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RETRACTIVE NIL-EXTENSIONS OF BANDS OF GROUPS*

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Abstract. The present paper is the continuation of [6,7] and it is devoted to study of retractive nil-extensions of bands of groups, in the general and in some more important particular cases. Especially, combining the methods from S. Bogdanović and M. Ćirić [6,7] and the ones of M. Petrich [12,13] and M. Ćirić and S. Bogdanović [10], retractive nil-extensions of bands of groups whose idempotents form a subsemigroup will be characterized via subdirect products.

1. Introduction

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. For a semigroup S, $Reg(S)$ ($Gr(S)$, $E(S)$) will denote the set of all regular (completely regular, idempotent) elements of S, for $a \in Reg(S)$, $V(a)$ will denote the set of all inverses of a, i.e. $V(a) = \{x \in S \mid a = axa,\ x = xax\}$, and for $a \in Gr(S)$, a^{-1} will denote the group inverse of a in the subgroup of S containing it. For a congruence ϱ of a semigroup S, $\varrho^\natural$ will denote the natural homomorphism determined by ϱ, and if the related factor is a semilattice of groups, then ϱ is a *semilattice-of-groups* congruence.

An element a of a semigroup S is π-*regular* if some its power is regular and it is *completely* π-*regular* if some its power is completely regular. It is well-known that for a completely π-regular element a of a semigroup S, all its completely regular powers lie in the same subgroup of S, and a^0 will denote the identity of this group and $\overline{a} = (aa^0)^{-1}$. Clearly, $a^0 = a\overline{a} = \overline{a}a$. A semigroup S is (*completely*) π-*regular* if each its element is (completely) π-regular. By a *nil-extension* of a semigroup we mean any its ideal extension by a nil-semigroup. A semigroup S is a π-*group* if it is a nil-extension of a group. A semigroup S is *completely Archimedean* if it is a nil-extension of

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a completely simple semigroup. A subsemigroup T of a semigroup S is a *retract* of S if there exists a homomorphism φ of S onto T such that $a\varphi = a$, for each $a \in T$, and such a homomorphism will be called a *retraction*. An ideal extension S of a semigroup T is a *retractive extension* of T if T is a retract of S.

Let B be a band. For $i \in B$, $[i]$ will denote the class of i with respect to the smallest semilattice congruence of B, and $\preccurlyeq$ will denote the quasi-order on B defined by: $j \preccurlyeq i \;\Leftrightarrow\; jij = j$ (or equivalently $[j] \leq [i]$), $(i, j \in B)$, where $\leq$ is the natural partial order on the greatest semilattice homomorphic image of B. To each $i \in B$ let we associate a semigroup S_i such that $S_i \cap S_j = \varnothing$ if $i \neq j$. Let $\varphi_{i,j}$ be homomorphisms of S_i onto S_j, defined for $i \succcurlyeq j$, such that $\varphi_{i,i}$ is the identity mapping on S_i, for each $i \in B$, and $\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k}$ whenever $i \succcurlyeq j \succcurlyeq k$. Define a multiplication $*$ on $S = \cup_{i\in B} S_i$ by: $a * b = (a\varphi_{i,ij})(b\varphi_{j,ij})$, for $a \in S_i,\ b \in S_j$. Then S is a band B of semigroups $S_i,\ i \in B$, in notation $S = [B; S_i, \varphi_{i,j}]$, called a *strong band of semigroups* $S_i,\ i \in B$, [10].

For undefined notions and notations we refer to [4] and [14].

2. Preliminary Results

In this section we will give several results that are needed in our further considerations. First we quote Lemma 3.1 [5], in the following slightly changed form:

Lemma 1. *Let S be a completely π-regular semigroup and let $Gr(S)$ be a subsemigroup of S. If φ is a retraction of S onto $Gr(S)$, then $a\varphi = aa^0$, for each $a \in S$.*

The following theorem, which is a generalization of Theorem 3.1 [13], will be very useful in the proofs of the main theorems of this paper.

Theorem 1. *Let a semigroup S be a band of π-groups. Then $E(S)$ is a subsemigroup of S if and only if a relation η on S defined by*

$$a\,\eta\,b \;\Leftrightarrow\; aa^0 = a^0bb^0a^0,\quad bb^0 = b^0aa^0b^0 \qquad (a, b \in S), \tag{1}$$

is a semilattice-of-groups congruence on S. In this case it is the smallest semilattice-of-groups congruence on S.

Proof. Assume that S is a band B of semigroups $S_i,\ i \in B$, and for $i \in B$, let S_i be a nil-extension of a group G_i with the identity e_i. Let $T = Reg(S)$. Clearly, $T = Gr(S) = \cup_{i\in B} G_i$.

Let $E(S)$ be a subsemigroup of S. Then T is also a subsemigroup of S, so by Theorem 3 [8], T is a retract of S and it is a band B of groups

$G_i,\ i \in B$. By Theorem 2 [16], $T = [B; G_i, \varphi_{i,j}]$ (see also Lemma 2 [10]). As it was noted in [10], $\varphi_{i,j}$ are uniquely determined by $a\varphi_{i,j} = e_j a e_j$, for $a \in S_i,\ i,j \in B,\ i \succcurlyeq j$. Let φ be a retraction of S onto T. By Lemma 1, $a\varphi = aa^0$, for each $a \in S$. Now, if for $i,j \in B,\ i \succcurlyeq j$, we define a homomorphism $\phi_{i,j}$ of S_i into G_j by $a\phi_{i,j} = (a\varphi)\varphi_{i,j},\ (a \in S_i)$, then

$$a\,\eta\,b \;\Leftrightarrow\; a \in S_i,\ b \in S_j,\ [i] = [j],\ a\phi_{i,i} = b\phi_{j,i} \text{ and } b\phi_{j,j} = a\phi_{i,j}.$$

Clearly, η is reflexive and symmetric. Let $a\,\eta\,b,\ b\,\eta\,c,\ a \in S_i,\ b \in S_j,\ c \in S_k,\ i,j,k \in B$. Then $a\phi_{i,k} = a\varphi\varphi_{i,k} = a\phi_{i,i}\varphi_{i,k} = b\phi_{j,i}\varphi_{i,k} = b\varphi\varphi_{j,i}\varphi_{i,k} = b\varphi\varphi_{j,k} = b\phi_{j,k} = c\phi_{k,k}$. Thus, $a\phi_{i,k} = c\phi_{k,k}$. Similarly we obtain that $c\phi_{k,i} = a\phi_{i,i}$. Therefore, η is transitive.

Assume $a,b,x \in S,\ a\,\eta\,b$. Let $a \in S_i,\ b \in S_j,\ x \in S_k,\ i,j,k \in B$. Then $ax \in S_{ik},\ bx \in S_{jk},\ [ik] = [i][k] = [j][k] = [jk]$ and

$$\begin{aligned}(ax)\phi_{ik,jk} &= (ax)\varphi\varphi_{ik,jk} = [(a\varphi)(x\varphi)]\varphi_{ik,jk}\\ &= [(a\varphi\varphi_{i,ik})(x\varphi\varphi_{k,ik})]\varphi_{ik,jk} = (a\varphi\varphi_{i,jk})(x\varphi\varphi_{k,jk})\\ &= (a\varphi\varphi_{i,j}\varphi_{j,jk})(x\phi_{k,jk}) = (a\phi_{i,j}\varphi_{j,jk})(x\phi_{k,jk})\\ &= (b\phi_{j,j}\varphi_{j,jk})(x\phi_{k,jk}) = (b\phi_{j,jk})(x\phi_{k,jk}) = (bx)\phi_{jk,jk}\end{aligned}$$

Similarly we prove that $(ax)\phi_{ik,ik} = (bx)\phi_{jk,ik}$. Thus, $ax\,\eta\,bx$, and similarly, $xa\,\eta\,xb$. Hence, η is a congruence on S.

Let $Q = S/\eta$ and let $u \in Q$. Then $u = a\eta^\natural$, for some $a \in S$, whence $u = (a\varphi)\eta^\natural$, since $(a, a\varphi) \in \eta$, so u is completely regular. Therefore, Q is a union of groups. Assume $p,q \in E(Q)$. By Corollary 2 [2], $p = e\eta^\natural,\ q = f\eta^\natural$, for some $e,f \in E(S)$. If $e \in S_i,\ f \in S_j,\ i,j \in B$, then $ef \in S_{ij},\ fe \in S_{ji},\ [ij] = [ji]$, so $ef\,\eta\,fe$, whence $pq = qp$. Hence, Q is a semilattice of groups.

Conversely, let η be a semilattice-of-groups congruence on S. Let $Q = S/\eta$ be a semilattice Y of groups $G_\alpha,\ \alpha \in Y$ and for $\alpha \in Y$, let e_α be an identity of G_α. Assume $e,f \in E(S)$. Then $e\eta^\natural = e_\alpha,\ f\eta^\natural = e_\beta$, for some $\alpha,\beta \in Y$, whence $(ef)\eta^\natural = e_{\alpha\beta} = (fe)\eta^\natural$. Therefore, $ef\,\eta\,fe$, so by (1)

$$ef = (ef)^0 fe (ef)^0 = \overline{ef}\ ef feef\ \overline{ef} = \overline{ef}\ efef\ \overline{ef} = \big((ef)^0\big)^2 = (ef)^0.$$

Hence, $ef \in E(S)$, i.e. $E(S)$ is a subsemigroup of S.

Finally, let η be a semilattice-of-groups congruence on S, let μ be an arbitrary semilattice-of-groups congruence on S, let S/μ be a semilattice Y of groups $G_\alpha,\ \alpha \in Y$, for $\alpha \in Y$, let e_α be an identity of G_α, and let $(a,b) \in \eta$. i.e. let $aa^0 = a^0bb^0a^0$ and $bb^0 = b^0aa^0b^0$. Then $a\mu^\natural \in G_\alpha,\ b\mu^\natural \in G_\beta$, for some $\alpha,\beta \in Y$, and it is easy to verify that $a^0\mu^\natural = e_\alpha,\ b^0\mu^\natural = e_\beta$, whence $a\mu^\natural = (aa^0)\mu^\natural = (a^0bb^0a^0)\mu^\natural e_\alpha (b\mu^\natural) e_\beta e_\alpha \in G_{\alpha\beta}$, so $\alpha\beta = \alpha$, and similarly

$\alpha\beta = \beta$. Thus, $\alpha = \beta$, whence $a\mu^\natural = e_\alpha(b\mu^\natural)e_\alpha = b\mu^\natural$, i.e. $(a,b) \in \mu$. Therefore, $\eta \subseteq \mu$, so η is the smallest semilattice-of-groups congruence on S. □

Remark 1. Note that if a semigroup S is a band of π-groups, with ξ as the related band congruence, if $E(S)$ is a subsemigroup of S, η is a relation on S defined by (1) and if φ is a retraction of S onto $Reg(S)$, then $\xi \cap \eta = \ker\varphi$.

It is easy to prove the following

Lemma 2. *A completely π-regular semigroup S is a band of π-groups if and only if $(ab)^0 = a^0b^0$, for all $a, b \in S$.*

Lemma 3. *Let S be a π-regular semigroup. Then for every regular element of S, each its inverse is a group inverse if and only if S is a semilattice of π-groups.*

Proof. This follows by Theorem 3.2 [3]. □

Corollary 1. *Let S be a regular semigroup. Then for every element of S, each its inverse is a group inverse if and only if S is a semilattice of groups.*

Lemma 4. *Let a semigroup S be a subdirect product of semilattices of groups. Then the following conditions are equivalent:*

(i) *S is π-regular;*
(ii) *S is regular;*
(iii) *S is completely regular;*
(iv) *S is a semilattice of groups.*

Proof. Let $S \subseteq \prod_{i\in I} S_i$ be a subdirect product of semigroups S_i, $i \in I$, that are semilattices of groups.

(iv) ⇒ (iii) ⇒ (ii) ⇒ (i). This follows immediately.

(i) ⇒ (iii). Assume $a = (a_i) \in S$. Then $a^n \in Reg(S)$, for some $n \in \mathbf{Z}^+$. Assume $x = (x_i) \in V(a^n)$. Then for each $i \in I$, $x_i \in V(a_i^n)$, so by Corollary 1, x_i is a group inverse of a_i^n in some subgroup G of S_i. Now, $y_i = a_i^{n-2}x_ia_i$ is a group inverse of a_i in G, for $y = (y_i)$, $y = a^{n-2}xa \in S$ and it is a group inverse of a. Therefore, S is completely regular.

(iii) ⇒ (iv). This follows by the fact that the idempotents of S commutes. □

3. The Main Theorems

In [8] the authors proved that a π-regular semigroup is a band of π-groups and $Reg(S)$ is a subsemigroup of S if and only if $Reg(S)$ is a band of groups and a retract of S. Here we apply this result to describe retractive nil-extensions of bands of groups. Combining the results from [8] and [7] we go to the main theorem of this paper.

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a band of groups;*

(ii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that*

(2) $$xa^n y \in x^2 aSay^2;$$

(iii) *S is a band of π-groups and $Reg(S)$ is an ideal of S.*

Proof. (i) $\Rightarrow$ (ii). Let S be a retractive nil-extension of a semigroup K that is a band of groups, let ξ be the related band congruence on K and let φ be the related retraction of S onto K. Clearly, S is π-regular. Assume $x, a, y \in S$. Then $a^n \in K$, for some $n \in \mathbf{Z}^+$, so $xa^n y, x^2a^n y^2 \in K$ and

$$xa^n y = (xa^n y)\varphi = (x\varphi)a^n(y\varphi)\,\xi\,(x\varphi)^2 a^n (y\varphi)^2 = x^2 a^n y^2.$$

Therefore, $xa^n y, x^2a^n y^2 \in G$, where G is a subgroup of K, whence $xa^n y \in x^2a^n y^2 G x^2 a^n y^2 \subseteq x^2aSay^2$.

(ii) $\Rightarrow$ (iii). By Theorem 1 [7], $Reg(S)$ is an ideal of S. Further, for $a, b \in S$, by (2), there exists $n \in \mathbf{Z}^+$ such that $(ab)^{n+1} = a(ba)^n b \in a^2baSbab^2$, so by Theorem 4 [8], S is a band of π-groups.

(iii) $\Rightarrow$ (i). By Theorem 3 [8], $Reg(S)$ is a band of groups and a retract of S, whence we obtain (i). □

As we noted above, retractive nil-extensions of bands of groups whose idempotents form a subsemigroup will be characterized via subdirect products.

Theorem 3. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a band of groups and $E(S)$ is a subsemigroup of S;*

(ii) *S is π-regular and a subdirect product of a nil-semigroup, a band and a semilattice of groups;*

(iii) *S is π-regular and a subdirect product of a nil-semigroup, a band and of groups with a zero possibly adjoined.*

Proof. (i) $\Rightarrow$ (ii). Let S be a retractive nil-extension of a semigroup K that is a band of groups and let $E(S)$ ($= E(K)$) be a subsemigroup. By Theorem 2, S is a band of π-groups, and by this and by Theorem 1, a relation η on S, defined by (1), is a semilattice-of-groups congruence on S. Let ξ denote the related band congruence on S whose classes are π-groups and let ϱ denote the Rees congruence on S induced by K. Assume $(a,b) \in \varrho \cap \xi \cap \eta$. Clearly, if $a, b \notin K$, then $a = b$. Let $a, b \in K$. By $(a,b) \in \xi$, $a^0 = b^0$, so by $(a,b) \in \eta$, i.e. by (1), $a = aa^0 = a^0bb^0a^0 = bb^0 = b$. Therefore, $\varrho \cap \xi \cap \eta$ is the identity relation on S, so S is a subdirect product of S/ϱ, S/ξ and S/η, i.e. of a nil-semigroup, a band and a semilattice of groups.

(ii) ⇒ (i). Let S be a π-regular semigroup and let $S \subseteq N \times B \times T$ be a subdirect product of N, B and T, where N is a nil-semigroup, B is a band and T is a semilattice of groups. Let $K = (\{0\} \times B \times T) \cap S$. For $a = (u,i,p) \in Reg(S)$, $x = (v,j,q) \in V(a)$ we obtain $v \in V(u)$, $j \in V(i)$, $q \in V(p)$, whence $u = v = 0$, so $Reg(S) \subseteq K$. Assume $a = (0,i,p) \in K$. Then $a^n \in Reg(S)$, for some $n \in \mathbf{Z}^+$, and for $x = (0,j,q) \in V(a^n)$, $j \in V(i)$ and $q \in V(p^n)$. By Corollary 1, $q = (p^n)^{-1}$, $p^{n-1}qp = p^0 = p^{n-2}qp^2$ and $p^{n-1}qp^2 = p$. Now, $y = a^{n-2}xa \in S$ and

$$\begin{aligned} aya &= a^{n-1}xa^2 = (0,i,p^{n-1}) \cdot (0,j,q) \cdot (0,i,p^2) \\ &= (0,iji,p^{n-1}qp^2) = (0,i,p) = a, \\ ay &= a^{n-1}xa = (0,i,p^{n-1}) \cdot (0,j,q) \cdot (0,i,p) \\ &= (0,iji,p^{n-1}qp) = (0,i,p^0) = (0,iji,p^{n-2}qp^2) \\ &= (0,i,p^{n-2}) \cdot (0,j,q) \cdot (0,i,p^2) = a^{n-2}xa^2 = ya. \end{aligned}$$

Thus, $K = Gr(S) = Reg(S)$, and clearly, it is an ideal of S.

Further, if $a = (u,i,p) \in S$, then $u^n = 0$, for some $n \in \mathbf{Z}^+$, so

$$a^0 = (a^n)^0 = (0,i,p^n)^0 = (0,i,(p^n)^0) = (0,i,p^0).$$

Since T is a semilattice of groups, then $(pq)^0 = p^0q^0$, for all $p, q \in T$, whence $(ab)^0 = a^0b^0$, for all $a, b \in S$, so by Lemma 2, S is band of π-groups. Now, by Theorem 2, S is a retractive nil-extension of a band of groups. Clearly, $E(S)$ is a subsemigroup of S.

(ii) ⇒ (iii). This follows by Corollary 2.3 [13].

(iii) ⇒ (ii). This follows by Lemma 4. □

Since the idempotents of a left regular band of groups always form a subsemigroup, then we immediately obtain the following

Corollary 2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a left regular band of groups;*

(ii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in x^2aSx$.*

(iii) *S is π-regular and a subdirect product of a nil-semigroup, a left regular band and a semilattice of groups;*

(iv) *S is π-regular and a subdirect product of a nil-semigroup, a left regular band, and of groups with a zero possibly adjoined.*

Corollary 3. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of groups;*

(ii) *S is a retractive nil-extension of a semilattice of groups;*

(iii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^n y \in ySx$.*

(iv) *S is π-regular and a subdirect product of a nil-semigroup and a semilattice of groups;*

(v) *S is π-regular and a subdirect product of a nil-semigroup and of groups with a zero possibly adjoined.*

4. Retractive Nil-Extensions of Normal Bands of Groups

In this section we will describe retractive nil-extensions of normal bands of groups, as very important particular types of the above considered semigroups.

Theorem 4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a normal band of groups;*

(ii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$xa^n y \in xyaSxy. \tag{3}$$

(iii) *S is a semilattice of completely Archimedean semigroups and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$xa^n y \in xySxy. \tag{4}$$

(iv) *S is completely π-regular and a subdirect product of a nil-semigroup and of completely simple semigroups with a zero possibly adjoined.*

Proof. (i) $\Rightarrow$ (ii). This can be proved similarly as the related part of Theorem 2.

(ii) $\Rightarrow$ (iii). For $a, b \in S$, there exists $n \in \mathbf{Z}^+$ such that $(ab)^{n+1} = a(ba)^n b \in abbaSab \subseteq Sb^2S$, so by Theorem 1 [11], S is a semilattice of Archimedean semigroups. Further, for $a \in S$, $a^n = a^n x a^n$, for some $n \in \mathbf{Z}^+$, $x \in S$, whence $a^n = (a^n x)(a^n x)a^n \in (a^n x)a^n(a^n x)S(a^n x)a^n \subseteq a^{2n}Sa^n$, by (3). Thus, by Proposition 3.2 [3], S is completely π-regular. Finally, by Theorem 2.13 [15], S is a semilattice of completely Archimedean semigroups.

(iii) $\Rightarrow$ (i). By Theorem [16], $Reg(S) = Gr(S)$. For $x \in S$, $e \in E(S)$, by (4), $xe \in xeSxe$ and $ex \in exSex$, so $K = Reg(S)$ is an ideal of S. Thus, S is a nil-extension of a semigroup K that is a union of groups.

Assume $x \in S$, $e \in E(S)$. We will prove that

$$(xe)^m \in x^m eS, \qquad \text{for each } m \in \mathbf{Z}^+, \tag{5}$$

Assume $m \in \mathbf{Z}^+$ such that $(xe)^m = x^m eu$, for some $u \in S$. Then by (4),

$$(xe)^{m+1} = xe(xe)^m = xe(x^m eu) \in x(x^m eu)Sx(x^m eu) \subseteq x^{m+1}eS.$$

Now by induction we obtain (5). On the other hand, since K is completely regular, $xe = (xe)^2 v$, for some $v \in S$, whence $xe = (xe)^{m+1}v^m$, for any $m \in \mathbf{Z}^+$, so by (5), $xe = (xe)^{m+1}v^m e \in x^m eSxev^m e \subseteq x^m Se$. Hence, $xe \in x^m Se$, for each $m \in \mathbf{Z}^+$, and similarly we obtain that $ex \in eSx^m$, for each $m \in \mathbf{Z}^+$. Now as in the proof of Theorem 1 [7] we obtain that K is a retract of S.

Further, K is a semilattice Y of completely simple semigroups $K_\alpha,\ \alpha \in Y$, and for $a, b \in K$, ab, a^2b, $ab^2 \in K_\alpha$, for some $\alpha \in Y$. By (4), there exists $n \in \mathbf{Z}^+$, $u \in S$, such that $a(ab)^n b = abuab$. Without loss of generality we can assume that $u \in K_\alpha$. Let G be the maximal subgroup of K_α containing ab. Since K_α is completely simple, G is a bi-ideal of S, so $a(ab)^n b \in G$. Therefore, $ab \in a(ab)^n bGa(ab)^n b \subseteq a^2bKab^2$, so by Theorem 4 [8] and by the regularity of K, K is a band of groups. Let B be the related band homomorphic image of S. Then B satisfies (4), so it is easy to check that it satisfies the identities $xyz = xzxyz$ and $xyz = xyzxz$. Now by Proposition II.3.10 [14], B is a normal band.

(i) $\Rightarrow$ (iv). This follows by Theorem 1 [6] and Theorem 4.1 [12].

(iv) $\Rightarrow$ (i). By the transitivity of subdirect products, S is a subdirect product of a nil-semigroup and of a semigroup $T \subseteq \prod_{i\in I} T_i$ that is a subdirect product of semigroups T_i, $i \in I$, where T_i are completely simple semigroups with a zero possibly adjoined. Since T is a homomorphic image of S, then it is completely π-regular.

Assume $a = (a_i) \in T$. Then $a^n \in Gr(T)$, for some $n \in \mathbf{Z}^+$. Let x be the group inverse of a^n and $x = (x_i)$. It is not hard to show that for each $i \in I$, x_i is the group inverse of a_i^n. For each $i \in I$, T_i is a union of groups, so $a_i^{n-2}x_i a_i$ is the group inverse of a_i in T_i. Thus, $y = a^{n-2}xa \in T$ and it is a group inverse of a in T, so T is completely regular. Now by Theorem 4.1 [12] and by Theorem 1 [7] we obtain (i). □

Remark 2. It is easy to verify that the condition (3) can be replaced by each condition of the form $xa^n y \in xyuSvxy$, where u and v are any words from the free monoid over an alphabet $\{x, a, y\}$ such that one of u and v is a non-empty word.

Similarly we prove the following two corollaries

Corollary 4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a normal band of groups and $E(S)$ is a subsemigroup of S;*

(ii) *S is π-regular and a subdirect product of a nil-semigroup, a normal band and a semilattice of groups;*
(iii) *S is π-regular and a subdirect product of a nil-semigroup, a normal band and of groups with a zero possibly adjoined;*
(iv) *S is completely π-regular and a subdirect product of a nil-semigroup and of rectangular groups with a zero possibly adjoined.*

Corollary 5. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a left normal band of groups;*
(ii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^n y \in xySx$.*
(iii) *S is π-regular and a subdirect product of a nil-semigroup, a left normal band and a semilattice of groups;*
(iv) *S is completely π-regular and a subdirect product of a nil-semigroup, a left normal band and of groups with a zero possibly adjoined;*
(v) *S is completely π-regular and a subdirect product of a nil-semigroup and of left groups with a zero possibly adjoined.*

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RETRAKTIVNE NIL-EKSTENZIJE TRAKA GRUPA

Stojan Bogdanović i Miroslav Ćirić

Ovaj rad je nastavak radova [6,7] i posvećen je proučavanju retraktivnih nil-ekstenzija traka grupa, u opštem i nekim važnijim specijalnim slučajevima. Posebno, kombinujući metode iz radova S. Bogdanovića i M. Ćirića [6,7] i one M. Petricha [12,13] i M. Ćirića i S. Bogdanovića [10], dajemo karakterizaciju retraktivnih nil-ekstenzija traka grupa čiji idempotenti čine podpolugrupu, preko poddirektnih proizvoda.