)^n$$

if $E_{\alpha\beta}$ is a right zero band. Thus, (ii) holds. (ii) $\Rightarrow$ (iii). Let (ii) holds. Let $x \in S$. Then there exists $n \in \mathbb{Z}^+$ such that $x^{2n} = x^{2n+1}$, and by this it follows that S is $\pi-$regular. Let $x, y \in S$. Then we have that

$$(xy)^n \;\in\; \{(xy)^n x, y(xy)^n\} \;\subseteq\; Sx \cup yS \;,$$

for some $n \in \mathbb{Z}^+$, so S is an LR-semigroup. Let $a \in \mathrm{Reg}(S)$. Then there exists $x \in S$ such that $a = axa$ and $x = xax$. Now we have that there exists $n \in \mathbb{Z}^+$ such that

$$ax \;=\; (ax)^n \;=\; (ax)^n a \;=\; axa \;=\; a$$

or

$$ax \;=\; (ax)^n \;=\; x(ax)^n \;=\; xax \;=\; x \;.$$

If $ax = a$, then $a = axa = a(ax)a = a^2$. Therefore, $\mathrm{Reg}(S) = \mathrm{E}(S)$.

(iii) $\Rightarrow$ (i). Let S is a $\pi-$regular LR-semigroup and $\mathrm{Reg}(S) = \mathrm{E}(S)$. Then by Theorem 3.1 we have that S is a semilattice of nil- extensions of left and right groups. Since $\mathrm{Reg}(S) = \mathrm{E}(S)$, then S is a semilattice of nil-extensions of singular bands.

(ii) $\Rightarrow$ (iv). Let (ii) holds and let $a = axa$. As in the proof of (ii)$\Rightarrow$(iii) we obtain that $ax = a$ or $ax = x$ and that S is $\pi-$ regular. If $ax = x$, then we have that $a = axa = xa$, so (3.6) holds.

(iv) $\Rightarrow$ (i). Let S be $\pi-$regular and let (3.6) holds. Let $a \in \mathrm{Reg}(S)$. then $a = axa$ and $x = xax$ for some $x \in S$. By (3.6) we have that $a = ax$ or $ax = x$, i.e. $ax = a$ or $xa = axa = a$, whence $a = axa = a^2$. Therefore, $\mathrm{Reg}(S) = \mathrm{E}(S)$. By Theorem 6 [2] we have that S is a semilattice Y of semigroups S_α , $\alpha \in Y$ and S_α is a nil-extension of a rectangular band E_α , $\alpha \in Y$ and $e, f \in E_\alpha$. Then $efe = e$, so by (3.6) it follows that $ef = e$ or $fe = e$. Thus, E_α is a rectangular WR-band, so by Lemma 2.1 it follows that E_α is a singular band. Thus, S is a semilattice of nil-extensions of singular bands. □

Theorem 3.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left and right groups ;*

(ii) *S is regular and*

(3.7) $$(\forall x, y \in S) \;\; xy \in Sx \cup yS \;;$$

(iii) *S is regular and*

$$(3.8) \qquad (\forall x, y \in S) \quad xy \in L(x) \cup R(y) \ ;$$

(iv) *S is a regular LR-semigroup ;*
(v) *S is completely regular and* $\mathrm{E}(S)$ *is an LR-band ;*
(vi) *S is regular and*

$$(3.9) \qquad a = axa \ \Rightarrow \ ax = ax^2a \ \vee \ ax = xa^2x \ .$$

Proof. (i) ⇒ (ii). Let S be a semilattice Y of semigroups $S_\alpha,\ \alpha \in Y$ where S_α is a left or right group, $\alpha \in Y$. Let $x \in S_\alpha,\ y \in\ S_\beta,\ \alpha, \beta \in Y$. Then $xy, yx \in S_{\alpha\beta}$ and by Lemma 3.3 we have that

$$xy \in xyS_{\alpha\beta}yx \subseteq Sx,$$

if $S_{\alpha\beta}$ is a left group, and

$$xy \in yxS_{\alpha\beta}xy \subseteq yS,$$

if $S_{\alpha\beta}$ is a right group. Thus (3.7) holds. It is clear that S is regular.

(ii) ⇒ (iii) and (iii) ⇒ (iv) follows immediately.

(iv) ⇒ (i). Let S be a regular LR-semigroup. By Theorem 3.1 S is a semilattice Y of semigroups $S_\alpha,\ \alpha \in Y$, S_α is a nil-extension of T_α and T_α is a left or a right group, $\alpha \in Y$. Let $\alpha \in Y$ and $a \in S_\alpha$. Since S is regular, then there exists $x \in S$ such that $a = axa$ and $x = xax$. Then $x \in S_\alpha$ and $ax, xa \in S_\alpha \cap \mathrm{E}(S) = \mathrm{E}(S_\alpha) = \mathrm{E}(T_{al}) \subseteq T$. Since T_α is an ideal of S_α, then $a = axa \in S_\alpha T_\alpha \subseteq T_\alpha$. Thus $S_\alpha = T_\alpha$ for all $\alpha \in Y$, whence S is a semilattice of left and right groups.

(ii) ⇒ (v). Let S be regular and let (3.7) holds. As in the proof of Theorem 3.1 we obtain that S is completely regular. Let $e, f \in \mathrm{E}(S)$. Then $ef \in Se \cup fS$. Let $ef = ue$ for some $u \in S$. Then

$$efe = uee = ue = ef.$$

Similarly, by $ef = fv$ for some $v \in S$ it follows that $fef = ef$. Now $(ef)^2 = ef$ and by Theorem 2.2 we have that $\mathrm{E}(S)$ is an LR-band.

(v) ⇒ (i). Let S be completely regular and $\mathrm{E}(S)$ be an LR-band. By Proposition IV 3.7 [8] S is a semilattice Y of rectangular groups $S_\alpha, \alpha \in Y$. Let $\alpha \in Y$. Since $\mathrm{E}(S)$ is an LR-band, then $\mathrm{E}(S_\alpha)$ is an LR-band. Thus, $\mathrm{E}(S_\alpha)$ is a rectangular LR-band and by Lemma 2.2 $\mathrm{E}(S_\alpha)$ is a singular band. Therefore, S_α is a left or right group, so S is a semilattice of left and right groups.

(iv) ⇒ (vi) and (vi) ⇒ (iv) follows by Theorem 3.1. □

Corollary 3.2 [8]**.** *A semigroup S is a semilattice of left groups if and only if S is regular and*

$$(\forall x, y \in S) xy \in Sx.$$

Corollary 3.3. *A band S is an LR-band if and only if S is an LR-semigroup.*

4. Chain of nil-extensions of rectangular groups

Chains of nil-extensions of rectangular groups are considered in [1] (Theorem 5.2 [1]). Now we have the following theorem:

Theorem 4.1. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of nil-extensions of rectangular groups;*
(ii) *S is completely π regular and $\mathrm{E}(S)$ is a chain of rectangular bands;*
(iii) *S is a GV-semigroup and $\mathrm{E}(S)$ is a chain of rectangular bands.*

Proof. (i) $\Rightarrow$ (ii). Let S be a chain Y of semigroups $S_\alpha, \alpha \in Y$ and let S_α is a nil-extension of a rectangular group $T_\alpha, \alpha \in Y$. By Theorem IV 2 [5] S is completely π-regular. Let $e, f \in \mathrm{E}(S)$. Then by Theorem 5.2 [1] it follows that there exists $n \in \mathbb{Z}^+$ such that $e = (efe)^n$ or $f = (fef)^n$, and by Bogdanović's theorem we have that there exists $m \in \mathbb{Z}^+$ such that $(ef)^m = (ef)^{m+1}$, whence $(ef)^k = (ef)^m$ for every $k \in \mathbb{Z}^+, k \geq m$ and $(ef)^m \in \mathrm{E}(S)$. Now

$$ef = e^m f = (efe)^{mn} f = (ef)^{mn} ef = (ef)^{mn+1} = (ef)^m \in \mathrm{E}(S)$$

or

$$ef = ef^m = e(fef)^{mn} = ef(ef)^{mn} = (ef)^{mn+1} = (ef)^m \in \mathrm{E}(S).$$

Therefore, $\mathrm{E}(S)$ is a band, whence $efe, fef \in \mathrm{E}(S)$ for all $e, f \in \mathrm{E}(S)$. Now, by Theorem 5.2 [1] it follows that $e = efe$ or $f = fef$ for all $e, f \in \mathrm{E}(S)$, so by Theorem 2.3 it follows that $\mathrm{E}(S)$ is a chain of rectangular bands.

(ii) $\Rightarrow$ (iii). Let S be completely π-regular and let $\mathrm{E}(S)$ be a chain of rectangular bands. Since $\mathrm{E}(S)$ is a subsemigroup of S, then by Proposition 1 [4], $T = \mathrm{Reg}(S)$ is a regular subsemigroup of S, Let $a \in T = \mathrm{Reg}(s)$. Then $a = axa$ and $x = xax$ for some $x \in T$. Since $ax, xa \in \mathrm{E}(S)$ and $\mathrm{E}(S)$ is a chain of rectangular bands, then by Theorem 2.3 we have that

$$a = axa = (ax)(xa)(ax)a = ax^2a^2xa \in Ta^2T$$

or

$$a = axa = a(xa)(ax)(xa) = axa^2x^2a \in Ta^2T.$$

Therefore, T is intra-regular so by Theorem II 4.5 [8] T is a semilattice Y of simple semigroups $T_\alpha, \alpha \in Y$. Let $\alpha \in Y$ and $a \in T_\alpha$. Since S is completely π-regular, then there exist $n \in \mathbb{Z}^+$ and $x \in S$ such that

$$a^n = a^n x a^n \text{ and } a^n x = x a^n.$$

Let $y = xa^n x$. Then $a^n y a^n = a^n$, $ya^n y = y$ and $a^n y = ya^n$. Thus, $y \in T$ and it is not hard to verify that $y \in T_\alpha$. Hence $T_\alpha$is a simple completely π-regular semigroup, so by Theorem VI 2.1.1 [5] we have that T_α is completely simple. Therefore, T is a semilattice of completely simple semigroups, so $T = \mathrm{Reg}(S)$ is completely regular, i.e. S is a GV-semigroup.

(iii) $\Rightarrow$ (i). This follows by Theorem 5.2 [1] and Theorem 4.3. □

Corollary 4.1. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of nil-extensions of groups;*
(ii) *S is completely π-regular and $\mathrm{E}(S)$ is a chain;*
(iii) *S is a GV-semigroup and $\mathrm{E}(S)$ is a chain.*

Corollary 4.2. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of rectangular groups;*
(ii) *S is regular, completely π-regular and $\mathrm{E}(S)$ is a chain of rectangular bands;*
(iii) *S is completely regular and $\mathrm{E}(S)$ is a chain of rectangular bands.*

Corollary 4.3. *A semigroup S is a chain of nil-extensions of periodic rectangular groups if and only if S is periodic and $\mathrm{E}(S)$ is a chain of rectangular bands.*

Corollary 4.4. *A semigroup S is a chain of periodic rectangular groups if and only if S is regular, periodic and $\mathrm{E}(S)$ is a chain of rectangular bands.*

Theorem 4.2. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of nil-extensions of rectangular bands;*
(ii) $(\forall x, y \in S)(\exists n \in \mathbb{Z}^+)\ x^{2n} = x^n y x^n \ \vee\ y^{2n} = y^n x y^n$*;*
(iii) *S is π-regular and $\mathrm{Reg}(S)$ is a chain of rectangular bands.*

Proof. (i) ⇒ (ii). Let S be a chain Y of semigroups $S_\alpha, \alpha \in Y$ and let S_α is a nil-extension of a rectangular band $E_\alpha, \alpha \in Y$. Let $x \in S_\alpha, y \in S_\beta, \alpha, \beta \in Y$. Assume that $\alpha \leq \beta$. Then $x^n \in E_\alpha$ for some $n \in \mathbb{Z}^+$, $x^n y \in S_\alpha$. Since E_α is an ideal of S_α, then $x^n y x^n \in S_\alpha E_\alpha \subseteq E_\alpha$. Now we have that

$$x^{2n} = x^n (x^n y x^n) x^n = x^{2n} y x^{2n} = x^n y x^n.$$

Similarly, if $\alpha \leq \beta$, then we obtain that $y^{2n} = y^n x y^n$ for some $n \in \mathbb{Z}^+$. Thus (ii) holds.

(ii) ⇒ (iii). Let (ii) holds. For every $x \in S$ there exists $n \in \mathbb{Z}^+$ such that $x^{2n} = x^{2n+1}$ so by Lemma 4.2 [1] S is a union of nil-semigroups. Let $e, f \in \mathrm{E}(S)$. Then we have that $e = efe$ or $f = fef$, so by Theorem 2.3 $\mathrm{E}(S)$ is a chain of rectangular bands. Now, by Theorem 4.1 S is a GV-semigroup, whence $\mathrm{Reg}(S) = \mathrm{Gr}(S)$ and by Theorem 5 [2] we have that $\mathrm{Gr}(S) = \mathrm{E}(S)$. Thus, $\mathrm{Reg}(S) = \mathrm{E}(S)$ is a chain of rectangular bands.

(iii) ⇒ (i). Let S be π-regular and ler $\mathrm{Reg}(S)$ be a chain of rectangular bands. It is clear that $\mathrm{E}(S) = \mathrm{Reg}(S)$. Thus S is periodic, so by Theorem 4.1 S is a chain of nil-extensions of rectangular groups. Since $\mathrm{E}(S) = \mathrm{Reg}(S)$, then S is a chain of nil-extensions of rectangular bands. □

5. Chain of nil-extensions of left and right groups

Theorem 5.1. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of nil-extensions of left and right groups;*
(ii) *for every $x, y \in S$ there exists $n \in \mathbb{Z}^+$ such that*

$$x^n \in x^{2n} S (xy)^n \cup (yx)^n S x^{2n} \ \text{or}\ y^n \in y^{2n} S (yx)^n \cup (xy)^n S y^{2n};$$

(iii) *S is completely π-regular and* $\mathrm{E}(S)$ *is a* WR*-band;*
(iv) *S is a* GV*-semigroup and* $\mathrm{E}(S)$ *is a* WR*-band.*

Proof. (i) $\Rightarrow$ (ii). Let S be a chain Y of semigroups $S_\alpha, \alpha \in Y$ S_α is a nil-extension of T_α and T_α is a left or a right group, $\alpha \in Y$. Let $x \in S_\alpha, y \in S_\beta, \alpha, \beta \in Y$. Assume that $\alpha \leq \beta$. Then $x, xy, yx \in S_\alpha$ and there exists $n \in \mathbb{Z}^+$ such that $x^n, (xy)^n, (yx)^n \in T_\alpha$. Let T_α be a left group. Let $x^n \in G_e$ (this is the maximal subgroup of S with the identity element e), $(xy)^n \in G_f$, $e, f \in \mathrm{E}(T_\alpha)$, then $ef = e$, whence $x^n = x^n e = x^n ef = x^{2n}(x^n)^{-1}((xy)^n)^{-1}(xy)^n$, so $x^n \in x^{2n}S(xy)^n$. Similarly, if T_α is a right group, then $x^n \in (yx)^n S x^{2n}$. In a similar way we consider the case $\beta \leq \alpha$. Thus, (ii) holds.

(ii) $\Rightarrow$ (iii). Let $x \in S$. Then there exists $n \in \mathbb{Z}^+$ such that $x^n \in x^{2n}Sx^{2n}$, so S is completely π-regular. Let $e, f \in \mathrm{E}(S)$. Then there exists $n \in \mathbb{Z}^+$ such that

$$e \in eS(ef)^n \cup (fe)^n Se \text{ or } f \in fS(fe)^n \cup (ef)^n Sf,$$

whence it follows that $\mathrm{E}(S)$ is a WR-band.

(iii) $\Rightarrow$ (iv). This follows by Theorem 2.1 and by Theorem 4.1.

(iv) $\Rightarrow$ (i). Let S be a GV-semigroup and let $\mathrm{E}(S)$ be a WR-band. By Theorem 2.1 and by Theorem 4.1 we have that S is a chain Y of semigroups $S_\alpha, \alpha \in Y$ and S_α is a nil-extension of a rectangular group $T_\alpha, \alpha \in Y$. Let $\alpha \in Y$. Let $\alpha \in Y$. Then $\mathrm{E}(T_\alpha)$ is a rectangular WR-band, so by Lemma 2.1 it follows that $\mathrm{E}(T_\alpha)$ is a singular band. Thus, T_α is a left or a right group, so S is a chain of nil-extensions of left and right groups. □

Corollary 5.1. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of left and right groups;*
(ii) $(\forall x, y \in S)$ $x \in xySxy \cup yxSyx$ *or* $y \in xySxy \cup yxSyx$;
(iii) *S is regular, comletely π-regular and* $\mathrm{E}(S)$ *is a* WR*-band;*
(iv) *S is a comletely regular and* $\mathrm{E}(S)$ *is a* WR*-band.*

Proof. (i) $\Rightarrow$ (ii). Let S be a chain Y of left and right groups $S_\alpha, \alpha \in Y$. Let $x \in S_\alpha, y \in S_\beta, \alpha, \beta \in Y$. Assume that $\alpha \leq \beta$ (the similar proof we have in the case $\beta \leq \alpha$). Then $x, xy, yx \in S_\alpha$ Assume that $x \in G_e,\ e \in \mathrm{E}(S_\alpha)$. If S_α is a left group. then $xy = exy \in G_e S_\alpha \subseteq G_e$, whence

$$x \in xyG_e xu \subseteq xySxy.$$

If S_α is a right group, then, similarly, $yx \in G_e$, whence $x \in yxSyx$.

(ii) $\Rightarrow$ (iv). This follows immediately.

(i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). This follows by Theorem 5.1. □

Corollary 5.2. *S is a chain of nil-extensions of periodic left and right groups if and only if S is periodic and* $\mathrm{E}(S)$ *is a* WR*-band.*

Corollary 5.3. *S is a chain of periodic left and right groups if and only if S is a regular periodic semigroup and* $\mathrm{E}(S)$ *is* WR*-band.*

Theorem 5.2. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of nil-extensions of left groups;*
(ii) $(\forall x, y \in S)$ $(\exists n \in \mathbb{Z}^+) x^n \in x^{2n}S(xy)^n$ *or* $y^n \in y^{2n}S(yx)^n$;
(iii) *S is completely π-regular and $\mathrm{E}(S)$ is an LWR-band;*
(iv) *S is a GV-semigroup and $\mathrm{E}(S)$ is an LWR-band.*

Proof. This follows by Theorem 5.1. □

Corollary 5.4. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of left groups;*
(ii) $(\forall x, y \in S)$ $x \in xSxy \vee y \in ySyx$;
(iii) *S is regular, comletely π-regular and $\mathrm{E}(S)$ is an LWR-band;*
(iv) *S is a completely regular and $\mathrm{E}(S)$ is an LWR-band.*

Theorem 5.3. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of nil-extensions of singular bands;*
(ii) $(\forall x, y \in S)$ $(\exists n \in \mathbb{Z}^+) x^n = x^n y \vee x^n = yx^n \vee y^n = y^n x \vee y^n = xy^n$;
(iii) *S is π-regular and* $\mathrm{Reg}(S)$ *is a WR-band.*

Proof. (i) ⇒ (ii). Let S be a chain Y of semigroups $S_\alpha, \alpha \in Y$ and S_α be a nil-extension of a singular band $E_\alpha, \alpha \in Y$. Let $x \in S_\alpha, y \in S_\beta, \alpha, \beta \in Y$. Assume that $\alpha \leq \beta$(the similar proof we have in the case $\beta \leq \alpha$). Then $x^n \in S_\alpha$ for some $n \in \mathbb{Z}^+$ and $x^n y, yx^n \in S_\alpha$. Since $E\alpha$ is an ideal of S_α, then we have that $x^n y = x^n x^n y \in E_\alpha S_\alpha \subseteq E_\alpha$ and similarly $yx^n \in E_\alpha$. Now we have that

$$x^n = x^n x^n y = x^n y,$$

if E_α is a left zero band, and

$$x^n = yx^n x^n = yx^n,$$

if E_α is a right zero band. Thus, (ii) holds.

(ii) ⇒ (iii). Let (ii) holds. Then for every $x \in S$ there exists $n \in \mathbb{Z}^+$ such that $x^n = x^{n+1}$, so by Lemma 4.2 [1] we have that S is a union of nil-semigroups. By theorem 5 [2] it follows that S is completely π-regular and $\mathrm{Gr}(S) = \mathrm{E}(S)$. By (ii) it follows that $\mathrm{E}(S)$ is a WR-band, so by Theorem 5.1 we have that S is a GV-semigroup, i.e. $\mathrm{Reg}(S) = \mathrm{Gr}(S)$. Thus, $\mathrm{Reg}(S) = \mathrm{E}(S)$ is a WR-band, i.e. (iii) holds.

(iii) ⇒ (i). Let S be π-regular and let $\mathrm{Reg}(S)$ be a WR-band. Then $\mathrm{Reg}(S) = \mathrm{E}(S) = \mathrm{Gr}(S)$, so S is a GV-semigroup and $\mathrm{E}(S)$ is a WR-band and by Theorem 5.1 it follows that S is a chain of nil-extensions of left and right groups. Since $\mathrm{Reg}(S) = \mathrm{E}(S)$, then S is a chain of nil-extensions of singular bands. □

Corollary 5.5. *The following conditions on a semigroup S are equivalent:*
(i) *S is a chain of nil-extensions of left zero bands;*
(ii) $(\forall x, y \in S)$ $(\exists n \in \mathbb{Z}^+) x^n = x^n y \ \vee y^n = y^n x$;

(iii) *S is* π*-regular and* $\mathrm{Reg}(S)$ *is an* LWR*-band;*

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POLUGRUPE GALBIATI–VERONESI III (POLUMREŽE NIL-EKSTENZIJA LEVIH I DESNIH GRUPA)

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Ovaj rad je nastavak radova [1] i [2]. Ovde razmatramo polugrupe koje su polumreže nil-ekstenzija levih i desnih grupa.