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THE LATTICE OF POSITIVE QUASI-ORDERS ON A SEMIGROUP II*

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Abstract. Positive quasi-orders on semigroups have been studied from different points of view by B. M. Schein, T. Tamura, M. S. Putcha and the authors. T. Tamura [11] related them to the study of semilattice decompositions of semigroups. Yet other aspects of the role of positive quasi-orders in semilattice decompositions of semigroups were studied by the authors in [5] and [7]. In this paper we investigate some properties of positive quasi-orders on semigroups with zero. The obtained results will be a very useful tool in studying of quasi-semilattice decompositions of semigroups with zero, which will be introduced and studied in the next paper of the authors [8].

1. Introduction

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers, and S^1 will denote a semigroup S with an identity possibly adjoined. Further, $S = S^0$ means that S is a semigroup with zero 0. If $S = S^0$, we will write 0 instead $\{0\}$, and if A is a subset of S, then $A^\bullet = A - 0$, $A^0 = A \cup 0$ and $A' = (S - A)^0$.

A subset A of a semigroup S is: *consistent*, if for $x, y \in S$, $xy \in A$ implies $x, y \in A$, *completely semiprime*, if for $x \in S$, $n \in \mathbf{Z}^+$, $x^n \in A$ implies $x \in A$, and it is *completely prime*, if for $x, y \in S$, $xy \in A$ implies $x \in A$ or $y \in A$. For semigroups with zero we introduce also the following more general notions: a subset A of a semigroup $S = S^0$ is: *0-consistent*, if $A^\bullet$ is consistent, *completely 0-semiprime*, if $A^\bullet$ is completely semiprime, *completely 0-prime*, if $A^\bullet$ is completely prime, and A is a *0-filter*, if it is a 0-consistent subsemigroup of S.

If ξ is a binary relation on a set A, ξ^{-1} will denote the relation defined by: $a\,\xi^{-1}\,b \;\Leftrightarrow\; b\,\xi\,a$, for $a \in A$, $a\xi = \{x \in A \mid a\,\xi\,x\}$, $\xi a = \{x \in A \mid x\,\xi\,a\}$, for

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$X \subseteq A$, $X\xi = \bigcup_{x\in X} x\xi$, $\xi X = \bigcup_{x\in X} \xi x$, and ξ^∞ will denote the transitive closure of ξ. By a *quasi-order* we mean a reflexive and transitive binary relation. The poset of quasi-orders on a set A is a complete lattice and it will be denoted by $\mathcal{Q}(A)$. For a quasi-order ξ on a set A, $\widetilde{\xi}$ will denote the *natural equivalence* of ξ, i.e. an equivalence relation on A defined by: $\widetilde{\xi} = \xi \cap \xi^{-1}$.

A binary relation ξ on a semigroup S is: *positive*, if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$, *lower-potent*, if $a^n\,\xi\,a$, for all $a \in S$, $n \in \mathbf{Z}^+$, and it satisfies the *cm-property*, if for $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$, [11]. For semigroups with zero we introduce the following, more general notions: a binary relation ξ on a semigroup $S = S^0$ is: *0-positive*, if for $a, b \in S$, $ab \neq 0$ implies $a\,\xi\,ab$ and $b\,\xi\,ab$, *0-lower-potent*, if for $a \in S$, $n \in \mathbf{Z}^+$, $a^n \neq 0$ implies $a^n\,\xi\,a$, and it satisfies the *0-cm-property*, if for $a, b, c \in S$, $ab \neq 0$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$.

A mapping φ of a poset P into a poset Q is *isotone* (*antitone*) if for $x, y \in P$, $x \leq y$ implies $x\varphi \leq y\varphi$ ($x \leq y$ implies $y\varphi \leq x\varphi$), and φ is an *order isomorphism* (*dual order isomorphism*) if it is an isotone (antitone) bijection with isotone (antitone) inverse. Note that a poset isomorphic or dually isomorphic to a (complete) lattice is also a (complete) lattice, and by Lemma II 3.2 [1] and its dual, any (dual) order isomorphism between lattices is a (dual) lattice isomorphism. A mapping φ of a lattice L into itself is: *extensive*, if $x \leq x\varphi$, for any $x \in L$, *contractive*, if $x\varphi \leq x$, for any $x \in L$, and *idempotent*, if $(x\varphi)\varphi = x\varphi$, for any $x \in L$. An extensive, idempotent and isotone mapping of a lattice L into itself will be called a *closure operation* on L, and elements $x \in L$ for which $x\varphi = x$ will be called *closed elements* of L (with respect to φ). Similarly, a contractive, idempotent and isotone mapping of a lattice L into itself will be called an *interior operation* on L, and elements $x \in L$ for which $x\varphi = x$ will be called *open elements* of L (with respect to φ).

Let L be a complete lattice with the zero 0 and the unity 1. A subset K of L is *closed for meets* (*closed for joins*) if it contains the meet (the join) of any its non-empty subset, and it is a *closed subset* of L if it is closed both for meets and joins. Clearly, any closed subset of L is its complete sublattice. A sublattice K of L is a *1-sublattice* (*0-sublattice*) of L if $1 \in K$ ($0 \in K$), and it is a *0,1-sublattice* of L if $0, 1 \in K$. A subset A of a lattice L is *meet-dense* in L if any element of L can be represented as a meet of some subset of A.

By $\mathcal{I}d(S)$ we will denote the lattice of all ideals of a semigroup S. For a semigroup with zero, it is a complete lattice. Let K be a subset of $\mathcal{I}d(S)$ closed for meets, containing the unity of $\mathcal{I}d(S)$. Then for any $a \in S$, there exists a smallest element of K containing a, in notation $K(a)$, called the *principal element* of K generated by a. For a semigroup $S = S^0$, the set of completely 0-semiprime ideals is a complete 0,1-sublalttice of $\mathcal{I}d(S)$, and it will be denoted by $\mathcal{I}d^{\mathbf{c0s}}(S)$. For a sublattice K of $\mathcal{I}d^{\mathbf{c0s}}(S)$ we say that it satisfies the *c-0-pi-property* (*completely 0-prime ideal-property*) if the set of completely 0-prime ideals from K is meet-dense in K.

For undefined notions and notations we refer to [1], [2], [9] and [10].

2. Restriction Operations on Positive Quasi-Orders

In studying of semigroups with zero, it is often of interest to use relations and subsets with some "restrictions" and "weakenings" on the zero. On a relation ξ on a semigroup $S = S^0$ we define the following restriction operations: ${}^\bullet\xi = \xi - (0 \times S^\bullet)$, $\xi^\bullet = \xi - (S^\bullet \times 0)$, ${}^\bullet\xi^\bullet = \xi - (0 \times S^\bullet \cup S^\bullet \times 0)$. Clearly, ${}^\bullet\xi^\bullet = ({}^\bullet\xi)^\bullet = {}^\bullet(\xi^\bullet)$. Especially, $\gamma_l = {}^\bullet\omega$, $\gamma_r = \omega^\bullet$ and $\gamma = {}^\bullet\omega^\bullet$, where ω denotes the universal relation on S. A relation ξ on a semigroup $S = S^0$ is *left 0-restricted* (*right 0-restricted*) if $0\xi = 0$ ($\xi 0 = 0$), and it is *0-restricted* if it is both left and right 0-restricted, i.e. if $0\xi = \xi 0 = 0$.

The following lemma, that can be proved immediately, gives some basic properties of operations defined above:

Lemma 1. *For any semigroup $S = S^0$, the mappings $\xi \mapsto {}^\bullet\xi$, $\xi \mapsto \xi^\bullet$ and $\xi \mapsto {}^\bullet\xi^\bullet$ are interior operators on the lattice $\mathcal{Q}(S)$ and the related sets of open elements are sets of left 0-restricted, of right 0-restricted and of 0-restricted quasi-orders on S, respectively.*

A characterization of restricted quasi-orders is given by the following:

Lemma 2. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is left 0-restricted (right 0-restricted, 0-restricted);*
(ii) *for each $a \in S^\bullet$, $0 \notin \xi a$ ($0 \notin a\xi$, $0 \notin \xi a \cup a\xi$);*
(iii) *$\xi \subseteq \gamma_l$ ($\xi \subseteq \gamma_r$, $\xi \subseteq \gamma$).*

Proof. It is enough to prove the one part of the lemma that characterizes left 0-restricted quasi-orders.

(i) $\Leftrightarrow$ (ii). This follows immediately.

(i) $\Rightarrow$ (iii). If $(a, b) \in \xi$, then $(a, b) \notin 0 \times S^\bullet$, whence $(a, b) \in \gamma_l$.

(iii) $\Rightarrow$ (i). By Lemma 1, γ_l is left 0-restricted, so $0\xi \subseteq 0\gamma_l = 0$. $\square$

Now we obtain the following:

Theorem 1. *The sets of left 0-restricted quasi-orders, right 0-restricted quasi-orders and 0-restricted quasi-orders on a semigroup $S = S^0$ are the principal ideals of $\mathcal{Q}(S)$ generated by γ_l, γ_r and γ, respectively.*

Proof. This follows by Lemma 2. □

On a relation ξ on a semigroup $S = S^0$ we also define the operation $\xi^0 = \xi \cup S^\bullet \times 0$. It is easy to prove the following two lemmas that give important properties of this operation.

Lemma 3. *For a relation ξ on a semigroup $S = S^0$, $a\xi^0 = (a\xi)^0$, for each $a \in S$, and $\xi^0 a = \xi a$, for each $a \in S^\bullet$.*

Lemma 4. *For any semigroup $S = S^0$, the mapping $\xi \mapsto \xi^0$ is a closure operation on the lattice of left 0-restricted quasi-orders on S, and $\widetilde{\xi} = \widetilde{\xi^0}$, for any left 0-restricted quasi-order ξ on S.*

Further we will consider restriction operations on positive quasi-orders.

Lemma 5. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is positive;*
(ii) *${}^\bullet\xi$ is positive;*
(iii) *$\xi^\bullet$ is 0-positive.*

Proof. (i) ⇒ (ii). For $a, b \in S$, $(a, ab), (b, ab) \in \xi$, and clearly $(a, ab), (b, ab) \notin 0 \times S^\bullet$, so $(a, ab), (b, ab) \in {}^\bullet\xi$.

(ii) ⇒ (i). This follows by the proof of Theorem 1 [7], since ${}^\bullet\xi \subseteq \xi$.

(i) ⇒ (iii). For $a, b \in S$, $ab \neq 0$, $(a, ab), (b, ab) \in \xi$, and $(a, ab), (b, ab) \notin S^\bullet \times 0$, whence $(a, ab), (b, ab) \in \xi^\bullet$.

(iii) ⇒ (i). For $a, b \in S$, if $ab = 0$, then clearly $(a, ab), (b, ab) \in \xi$, and if $ab \neq 0$, then $(a, ab), (b, ab) \in \xi^\bullet \subseteq \xi$. □

Recall that the *division relation* $\mid$ on a semigroup S is defined by:

$$a \mid b \;\Leftrightarrow\; (\exists x, y \in S^1)\; b = xay.$$

On a semigroup $S = S^0$ we also define the relation $\parallel$ by:

$$a \parallel b \;\Leftrightarrow\; a = b = 0 \;\text{ or }\; \big((\exists x, y \in S^1)\; b = xay \neq 0\big)\,.$$

Lemma 6. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is 0-positive;*
(ii) *${}^\bullet\xi$ is 0-positive;*
(iii) *ξ^0 is positive;*
(iv) *$\parallel \subseteq \xi$.*

Proof. (i) $\Rightarrow$ (iii). For $a, b \in S$, if $ab = 0$, then clearly $(a, ab), (b, ab) \in \xi^0$, and if $ab \neq 0$, then $(a, ab), (b, ab) \in \xi \subseteq \xi^0$.

(iii) $\Rightarrow$ (i). For $a, b \in S$, if $ab \neq 0$, then by $(a, ab), (b, ab) \in \xi^0$ it follows $(a, ab), (b, ab) \in \xi$.

(i) $\Leftrightarrow$ (iv). Clearly, $\| \subseteq \xi$ if and only if $\|^0 \subseteq \xi^0$, and $\|^0 = |$, so by (i) $\Leftrightarrow$ (iii) and by the proof of Theorem 1 [7] we obtain that (i) $\Leftrightarrow$ (iv).

(i) $\Leftrightarrow$ (ii). This follows since (i) $\Leftrightarrow$ (iv). $\square$

The following theorem characterizes left 0-restricted positive quasi-orders on a semigroup in terms of closed 0,1-sublattices of ideal lattices.

Theorem 2. *The poset of left 0-restricted positive quasi-orders on a semigroup $S = S^0$ is a complete lattice and it is dually isomorphic to the lattice of closed 0,1-sublattices of $\mathcal{I}d(S)$.*

Proof. By the proof of Theorem 1 [7], the set of positive quasi-orders on S is a principal dual ideal of $\mathcal{Q}(S)$ generated by $|$. It is easy to check that $|$ is left 0-restricted, and by Theorem 1, the set of left 0-restricted positive quasi-orders on S is equal to the interval $[|, \gamma_l]$ of $\mathcal{Q}(S)$, so it is a complete lattice.

In the proof of Theorem 1 [7], the authors proved that the mapping $\xi \mapsto K_\xi$, where

$$K_\xi = \{I \in \mathcal{I}d(S) \mid I\xi = I\}, \tag{1}$$

is an isomorphism between the lattice of positive quasi-orders on S and the lattice of closed 1-sublattices of $\mathcal{I}d(S)$. Clearly, ξ is left 0-restricted if and only if $0 \in K_\xi$, whence we obtain the assertions of the theorem. $\square$

The following theorem will be also very useful:

Theorem 3. *The poset of 0-restricted 0-positive quasi-orders on a semigroup $S = S^0$ is a complete lattice and it is isomorphic to the lattice of left 0-restricted positive quasi-orders on S.*

Proof. By Lemma 6, the set of 0-positive quasi-orders on S is the principal dual ideal of $\mathcal{Q}(S)$ generated by $\|$, and by Theorem 1, the set of 0-restricted 0-positive quasi-orders on S is equal to the interval $[\|, \gamma]$ of $\mathcal{Q}(S)$, so it is a complete lattice.

Let ξ be a left 0-restricted positive quasi-order on S. Then $S^\bullet \times 0 \subseteq \xi$, since ξ is positive, whence

$$(\xi^\bullet)^0 = (\xi - S^\bullet \times 0) \cup S^\bullet \times 0 = \xi \cup S^\bullet \times 0 = \xi.$$

On the other hand, let η be a 0-restricted 0-positive quasi-order on S. Since η is right 0-restricted, then $\eta^\bullet = \eta$, by Lemma 1, whence

$$(\eta^0)^\bullet = (\eta \cup S^\bullet \times 0) - S^\bullet \times 0 = \eta - S^\bullet \times 0 = \eta^\bullet = \eta.$$

By this and by Lemmas 1 and 4 we obtain that the mappings $\xi \mapsto \xi^\bullet$ and $\eta \mapsto \eta^0$ are mutually inverse bijections between the lattice of left 0-restricted positive quasi-orders on S and the lattice of 0-restricted 0-positive quasi-orders on S. By Lemmas 1 and 4, these mappings are isotone, and by Lemma II 3.2 [1], these are lattice isomorphisms. □

3. The 0-Lower-Potency and the 0-*cm*-Property

As we seen in [7] and [11], in the study of semilattice decompositions of semigroups, the lower-potency and the *cm*-property play a crucial role. In this section we consider quasi-orders with more general properties: the 0-lower-potency and the 0-*cm*-property.

First we prove the following:

Lemma 7. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is 0-lower-potent;*
(ii) *${}^\bullet\xi$ is 0-lower-potent;*
(iii) *$\xi^\bullet$ is 0-lower-potent;*
(iv) *ξ^0 is 0-lower-potent.*

Proof. (i) ⇒ (ii). Let $a \in S$, $n \in \mathbf{Z}^+$ and $a^n \neq 0$. Then $(a^n, a) \in \xi$ and by $a^n \neq 0$ it follows $(a^n, a) \in {}^\bullet\xi$.

(i) ⇒ (iii). Let $a \in S$, $n \in \mathbf{Z}^+$ and $a^n \neq 0$. Then $(a^n, a) \in \xi$ and $a \neq 0$, whence $(a^n, a) \in \xi^\bullet$.

(iv) ⇒ (i). Let $a \in S$, $n \in \mathbf{Z}^+$ and $a^n \neq 0$. Then $(a^n, a) \in \xi^0$ and $a \neq 0$, whence $(a^n, a) \in \xi$.

(ii) ⇒ (i), (iii) ⇒ (i) and (i) ⇒ (iv). This follows by ${}^\bullet\xi \subseteq \xi$, $\xi^\bullet \subseteq \xi$ and $\xi \subseteq \xi^0$. □

On a semigroup $S = S^0$ we define the following relations:

$$a \rightsquigarrow b \quad \Leftrightarrow \quad b = 0 \quad \text{or} \quad \big((\exists n \in \mathbf{Z}^+)(\exists x, y \in S^1)\ b^n = xay \neq 0\big),$$
$$a \twoheadrightarrow b \quad \Leftrightarrow \quad a = b = 0 \quad \text{or} \quad \big((\exists n \in \mathbf{Z}^+)(\exists x, y \in S^1)\ b^n = xay \neq 0\big).$$

Lemma 8. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is positive and 0-lower-potent;*
(ii) *$\xi^\bullet$ is 0-positive and 0-lower-potent;*
(iii) *$a\xi$ is a completely 0-semiprime ideal of S, for each $a \in S$;*
(iv) $\rightsquigarrow^\infty \subseteq \xi$.

Proof. (i) ⇒ (ii). This follows by Lemmas 5 and 7.

(i) ⇒ (iii). Let $a \in S$. By Lemma 1 [7], $a\xi$ is an ideal of S. Let $x \in S$, $n \in \mathbf{Z}^+$, $x^n \neq 0$ and $x^n \in a\xi$. Then $a \,\xi\, x^n \,\xi\, x$, whence $x \in a\xi$. Thus, $a\xi$ is completely 0-semiprime.

(iii) ⇒ (i). By Lemma 1 [7], ξ is positive. Let $x \in S$, $n \in \mathbf{Z}^+$ and $x^n \neq 0$. Then $x^n \in (x^n)\xi$, whence $x \in (x^n)\xi$, i.e. $x^n \,\xi\, x$.

(i) ⇒ (iv). Let $a, b \in S$ and $a \rightsquigarrow b$. If $b = 0$, then $a \,\xi\, b$, since ξ is positive, and if $b \neq 0$, then $b^n = xay \neq 0$, for some $n \in \mathbf{Z}^+$, $x, y \in S^1$, whence $a \,\xi\, xa \,\xi\, xay = b^n \,\xi\, b$, i.e. $a \,\xi\, b$. Thus, $\rightsquigarrow \subseteq \xi$, whence $\rightsquigarrow^\infty \subseteq \xi^\infty = \xi$.

(iv) ⇒ (i). Since $\rightsquigarrow^\infty$ is positive and 0-lower-potent, then ξ is also positive and 0-lower-potent. □

Similarly we prove the following

Lemma 9. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is 0-positive and 0-lower-potent;*
(ii) *ξ^0 is positive and 0-lower-potent;*
(iii) *$(a\xi)^0$ is a completely 0-semiprime ideal of S, for each $a \in S$;*
(iv) $\twoheadrightarrow^\infty \subseteq \xi$.

Now we obtain the following two theorems:

Theorem 4. *The poset of left 0-restricted positive 0-lower-potent quasi-orders and the poset of 0-restricted 0-positive 0-lower-potent quasi-orders on a semigroup $S = S^0$ are isomorphic complete lattices.*

Proof. By Lemmas 8 and 9, the poset of left 0-restricted positive 0-lower-potent quasi-orders and the poset of 0-restricted 0-positive 0-lower-potent quasi-orders on S are the principal dual ideals of $\mathcal{Q}(S)$ generated by $\rightsquigarrow^\infty$ and $\twoheadrightarrow^\infty$, respectively, so these are complete lattices. By Lemma 7 and Theorem 3, the mapping $\xi \mapsto \xi^\bullet$ is an isomorphism of the first onto the second cited lattice. □

Theorem 5. *The lattice of left 0-restricted positive 0-lower-potent quasi-orders on a semigroup $S = S^0$ is dually isomorphic to the lattice of closed 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$.*

Proof. Consider a mapping $\xi \mapsto K_\xi$ from the proof of Theorem 2. By Lemma 8, ξ is 0-lower-potent if and only if K_ξ is a sublattice of $\mathcal{I}d^{\mathbf{c0s}}(S)$, whence we obtain the assertion of the theorem. □

Finally, we will consider quasi-orders satisfying the 0-*cm*-property.

Lemma 10. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is positive and it satisfies the 0-cm-property;*
(ii) *${}^\bullet\xi$ is positive and it satisfies the 0-cm-property;*
(iii) *$\xi^\bullet$ is 0-positive and it satisfies the 0-cm-property.*

Proof. (i) $\Rightarrow$ (iii). By Lemma 5, $\xi^\bullet$ is 0-positive. Let $a, b, c \in S$, $ab \neq 0$ and $(a, c), (b, c) \in \xi^\bullet$, i.e. $(a, c), (b, c) \in \xi$ and $(a, c), (b, c) \notin S^\bullet \times 0$. Then $c \neq 0$ and $(ab, c) \in \xi$, whence $(ab, c) \in \xi^\bullet$.

(iii) $\Rightarrow$ (i). By Lemma 5, ξ is positive. Let $a, b, c \in S$, $ab \neq 0$ and $(a, c), (b, c) \in \xi$. If $c = 0$, then $(ab, c) \in \xi$, since ξ is positive, and if $c \neq 0$, then $(a, c), (b, c) \in \xi^\bullet$, whence $(ab, c) \in \xi^\bullet \subseteq \xi$.

(i) $\Leftrightarrow$ (ii). This can be proved similarly as (i) $\Leftrightarrow$ (iii). □

Lemma 11. *The following conditions for a quasi-order ξ on a semigroup $S = S^0$ are equivalent:*

(i) *ξ is 0-positive and it satisfies the 0-cm-property;*
(ii) *${}^\bullet\xi$ is 0-positive and it satisfies the 0-cm-property;*
(iii) *$(\xi a)^0$ is a 0-filter of S, for each $a \in S$;*
(iv) $(\forall a, b \in S)\ ab \neq 0 \ \Rightarrow\ a\xi \cap b\xi = (ab)\xi$.

If ξ is right 0-restricted, then any of the previous conditions is equivalent to the following:

(v) *ξ^0 is positive and it satisfies the 0-cm-property.*

Proof. (i) $\Leftrightarrow$ (ii). This can be proved similarly as Lemma 10.

(i) $\Rightarrow$ (iii). Let $a \in S$. If $x, y \in S$ and $xy \in (\xi a)^0$, $xy \neq 0$, then $xy\,\xi\,a$, whence $x\,\xi\,a$ and $y\,\xi\,a$, i.e. $xy \in \xi a$, since ξ is 0-positive, so $(\xi a)^0$ is 0-consistent. Assume $x, y \in (\xi a)^0$. If $xy = 0$, then clearly $xy \in (\xi a)^0$, and if $xy \neq 0$, then $x, y \in \xi a$, whence $xy \in \xi a \subseteq (\xi a)^0$, by the 0-*cm*-property. Thus, $(\xi a)^0$ is a subsemigroup of S.

(iii) $\Rightarrow$ (i). Let $a, b \in S$ and $ab \neq 0$. Then $a, b \in \xi(ab)$, since $(\xi(ab))^0$ is 0-consistent, so ξ is 0-positive. Let $a, b, c \in S$, $ab \neq 0$ and $(a, c), (b, c) \in \xi$.

Then $a, b \in \xi c$, whence $ab \in (\xi c)^0$, and by $ab \neq 0$ we obtain $ab \in \xi c$. Thus, ξ satisfies the 0-*cm*-property.

(i) $\Rightarrow$ (iv). Let $a, b \in S$ and $ab \neq 0$. Since ξ is 0-positive, then $(ab)\xi \subseteq a\xi \cap b\xi$, and the opposite inclusion it follows by the 0-*cm*-property.

(iv) $\Rightarrow$ (i). This follows immediately.

Further, let ξ be right 0-restricted.

(i) $\Rightarrow$ (v). This can be proved similarly as Lemma 10.

(v) $\Rightarrow$ (i). By Lemma 6, ξ is 0-positive. Let $a, b, c \in S$, $ab \neq 0$ and $(a, c), (b, c) \in \xi$. Then $(a, c), (b, c) \in \xi^0$, whence $(ab, c) \in \xi^0$, and since $ab \neq 0$ and ξ is right 0-restricted, then $c \neq 0$ and $(ab, c) \in \xi$. □

Now we go to the main theorems of this paper:

Theorem 6. *The poset of left 0-restricted positive quasi-orders on a semigroup $S = S^0$ satisfying the 0-cm-property and the poset of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property are isomorphic complete lattices.*

Proof. It is easy to verify that the intersection of any non-empty family of quasi-orders on S satisfying the 0-*cm*-property satisfies also the 0-*cm*-property, and that γ_l and γ satisfy the 0-*cm*-property. By this it follows that the above cited posets are complete lattices. By Lemmas 10 and 11 and Theorem 3, the mapping $\xi \mapsto \xi^\bullet$ is an isomorphism of the first onto the second cited lattice. □

Lemma 12. *A subset A of a semigroup $S = S^0$ is a 0-filter of S if and only if A' is a completely 0-prime ideal of S.*

Proof. This follows by the fact that a subset X of S is a consistent subset of S if and only if $S - X$ is an ideal of S, and X is a subsemigroup of S if and only if $S - X$ is a completely prime subset of S. □

Theorem 7. *The poset of closed 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi-property is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on a semigroup $S = S^0$ satisfying the 0-cm-property.*

Proof. Consider the mapping $\xi \mapsto K_\xi$ from the proof of Theorem 2. Assume that ξ satisfies the 0-*cm*-property. For $a \in S^\bullet$, let $P_a = \big((\xi a)^0\big)' = (\xi a)'$. By Lemma 12, P_a is a completely 0-prime ideal of S and $a \notin P_a$. Let $y \in P_a\xi$. Then $x\,\xi\,y$, for some $x \in P_a$, and if $y \in \xi a$, i.e. $y\,\xi\,a$, then $x\,\xi\,a$, i.e. $x \in \xi a$, so we obtain a contradiction. Therefore, $y \in P_a$, so $P_a\xi = P_a$ and $P_a \in K_\xi$, for each $a \in S^\bullet$. Now, for $I \in K_\xi$, $I = \bigcap_{a \in S-I} P_a$, so K_ξ satisfies the *c*-0-*pi*-property.

Conversely, let K_ξ satisfies the c-0-pi-property, let $a, b \in S$ and $ab \neq 0$. By Lemma 9, $((ab)\xi)^0$ is a completely 0-semiprime ideal of S, so there exists a familly $\{P_\alpha \mid \alpha \in Y\}$ of completely 0-prime ideals from K_ξ such that $((ab)\xi)^0 = \bigcap_{\alpha\in Y} P_\alpha$. Let $U = \{\alpha \in Y \mid a \in P_\alpha\}$, $V = \{\alpha \in Y \mid b \in P_\alpha\}$. Without loss of generality, we can assume that $U \neq \varnothing$ and $V \neq \varnothing$ (for example, we can assume that S is one of P_α). Since P_α is completely 0-prime and $ab \in P_\alpha$, for each $\alpha \in Y$, then $Y = U \cup V$. Now, for each $\alpha \in Y$, $a \in P_\alpha$ implies $(a\xi)^0 = K_\xi(a) \subseteq P_\alpha$, since $P_\alpha \in K_\xi$, so $(a\xi)^0 \subseteq \bigcap_{\alpha\in U} P_\alpha$. Similarly, $(b\xi)^0 \subseteq \bigcap_{\beta\in V} P_\beta$. Thus,

$$(a\xi)^0 \cap (b\xi)^0 \subseteq \Big(\bigcap_{\alpha\in U} P_\alpha\Big) \cap \Big(\bigcap_{\beta\in V} P_\beta\Big) = \bigcap_{\alpha\in Y} P_\alpha = ((ab)\xi)^0 .$$

By this and by Lemma 11, ξ satisfies the 0-cm-property.

Hence, there exists a dual order isomorphism between the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property and the poset of closed 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi-property, so these are dually isomorphic complete lattices. □

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MREŽA POZITIVNIH KVAZI-UREĐENJA NA POLUGRUPI II

Miroslav Ćirić i Stojan Bogdanović

Pozitivna kvazi-uređenja na polugrupama su sa različitih tačaka gledišta izučavane od strane B. M. Scheina, T. Tamurae, M. S. Putchae i autora ovog rada. T. Tamura ih je u [11] povezao sa izučavanjem polumrežnih razlaganja polugrupa. Neki drugi aspekti uloge pozitivnih kvazi-uređenja u polumrežnim razlaganjima polugrupa su proučavani u radovima autora [5] i [7]. U ovom radu istražujemo neka svojstva pozitivnih kvazi-uređenja na polugrupama sa nulom. Dobijeni rezultati biće veoma korisno oruđe u izučavanju kvazi-polumrežnih razlaganja polugrupa sa nulom, koja će biti uvedena i proučavana u narednom radu autora [8].