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SEMIGROUPS OF GALBIATI-VERONESI IV (BANDS OF NIL-EXTENSIONS OF GROUPS)*

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Abstract. This paper is the continuation of [1], [2] and [4]. Here we consider semigroups that are bands of nil-extensions of groups and various special types of them.

1. Introduction and Preliminaries

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. A semigroup S is *π-regular* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^n S a^n$. A semigroup S is *completely π- regular* if for every $a \in S$ there exist $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$ and $a^n x = x a^n$. Let us denote by $Reg(S)$ $(Gr(S),\ E(S))$ the set of all regular (completely regular, idempotent) elements of a semigroup S. A semigroup S is *Archimedean* (*left Archimedean, right Archimedean, t-Archimedean*) if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$ $(a^n \in Sb,\ a^n \in bS,\ a^n \in bS \cap Sb)$. A semigroup S is *completely Archimedean* if S is Archimedean and has a primitive idempotent. By *radical* of the subset A of a semigroup S we mean the set $\sqrt{A}$ defined by $\sqrt{A} = \{a \in S \mid (\exists n \in \mathbf{Z}^+)\ a^n \in A\}$. If e is an idempotent of a semigroup S, then by G_e we denote the maximal subgroup of S with e as its identity and by T_e we denote the set $T_e = \sqrt{G_e}$. On a semigroup S we define the relation τ by

$$a\,\tau\,b \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists e \in E(S))\ a, b \in T_e \qquad (a, b \in S)$$

The relation τ always is symmetric and transitive, and it is an equivalence if and only if S is completely π-regular.

A semigroup S with the zero 0 is a *nil-semigroup* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. An ideal extension S of T is a

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nil-extension if S/T is a nil-semigroup. A subsemigroup T of a semigroup S is a *retract* of S if there exists a homomorphism φ of S onto T such that $\varphi(t) = t$ for all $t \in T$. Such a homomorphism we call a *retraction.* An ideal extension S of T is a *retract extension* (or *retractive extension*) of T if T is a retract of S. A semigroup S is a π-*group* if S is a nil-extension of a group.

A semigroup S is a *Rédei's band* if $xy = x$ or $xy = y$ for all $x, y \in S$. A band S is *left regular*(*normal, left normal*) if S satisfies the identity $ax = axa$ ($axya = ayxa$, $axy = ayx$).

Veronesi's theorem. [13] *A semigroup S is a semilattice of completely Archimedean semigroups if and only if S is π-regular and $Reg(S) = Gr(S)$.*

Munn's lemma. [9] *Let a be an element of a semigroup S such that a^n lies in some subgroup G of S, for some $n \in \mathbf{Z}^+$. If e is the identity of G, then $ea = ae \in G_e$ and $a^m \in G_e$ for all $m \in \mathbf{Z}^+$, $m \geq n$.*

Bogdanović-Milić's lemma. [5] *If S is a π-regular semigroup all of whose idempotents are primitive, then S is completely π-regular with maximal subgroups given by $G_e = eSe$ $(e \in E(S))$.*

Define a relation $\overset{t}{\sim}$ on a semigroup S by:

$$a \overset{t}{\sim} b \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\exists m, n \in \mathbf{Z}^+) \quad a^m \in bS \cap Sb \ \text{ and } \ b^n \in aS \cap Sa.$$

Putcha's theorem. [11] *A semigroup S is a band of t-Archimedean semigroups if and only if*

$$xay \overset{t}{\sim} xa^2y,$$

for all $a \in S$, $x, y \in S^1$.

For undefined notions and notations we refer to [10].

2. Bands of Nil-extensions of Groups

Bands of nil-extensions of groups (π-groups) are studied by J.L. Galbiati and M.L. Veronesi [7], B.L. Madison, T.K. Mukherjee and M.K. Sen [8] and L.N. Shevrin [12]. In the present paper some new characterizations of bands of nil-extensions of groups are given.

By Theorems 1,2 and 3 three preliminary results of decompositions of π-regular semigroups into a semilattice of retractive nil-extensions of completely simple semigroups will be given. The main results are Theorems 4 and 5.

Theorem 1. *Let* S *be a* π*-regular semigroup and let for all* $a,b\in S$ *there exists* $n\in \mathbf{Z}^+$ *such that*

$$(ab)^n\in a^2Sb^2. \tag{1}$$

Then S *is a semilattice of retractive nil-extensions of completely simple semigroups.*

Proof. Let (1) hold and let $a\in Reg(S)$. Then there exists $x\in S$ such that $a=axa$, so

$$\begin{aligned} a&=axa=(ax)^na && \text{for every } n\in \mathbf{Z}^+,\\ &\in a^2Sx^2a && \text{by (1)},\\ &\subseteq a^2S\ ,\\ a&=axa=a(xa)^n && \text{for every } n\in \mathbf{Z}^+,\\ &\in ax^2Sa^2 && \text{by (1)},\\ &\subseteq Sa^2\ . \end{aligned}$$

Thus, $a\in Gr(S)$, so by Veronesi's theorem it follows that S is a semilattice Y of completely Archimedean semigroups S_α, $\alpha\in Y$.

Let $\alpha\in Y$. Then S_α is a nil-extension of a completely simple semigroup K_α. Let $a\in T_e\subseteq S_\alpha$ for some $e\in E(S_\alpha)$ and let $f\in E(S_\alpha)$. We will prove that

$$af=eaf \qquad \text{and} \qquad fa=fae. \tag{2}$$

First, we will prove that for every $m\in \mathbf{Z}^+$ there exist $n\in \mathbf{Z}^+$ and $u\in S$ such that

$$(af)^n=a^muf. \tag{3}$$

It is clear that this holds for $m=1$. Assume that $(af)^n=a^muf$ for some $n\in \mathbf{Z}^+$ and $u\in S$. Then by (1) it follows that there exists $k\in \mathbf{Z}^+$ and $v\in S$ such that

$$(af)^{nk}=((af)^n)^k=(a^muf)^k=a^{2m}v(uf)^2=a^{m+1}u_1f,$$

where $u_1=a^{m-1}vufu$. Therefore, for every $m\in \mathbf{Z}^+$ there exist $n\in \mathbf{Z}^+$ and $u\in S$ such that (3) holds.

Assume that $m\in \mathbf{Z}^+$ is such that $a^m\in G_e$ and let $n\in \mathbf{Z}^+$ and $u\in S$ be such that (3) holds. Since $af\in K_\alpha$, then $af\ \mathcal{H}\ (af)^n$, where $\mathcal{H}$ is the Green's relation on K_α, so $af=(af)^nv$ for some $v\in S$. Thus

$$af=(af)^nv=a^mufv=ea^mufv=eaf.$$

Hence, the first part of (2) holds. In a similar way it can be proved the second part of (2). Therefore, (2) holds.

Define a mapping $\varphi: S_\alpha\to K_\alpha$ by:

$$\varphi(a) = ea \qquad \text{if } \ a \in T_e, \ e \in E(S_\alpha).$$

Let $a \in T_e$, $b \in T_f$, $e, f \in E(S_\alpha)$. Assume that $ab \in T_g$ for some $g \in E(S_\alpha)$. Then by (2) and by Munn's lemma we have that

$$\begin{aligned}\varphi(ab) &= gab = gaeb = gaebf = gabf = abf = afb = eafb \\ &= \varphi(a)\varphi(b)\ .\end{aligned}$$

Therefore, φ is a homomorphism and since $\varphi(a) = a$ if $a \in K_\alpha$, then φ is a retraction, so S is a semilattice of retractive nil-extensions of completely simple semigroups. □

Theorem 2. *Let S be a π-regular semigroup and let for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(ab)^n \in a^2Sa.$$

Then S is a semilattice of retractive nil-extensions of left groups.

Proof. By Theorem 2.2 [1] it follows that S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for every $\alpha \in Y$, S_α is a nil-extension of a left group K_α.

Let $\alpha \in Y$, let $a \in T_e \subseteq S_\alpha$, for some $e \in E(S_\alpha)$ and let $f \in E(S_\alpha)$. As in the proof of Theorem 1 we obtain that $af = eaf$. On the other hand, there exists $n \in \mathbf{Z}^+$ such that $(fa)^n \in Sf$, so $(fa)^n = (fa)^n f$. Since $fa\ \mathcal{H}\ (fa)^n$, where $\mathcal{H}$ is the Green's relation on K_α, then $fa = faf$, so $(fa)^n = fa^n$ for every $n \in \mathbf{Z}^+$. Assume that $n \in \mathbf{Z}^+$ is such that $a^n \in G_e$. Then $fa = u(fa)^n$ (since $fa\ \mathcal{H}\ (fa)^n$), whence

$$fa = u(fa)^n = ufa^n = ufa^n e = fae.$$

Therefore $fa = fae$, so as in the proof of Theorem 1 we prove that S_α is a retractive nil-extension of K_α. □

Theorem 3. *Let S be a band of π-groups and let $Reg(S)$ be a subsemigroup of S. Then $Reg(S)$ is a band of groups and a retract of S.*

Conversely, if S has a retract K which is a band of groups and $S = \sqrt{K}$, then S is a band of π-groups.

Proof. Let S be a band I of π-groups S_i, $i \in I$, and let $Reg(S)$ be a subsemigroup of S. For $i \in I$, let S_i be a nil-extension of a group G_i with the identity e_i. Then $Reg(S) = Gr(S) = \cup\{G_i \mid i \in I\}$. Then it is clear that $Reg(S)$ is a band I of groups G_i, $i \in I$. Define a mapping $\varphi : S \to Reg(S)$ by:

$$\varphi(x) = xe_i \qquad \text{if } \ x \in S_i, \ i \in I.$$

Let $x_i \in S_i$, $x_j \in S_j$, $i, j \in I$. Then $e_ie_{ij} = (e_ie_{ij})e_{ij} \in S_{ij}G_{ij} \subseteq G_{ij}$ and $e_{ij}e_j = e_{ij}(e_{ij}e_j) \in G_{ij}S_{ij} \subseteq G_{ij}$, so

$$(e_ie_{ij})^2 = e_i(e_{ij}(e_ie_{ij})) = e_i(e_ie_{ij}) = e_ie_{ij} \in S_{ij}\ ,$$
$$(e_{ij}e_j)^2 = ((e_{ij}e_j)e_{ij})e_j = (e_{ij}e_j)e_j = e_{ij}e_j \in S_{ij}\ .$$

Since S_{ij} contain exactly one idempotent e_{ij}, then

$$e_ie_{ij} = e_{ij}e_j = e_{ij}. \tag{4}$$

Now we obtain that

$$\begin{aligned}
\varphi(x_i)\varphi(x_j) &= (x_ie_i)(x_je_j) & \\
&= e_{ij}(x_ie_i)(x_je_j)e_{ij} & &(\text{since } x_ie_ix_je_j \in G_iG_j \subseteq G_{ij}) \\
&= e_{ij}e_ix_ix_je_je_{ij} & &(\text{by Munn's lemma}) \\
&= e_{ij}e_ix_ie_ix_je_{ij}e_je_{ij} & &(\text{since } e_{ij}e_ix_ix_j \in G_{ij}S_{ij} \subseteq G_{ij}) \\
&= e_{ij}e_ix_ix_je_{ij} & &(\text{by } (4)) \\
&= e_{ij}e_ie_{ij}x_ix_je_{ij} & &(\text{since } x_ix_je_{ij} \in S_{ij}G_{ij} \subseteq G_{ij}) \\
&= e_{ij}x_ix_je_{ij} & &(\text{by } (4)) \\
&= x_ix_je_{ij} & &(\text{since } x_ix_je_{ij} \in G_{ij}) \\
&= \varphi(x_ix_j)\ .
\end{aligned}$$

Therefore, φ is a homomorphism and since $\varphi(a) = a$ if $a \in Reg(S)$, then φ is a retraction of S onto $Reg(S)$.

Conversely, let S have a retract K which is a band I of groups G_i, $i \in I$ and let $\sqrt{K} = S$. Let $\varphi : S \to K$ be a retraction and let $S_i = \varphi^{-1}(G_i)$, $i \in I$. Then $S_i \cap K = G_i$ and $S_i = \sqrt{G_i}$ for every $i \in I$. Therefore, S_i is a π-group for all $i \in I$, so S is a band I of π-groups S_i, $i \in I$. □

Corollary 1. [7] (i) *A semigroup S is a retractive nil-extension of a completely simple semigroup if and only if S is a rectangular band of π-groups.*

(ii) *A semigroup S is a retractive nil-extension of a left (right) group if and only if S is a left (right) zero band of π-groups.*

Theorem 4. *The following conditions on a semigroup S are equivalent:*

(i) *S is π-regular and for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(ab)^n \in a^2bSab^2; \tag{5}$$

(ii) *S is completely π-regular and for all $a, b \in S$ is*

$$ab\,\tau\, a^2b\,\tau\, ab^2; \tag{6}$$

(iii) *S is a band of π-groups.*

Proof. $(i) \Rightarrow (ii)$. Let (i) hold. Then by Theorem 1 we have that S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, and for every $\alpha \in Y$, S_α is a retractive nil-extension of a completely simple semigroup K_α. Therefore,

S is completely π-regular and by Corollary 1 it follows that S_α is a rectangular band of π-groups for all $\alpha \in Y$.

Let $a, b \in S$. Then $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$ and $ab, a^2b, ab^2 \in S_{\alpha\beta}$. Moreover, $S_{\alpha\beta}$ is a rectangular band $I \times \Lambda$ of π-groups $T_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$. Let $ab \in T_{i\lambda}$, $a^2b \in T_{j\mu}$, $ab^2 \in T_{l\nu}$ for some $i, j, l \in I$, $\lambda, \mu, \nu \in \Lambda$. Let $e_{j\mu}$ be the idempotent of $T_{j\mu}$. Then $e_{j\mu}a^2b \in T_{j\mu}^2 \subseteq T_{j\mu}$ and

$$e_{j\mu}a^2b = e_{j\mu}e_{j\mu}aab \in T_{j\mu}S_{\alpha\beta}T_{i\lambda} \subseteq T_{j\lambda},$$

so $\mu = \lambda$. In a similar way it can be proved that $l = i$. Also, by (5) it follows that there exist $n \in \mathbf{Z}^+$ and $u \in S$ such that $(ab)^n = a^2buab^2$, whence $uab^2a^2bu \in S_{\alpha\beta}$, so

$$(ab)^{2n} = a^2b(uab^2a^2bu)ab^2 \in T_{j\lambda}S_{\alpha\beta}T_{i\nu} \subseteq T_{j\nu}.$$

Since $(ab)^{2n} \in T_{i\lambda}$, then $j = i$ and $\nu = \lambda$. Therefore, $ab, a^2b, ab^2 \in T_{i\lambda}$, so (6) holds.

$(ii) \Rightarrow (iii)$. Let (ii) hold and let $a, b \in S$. Assume that $a \in T_e$, $b \in T_f$ for some $e, f \in E(S)$. By (6) it follows that $ab\,\tau\,a^kb$ for every $k \in \mathbf{Z}^+$. Let $k \in \mathbf{Z}^+$ be such that $a^k \in G_e$. Then

$$eb = a^k(a^k)^{-1}b\,\tau\,(a^k)^2(a^k)^{-1}b = a^keb = a^kb\,\tau\,ab.$$

Thus, $ab\,\tau\,eb$. In a similar way it can be proved that $eb\,\tau\,ef$. Therefore, $ab\,\tau\,ef$, so τ is a congruence on S. It is clear that τ is a band congruence and every τ-class is a π-group. Hence, (iii) holds.

$(iii) \Rightarrow (i)$. Let S be a band I of π-groups S_i, $i \in I$. Let $a \in S_i$, $b \in S_j$, $i, j \in I$. Then $ab, a^2b, ab^2 \in S_{ij}$, so (i) holds. □

Remark. $(ii) \Leftrightarrow (iii)$ of the previous theorem is proved in [8], but here a new proof is given.

Theorem 5. *The following conditions on a semigroup S are equivalent:*

(i) S is π-regular and for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that

$$(ab)^n \in a^2bSa; \tag{7}$$

(ii) S is completely π-regular and for all $a, b \in S$

$$ab\,\tau\,a^2b\,\tau\,aba; \tag{8}$$

(iii) S is a left regular band of π-groups.

Proof. $(i) \Rightarrow (ii)$. Let (i) hold. Then by Theorem 2 it follows that S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for every $\alpha \in Y$, S_α is a retractive nil-extension of a left group K_α. By this and by Corollary

1 it follows that for all $\alpha \in Y$, S_α is a left zero band of π-groups. It is clear that S is completely π-regular.

Let $a, b \in S$. Then $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$ and $ab, a^2b, aba \in S_{\alpha\beta}$. Moreover, $S_{\alpha\beta}$ is a left zero band I of π-groups T_i, $i \in I$. Assume that $ab \in T_i$, $a^2b \in T_j$ and $aba \in T_k$ for some $i, j, k \in I$. Then $(aba)^2 \in T_k$ and

$$(aba)^2 = ab(a^2ba) \in T_i S_{\alpha\beta} \subseteq T_i,$$

so $k = i$. Thus, $ab\,\tau\,aba$. On the other hand, by (7) it follows that there exist $n \in \mathbf{Z}^+$ and $u \in S$ such that $(ab)^n = a^2bua$. Since $ua^2b \in S_{\alpha\beta}$, then

$$(ab)^{n+1} = a^2b(ua^2b) \in T_j S_{\alpha\beta} \subseteq T_j,$$

so $j = i$. Thus, $ab\,\tau\,a^2b$. Hence, (ii) holds.

$(ii) \Rightarrow (iii)$. Let (ii) hold and let $a, b \in S$. Assume that $a\,\tau\,e$ and $b\,\tau\,f$ for some $e, f \in E(S)$. As in the proof of Theorem 4 it can be proved that $ab\,\tau\,eb$, so τ is a right congruence. By this and by (8) it follows that $(aba)b\,\tau\,(ab)b$, whence

$$ab\,\tau\,(ab)^2 = (aba)b\,\tau\,(ab)b = ab^2.$$

Therefore, (6) holds, whence S is a band S/τ of π-groups. Let $a\tau, b\tau \in S/\tau$. Then by (8) it follows that $(a\tau)(b\tau) = (a\tau)(b\tau)(a\tau)$, so S/τ is a left regular band.

$(iii) \Rightarrow (i)$. Let S be a left regular band I of π-groups S_i, $i \in I$. Let $a, b \in S$. Then $a \in S_i$, $b \in S_j$ for some $i, j \in I$, whence $ab, a^2b \in S_{ij}$ and $aba \in S_{iji} = S_{ij}$. Since S_{ij} is a π-group, then it follows that (7) holds. Hence, (i) holds. $\square$

3. Normal Bands of t-Archimedean Semigroups

In [11] bands of t-Archimedean semigroups and various other (semilattice) band decompositions were also characterized. In this section we characterize normal bands of t-Archimedean semigroups and using this characterization we obtain some characterizations for normal bands of π-groups.

Theorem 6. *A semigroup S is a normal band of t-Archimedean semigroups if and only if for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(abc)^n \in acSac\,. \tag{9}$$

Proof. Let (9) hold and let $a \in S$, $x, y \in S^1$. Assume that $x, y \in S$ (in a similar way we can prove the cases with $x = 1$ or $y = 1$). Then by (9) it follows that there exists $n \in \mathbf{Z}^+$ such that

$$(xa^2y)^n = ((xa)ay)^n \in xaySxay.$$

Moreover, there exists $m \in \mathbf{Z}^+$ such that

$$(xay)^{2m} = ((xa)(yx)(ay))^m \in xa^2ySxa^2y.$$

Therefore, $xay \overset{t}{\sim} xa^2y$, so by Putcha's theorem we obtain that S is a band I of t-Archimedean semigroups S_i, $i \in I$. Let $i, j, k \in I$. Since I is a homomorphic image of S, then by (9) it follows that there exist $n \in \mathbf{Z}^+$ and $u \in I$ such that $ijk = (ijk)^n = ikuik$, whence $ijk = ikijkik$, so by Proposition II 3.10 [10] we obtain that I is a normal band.

Conversely, let S be a normal band I of t-Archimedean semigroups S_i, $i \in I$. Let $a, b, c \in S$. Then $a \in S_i$, $b \in S_j$, $c \in S_k$ for some $i, j, k \in I$, so

$$acabc \in S_{ikijk} = S_{ijk} \quad \text{and} \quad abcac \in S_{ijkik} = S_{ijk},$$

since I is a normal band. Since S_{ijk} is t-Archimedean, then there exists $m, n \in \mathbf{Z}^+$ such that

$$(abc)^n \in acabcSacabc \quad \text{and} \quad (abc)^m \in abcacSabcac,$$

whence $(abc)^{m+n} \in acSac$. $\square$

Theorem 7. *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a left normal band of t-Archimedean semigroups;*
(*ii*) *for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(abc)^n \in acSa;$$

(*iii*) *for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(abc)^n \in acSb.$$

Proof. (*i*) $\Rightarrow$ (*ii*). Let S be a left normal band I of t-Archimedean semigroups S_i, $i \in I$. Let $a, b, c \in S$. Then $a \in S_i$, $b \in S_j$, $c \in S_k$ for some $i, j, k \in I$, so $abc \in S_{ijk}$ and $acba \in S_{ijki} = S_{ijk}$, since I is a left normal band. Therefore, there exixts $n \in \mathbf{Z}^+$ such that

$$(abc)^n \in acbaSacba \subseteq acSa.$$

Hence, (*ii*) holds.

(*ii*) $\Rightarrow$ (*iii*). Let (*ii*) hold and let $a, b, c \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $(abc)^n \in acSa \subseteq acS$ and there exists $m \in \mathbf{Z}^+$ such that

$$(abc)^{2m} = ((ab)(ca)(bc))^m \in abbcSab \subseteq Sb.$$

so

$$(abc)^{2m+n} \in acSSb \subseteq acSb.$$

Therefore, (iii) holds.

$(iii) \Rightarrow (i)$. Let (iii) hold and let $a \in S$, $x, y \in S^1$. Assume that $x, y \in S$ (in a similar way we can prove the cases with $x = 1$ or $y = 1$). Then there exist $n, m \in \mathbf{Z}^+$ such that

$$(xay)^{2n} = ((xa)(yx)(ay))^n \in xaaySyx \subseteq xa^2yS,$$
$$(ayx)^{2m} = ((ay)(xa)(yx))^m \in ayyxSxa \subseteq Sxa,$$

whence

$$(xay)^{2n+2m+1} = (xay)^{2n}x(ayx)^{2m}ay \in xa^2ySxSxaay \subseteq xa^2ySxa^2y.$$

Moreover, there exist $k, t \in \mathbf{Z}^+$ such that

$$(xa^2y)^k = ((xa)ay)^k \in xayS,$$
$$(yxa^2)^t = (y(xa)a)^t \in yaSxa \subseteq Sxa,$$

so

$$(xa^2y)^{k+t+1} = (xa^2y)^k xa^2(yxa^2)^t y \in xaySxa^2Sxay \subseteq xaySxay.$$

Therefore, $xa^2y \overset{t}{\sim} xay$, so by Putcha's theorem it follows that S is a band I of t-Archimedean semigroups. Since I is a homomorphic image of S, then for $i, j, k \in I$ there exist $n \in \mathbf{Z}^+$ and $u \in I$ such that $ijk = (ijk)^n = ikuj$, whence $ijk = ikijkj$, so by Proposition II 3.13 [10] it follows that I is a left normal band. $\square$

Theorem 8. *The following conditions on a semigroup S are equivalent:*

(i) *S is π-regular and for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

(10) $$(abc)^n \in acSac;$$

(ii) *S is completely π-regular and for all $a, b, c, d \in S$*

(11) $$abcd\,\tau\,acbd;$$

(iii) *S is a normal band of π-groups.*

Proof. $(i) \Rightarrow (iii)$. Let (1) hold. Then by Theorem 6 it follows that S is a normal band I of t-Archimedean semigroups S_i, $i \in I$. Let $a \in Reg(S)$. Then $a = axa$ for some $x \in S$, so by (10) it follows that there exists $n \in \mathbf{Z}^+$ such that $ax = (axax)^n \in aaxSaax$, whence

$$a = axa \in a^2xSa^2xa \subseteq a^2Sa^2.$$

Therefore, $a \in Gr(S)$, so S is completely π-regular. It is easy to verify that S_i is completely π-regular for every $i \in I$, whence it follows that for every $i \in I$, S_i is a π-group.

$(iii) \Rightarrow (ii)$. This follows since $abca\,\tau\,acba$ implies $abcd\,\tau\,acbd$.

$(ii) \Rightarrow (i)$. Let (ii) hold. Then it is clear that S is π-regular. Let $a, b, c \in S$. Then by (11) we obtain that

$$(abc)^2 = ab(cab)c\,\tau\,a(cab)bc = acab^2c, \quad (abc)^2 = a(bca)bc\,\tau\,ab(bca)c = ab^2cac,$$

whence it follows that there exists $m, n \in \mathbf{Z}^+$ such that

$$(abc)^{2m} \in acS \qquad \text{and} \qquad (abc)^{2n} \in Sac,$$

so $(abc)^{2m+2n} \in acSac$. Hence, (i) holds. □

In a similar way it can be proved the following theorem:

Theorem 9. *The following conditions on a semigroup S are equivalent:*

(i) *S is π-regular and for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(abc)^n \in acSa;$$

(ii) *S is π-regular and for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(abc)^n \in acSb;$$

(iii) *S is completely π-regular and for all $a, b, c \in S$, $abc\,\tau\, acb$;*

(iv) *S is a left normal band of π-groups.*

4. Rédei's Bands of Nil-extensions of Groups

Rédei's bands of periodic π-groups are studied by the authors [6]. Here, we characterize Rédei's bands of π-groups in the general case.

Theorem 10. *The following conditions on a semigroup S are equivalent:*

(i) *S is a Rédei's band of π-groups;*

(ii) *S has a retract K that is a Rédei's band of groups and $\sqrt{K} = S$;*

(iii) *for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$a^n \in (ab)^n S (ab)^n \quad \text{or} \quad b^n \in (ab)^n S (ab)^n. \tag{12}$$

Proof. $(i) \Rightarrow (ii)$. Let S be a Rédei's band I of π-groups S_i, $i \in I$. For $i \in I$, let S_i be a nil-extension of a group G_i with the identity e_i. Then it is clear that $E(S) = \{e_i \mid i \in I\}$. Let $e_i, e_j \in E(S)$, $i, j \in I$. Then $e_i e_j \in S_{ij}$. If $ij = i$, then $e_i e_j \in S_i$, so $e_i e_j = e_i(e_i e_j) \in G_i S_i \subseteq G_i$, whence

$$(e_i e_j)^2 = ((e_i e_j)e_i)e_j = (e_i e_j)e_j = e_i e_j.$$

If $ij = j$, then in a similar way it can be proved that $(e_i e_j)^2 = e_i e_j$. Therefore, $E(S)$ is a subsemigroup of S, so by Proposition 1 [3] we obtain that $Reg(S)$ is a subsemigroup of S and by Theorem 3 it follows that (ii) holds.

$(ii) \Rightarrow (i)$. This follows by Theorem 3.

$(i) \Rightarrow (iii)$. Let S be a Rédei's band I of π-groups S_i, $i \in I$. For $i \in I$, let S_i be a nil-extension of a group G_i. Let $a, b \in S$. Then

$a \in S_i,\ b \in S_j$ for some $i, j \in I$. If $ij = i$, then $ab \in S_i$, so there exists $n \in \mathbf{Z}^+$ such that $(ab)^n, a^n \in G_i$, whence

$$a^n \in (ab)^n G_i (ab)^n \subseteq (ab)^n S (ab)^n.$$

If $ij = j$, then in a similar way it can be proved that

$$b^n \in (ab)^n S (ab)^n,$$

for some $n \in \mathbf{Z}^+$. Hence, (iii) holds.

$(iii) \Rightarrow (i)$. Let (iii) hold. Then it is clear that S is completely π-regular. Also, by (12) it follows that

$$e \in Sf \qquad \text{or} \qquad f \in eS,$$

for all $e, f \in E(S)$, so it is easy to verify that $E(S)$ is a Rédei's band. By Theorem 5.1 [4] it follows that S is a chain Y of $S_\alpha,\ \alpha \in Y$, and for every $\alpha \in Y$, S_α is a nil-extension of a left or a right group K_α.

Let $\alpha \in Y$ and let $a, b \in S_\alpha$. Assume that K_α is a left group. Let $a \in T_e,\ b \in T_f,\ e, f \in E(S_\alpha),\ e \neq f$. By (12) we obtain that there exists $n \in \mathbf{Z}^+$ such that

$$a^n \in (af)^n S (af)^n \qquad \text{or} \qquad f \in (af)^n S (af)^n.$$

Assume that $f \in (af)^n S (af)^n \subseteq afSaf$, i.e. $f = afuaf$ for some $u \in S$. Since $af \in S_\alpha K_\alpha \subseteq K_\alpha$, then $af \in G_g$ for some $g \in E(S_\alpha)$. Now, by Bogdanović-Milić's lemma we obtain that

$$f = afuaf = g(afuaf)g = gfg \in gS_\alpha g = G_g,$$

whence $f = g$, i.e. $af \in G_f$. Also, $fa = f(fa) \in G_f K_\alpha \subseteq G_f$, since K_α is a left group, so $af = f(af) = (fa)f = fa$. Since $a^k \in G_e$ for some $k \in \mathbf{Z}^+$ and since K_α is a left group, then

$$a^k = a^k e = a^k ef = a^k f = fa^k \in G_f G_e \subseteq G_f,$$

which is not possible. Therefore, $a^n \in (af)^n S (af)^n$, whence $a^n \in afS_\alpha af \subseteq afK_\alpha af$, so by Bogdanović-Milić's lemma we obtain that $a^n \mathcal{H} af$, where $\mathcal{H}$ is the Green's relation on K_α. Hence, $af \in G_e$. In a similar way it can be proved that $be \in G_f$, so by Munn's lemma it follows that

$$be = fbe = bfe = bf = fb \qquad \text{and} \qquad af = eaf = aef = ae = ea,$$

whence

$$abe = afb = eab.$$

Assume that $(ab)^m \in G_g$ for some $g \in E(S_\alpha),\ m \in \mathbf{Z}^+$. Then

$$(ab)^m e \in G_g G_e \subseteq G_g \qquad \text{and} \qquad (ab)^m e = e(ab)^m \in G_e G_g \subseteq G_e.$$

Thus, $g = e$, i.e. $(ab)^m \in G_e$, so $ab \in T_e = T_{ef}$. Hence, S_α is a left zero band $E(S_\alpha)$ of π-groups T_e, $e \in E(S_\alpha)$. If K_α is a right group, then in a similar way it can be proved that S_α is a right zero band of π-groups.

Let $a \in T_e \subseteq S_\alpha$, $b \in T_f \subseteq S_\beta$, $\alpha, \beta \in Y$, $\alpha \neq \beta$. Assume that $\alpha < \beta$, i.e. $\alpha\beta = \beta\alpha = \alpha$ (in a similar way we consider the case $\beta < \alpha$). Since $E(S)$ is a Rédei's band, $ef, fe, e \in S_\alpha$ and $f \notin S_\alpha$, then $ef = fe = e$. By (12) it follows that there exists $n \in \mathbf{Z}^+$ such that

$$b^n \in (be)^n S(be)^n \qquad \text{or} \qquad e \in (be)^n S(be)^n .$$

If $b^n = (be)^n u(be)^n$ for some $u \in S$, then $u \in S_\gamma$ for some $\gamma \in Y$, so $\alpha\beta\gamma = \beta$, whence $\alpha\beta = \beta$, which is not possible. Therefore, $e \in (be)^n S(be)^n$, whence

$$e \in beS_\alpha be.$$

Since $be = (be)e \in S_\alpha K_\alpha \subseteq K_\alpha$, then by Lemma 1 [5] it follows that $be \in G_e$. In a similar way it can be proved that $eb \in G_e$, so $eb = (eb)e = e(be) = be$ and $abe = aeb = eab$ (by Munn's lemma). Let $(ab)^m \in G_g$ for some $g \in E(S_\alpha)$ and $m \in \mathbf{Z}^+$. Then by Bogdanović-Milić's lemma we have that

$$\begin{aligned} (ab)^m &= (ab)^m g = (ab)^m geg = (ab)^m eg = e(ab)^m g \\ &= e(ab)^m = ee(ab)^m = e(ab)^m e \\ &\in eS_\alpha e \\ &= G_e \ . \end{aligned}$$

Therefore, $(ab)^m \in G_e$, i.e. $ab \in T_e = T_{ef}$. Hence, S is a Rédei's band $E(S)$ of π-groups T_e, $e \in E(S)$. $\square$

Corollary 2. *A semigroup S is a Rédei's band of groups if and only if $a \in abSab$ or $b \in abSab$, for all $a, b \in S$.*

Corollary 3. [6] *A semigroup S is a Rédei's band of periodic π-groups if and only if S is π-regular and for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in \langle a \rangle \cup \langle b \rangle$.*

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POLUGRUPE GALBIATI-VERONESI IV (TRAKE NIL-EKSTENZIJA GRUPA)

Stojan Bogdanović i Miroslav Ćirić

Ovaj rad je nastavak radova [1], [2] i [4]. Ovde razmatramo polugrupe koje su trake nil-ekstenzija grupa i neke posebne slučajeve takvih polugrupa.