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POWER SEMIGROUPS THAT ARE ARCHIMEDEAN – II

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Abstract. In the present paper we continue the study started in [6] of semigroups whose power semigroups are Archimedean. We prove that these semigroups are exactly the nilpotent extensions of rectangular bands, and that the class of all such semigroups is power closed. Some special kinds of Archimedean semigroups will be described in terms of their forbidden divisors. We also outline some consequences of the obtained results which concern the pseudovariety of locally trivial semigroups.

1. Introduction and preliminaries

Although there are different kinds of problems concerning power semigroups, the following two general questions can be outlined as the main ones:

(1) What are the structural properties of a semigroup S inherited by its power semigroup $P(S)$?

(2) To what extent does the structure of $P(S)$ determine that of S?

Both of these problems were discussed in the first part [6] of this paper, where we investigated certain conditions under which the power semigroup of a semigroup is Archimedean. We gave structural characterizations of semigroups whose power semigroups are completely Archimedean or some special types of them, but we did not solve the most general problem: Describe the structure of semigroups whose power semigroups are Archimedean (in the most general sense)? We are going to do this here. Namely, we are going to prove that the power semigroup of a semigroup S is Archimedean if and only if S is a nilpotent extension of a rectangular band, and also, that the class of all nilpotent extensions of rectangular bands is closed under formation of power semigroups. Other theorems show how the membership of a semigroup S to certain classes of semigroups can be determined through forbidden divisors of S and $P(S)$.

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The interest for power semigroups was considerably increased when H. Straubing [15], C. Reutenauer [14], J. E. Pin [11] and others established a connection between the operators of making power semigroups (and the related operators on pseudovarieties of semigroups and monoids) and certain natural operators on recognizable languages. This connection initiated an intensive study of pseudovarieties and "power pseudovarieties" of monoids and semigroups, especially of power closed pseudovarieties. More information concerning "power pseudovarieties" can be found in the book [10] by J. E. Pin, the survey article [3] by J. Almeida and other articles cited below. Some of the results obtained here have consequences concerning the pseudovariety of locally trivial semigroups, and these consequences will be also outlined in the paper.

Let S be a semigroup. On the set $P(S)$ of all non-empty subsets of S one defines a multiplication by: $AB = \{ab \,|\, a \in A,\, b \in B\}$. Then then $P(S)$ becomes a semigroup, called the *power semigroup* of S. In the same way one defines a multiplication on the set $\overline{P}(S)$ of all subsets of S (including the empty subset), and $\overline{P}(S)$ is a semigroup having $P(S)$ as its subsemigroup.

We say that a semigroup T *divides* a semigroup S, and that T is a *divisor* of S if T is a homomorphic image of some subsemigroup of S. A *pseudovariety* is defined as a class of finite semigroups closed under formation of homomorphic images, subsemigroups and finite direct products, or equivalently, under formation of divisors and finite direct products. For a pseudovariety $\boldsymbol{V}$ one defines $\boldsymbol{P}(\boldsymbol{V})$ and $\overline{\boldsymbol{P}}(\boldsymbol{V})$ as pseudovarieties generated by the sets $\{P(S) \,|\, S \in \boldsymbol{V}\}$ and $\{\overline{P}(S) \,|\, S \in \boldsymbol{V}\}$, respectively (note that J. Almeida in [3] used another notations). A class $\mathfrak{C}$ of semigroups is *power closed* if $P(S) \in \mathfrak{C}$, for any $S \in \mathfrak{C}$. For a pseudovariety $\boldsymbol{V}$ of semigroups we have that it is power closed if and only if $\boldsymbol{P}(\boldsymbol{V}) = \boldsymbol{V}$.

We denote by $\boldsymbol{LI}$, $\boldsymbol{K}$ and $\boldsymbol{D}$ pseudovarieties consisting of all finite semigroups which are nilpotent extensions of rectangular, left zero and right zero bands, respectively, and by $\boldsymbol{Nil}$ the pseudovariety of all finite nilpotent semigroups. Semigroups from $\boldsymbol{LI}$ are called *locally trivial* ([1], [3], [16]), while semigroups from $\boldsymbol{D}$ and $\boldsymbol{K}$ are known as *definite* and *reverse definite* semigroups, respectively ([16], [3]).

Throughout this paper, $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{C}_2$ will denote the two-element chain and for a prime p, $\mathbb{G}_p$ will denote the group of order p. For elements $a_1, \dots, a_n$ of a semigroup S, $\langle a_1, \dots, a_n \rangle$ will denote the subsemigroup of S generated by $a_1, \dots, a_n$.

A semigroup S is *Archimedean* (resp. *left Archimedean*, *right Archimedean*, t-Archimedean) if for all $a, b \in S$ there exists $n \in \mathbb{N}$ such that $b^n \in SaS$ (resp. $b^n \in Sa$, $b^n \in aS$, $b^n \in aSa$). Finite Archimedean semigroups form a pseudovariety which will be denoted by $\boldsymbol{FA}$.

For undefined notions and notations we refer to [5], [8] and [10].

2. The main results

We start with the following two lemmas:

Lemma 1. *A semigroup S is Archimedean if and only if any its bi-ideal is Archimedean.*

Proof. Let S be an Archimedean semigroup and let B be a bi-ideal of S. For all $a, b \in B$ there exists $n \in \mathbb{N}$ such that $a^n \in Sb^3S$, so $a^{n+2} \in (aSb)b(bSa) \subseteq BSBbBSB \subseteq BbB$. Thus, B is an Archimedean semigroup.

The converse is obvious. □

Lemma 2. *If B is a bi-ideal of a semigroup S, then $P(B)$ is a bi-ideal of $P(S)$.*

Proof. For all $X, Y \in P(B)$ and $A \in P(S)$ we have that $XAY \subseteq BSB \subseteq B$, whence $P(B)P(S)P(S) \subseteq P(B)$. □

Now we prove tha main result of this paper:

Theorem 1. *The following conditions on a semigroup S are equivalent:*

(i) *$P(S)$ is Archimedean;*
(ii) *$P(S)$ is a nilpotent extension of a rectangular band;*
(iii) *S is a nilpotent extension of a rectangular band.*

Proof. (i) ⇒ (iii). Let $P(S)$ be an Archimedean semigroup. By Corollary 1 [6], there exists $n \in \mathbb{N}$ such that S^n is a simple semigroup. Let $S^n = K$. By Lemmas 1 and 2, $P(K)$ is an Archimedean semigroup. For any $a \in K$ there exist $n \in \mathbb{N}$ and $B, C \in P(K)$ such that $\{a\}^n = BKC$. Since K is a simple semigroup, we then have that $\{a\}^n K = B(KCK) = BK$ and $K\{a\}^n = (KBK)C = KC$. Now we obtain $\{a^n\} = (BK)C = \{a^n\}KC = \{a^n\}K\{a^n\}$, whence $a^n = a^{n+1}$. By this and by the known Munn Theorem (Theorem 2.55 [8]) we conclude that K is a completely simple semigroup whose all maximal subgroups are trivial, and hence, K is a rectangular band. Therefore, S is a nilpotent extension of a rectangular band.

(iii) ⇒ (ii). Let S^n be a rectangular band, for some $n \in \mathbb{N}$. By Corollary 2 [6], $P(S^n)$ is an inflation of a rectangular band T, and hence, $\big(P(S^n)\big)^2 = T$. Since $\big(P(S)\big)^{2n} \subseteq \big(P(S^n)\big)^2 = T = T^2 \subseteq \big(P(S)\big)^{2n}$, we then have $\big(P(S)\big)^{2n} = T$. Therefore, $P(S)$ is a nilpotent extension of a rectangular band.

(ii) ⇒ (i). This is obvious. □

The first of the following two corollaries was given by the authors in [6], but in a slightly different form. The second one was obtained by M. S. Putcha in [12] and S. Bogdanović in [4].

Corollary 1. *The following conditions on a semigroup S are equivalent:*

(i) $P(S)$ *is left* (*resp. right*) *Archimedean;*
(ii) $P(S)$ *is a nilpotent extension of a left* (*resp. right*) *zero band;*
(iii) S *is a nilpotent extension of a left* (*resp. right*) *zero band.*

Corollary 2. *The following conditions on a semigroup S are equivalent:*

(i) $P(S)$ *is t-Archimedean;*
(ii) $P(S)$ *is nilpotent;*
(iii) S *is nilpotent.*

Theorem 1 and Corollaries 1 and 2 assert that the classes of nilpotent extensions of rectangular bands, of left zero bands and of right zero bands are power closed. By this it follows the same property for pseudovarieties $\boldsymbol{LI}$, $\boldsymbol{K}$, $\boldsymbol{D}$ and $\boldsymbol{Nil}$.

Corollary 3. *Pseudovarieties $\boldsymbol{LI}$, $\boldsymbol{K}$, $\boldsymbol{D}$ and $\boldsymbol{Nil}$ are power closed.*

On a semigroup S we define a relation $\uparrow$ by: $a \uparrow b \quad \Leftrightarrow \quad (\exists n \in \mathbb{N})\, b^n \in \langle a, b\rangle\, b\, \langle a, b\rangle$. A semigroup S is called *hereditary Archimedean* if $a \uparrow b$, for all $a, b \in S$, or equivalently, if any its subsemigroup is Archimedean [7]. The class of such semigroups will be denoted by $\mathcal{HA}$. They will be characterized in terms of forbidden divisors as follows:

Theorem 2. *A semigroup S is hereditary Archimedean if and only if $\mathbb{C}_2$ does not divide S.*

Proof. The class $\mathcal{HA}$ is closed under formation of divisors and it does not contain $\mathbb{C}_2$, whence we have that $\mathbb{C}_2$ does not divide any semigroup from $\mathcal{HA}$.

Conversely, let $\mathbb{C}_2$ does not divide S. Suppose that S is not hereditary Archimedean. Then there exist $a, b \in S$ such that $a \uparrow b$ does not hold, i.e. such that $b^n \notin T^1 a T^1$, for any $n \in \mathbb{N}$, where $T = \langle a, b\rangle$. But, now we have that the sets $A_0 = T^1 a T^1$ and $A_1 = \langle b\rangle$ form a partition of T which determines a congruence relation on S whose related factor is isomorphic to $\mathbb{C}_2$. This means that $\mathbb{C}_2$ divides S, which contradicts our starting hypothesis. Therefore, we conclude that $S \in \mathcal{HA}$. This completes the proof of the theorem. □

Corollary 4. *Let $\boldsymbol{V}$ be a pseudovariety of semigroups. Then $\boldsymbol{V} \subseteq \boldsymbol{FA}$ if and only if $\mathbb{C}_2 \notin \boldsymbol{V}$.*

In terms of forbidden divisors we also characterize nil-extensions of rectangular bands.

Theorem 3. *A semigroup S is a nil-extension of a rectangular band if and only if $\mathbb{C}_2$ and $\mathbb{G}_p$, for any prime p, do not divide S.*

Proof. The class of all semigroups which are nil-extensions of rectangular bands is closed under formation of divisors and it does not contain semigroups $\mathbb{C}_2$ and $\mathbb{G}_p$, for any prime p, so $\mathbb{C}_2$ and $\mathbb{G}_p$ do not divide any semigroup from this class.

Conversely, let $\mathbb{C}_2$ and $\mathbb{G}_p$, for any prime p, do not divide S. By Theorem 2, $S \in \mathcal{HA}$. Assume an arbitrary $a \in S$. If $\langle a \rangle$ is infinite, then it is isomorphic to the additive semigroup of positive integers, and any of the groups $\mathbb{G}_p$ is a homomorphic image of $\langle a \rangle$. Thus, $\mathbb{G}_p$ divide S, which contradicts our starting hypothesis. Hence, $\langle a \rangle$ is finite, for any $a \in S$, so S is periodic, and it is a nil-extension of a periodic completely simple semigroup K. In view of the hypothesis, K does not have non-trivial subgroups, so K is a rectangular band. $\square$

Characterization of nilpotent extensions of rectangular bands through its forbidden divisors remains an open problem. But we characterize these semigroups through forbidden divisors of their power semigroups, as follows:

Theorem 4. *A semigroup S is a nilpotent extension of a rectangular band if and only if $\mathbb{C}_2$ does not divide $P(S)$.*

Proof. Let S be a nilpotent extension of a rectangular band. By Theorem 1, $P(S)$ is also a nilpotent extension of a rectangular band, and by Theorem 5 [7], $P(S)$ is a hereditary Archimedean semigroup. Now, by Theorem 2 we have that $\mathbb{C}_2$ does not divide $P(S)$.

Conversely, let $\mathbb{C}_2$ does not divide $P(S)$. Then by Theorem 2 we have that $P(S)$ is Archimedean, and by Theorem 1, S is a nilpotent extension of a rectangular band. $\square$

Using Theorems 3 and 4 we can obtain the following result concerning subpseudovarieties of the pseudovariety of locally trivial semigroups.

Corollary 5. *Let $\boldsymbol{V}$ be a pseudovariety of semigroups. Then the following conditions are equivalent:*

(i) $\boldsymbol{V} \subseteq \boldsymbol{LI}$;
(ii) $\boldsymbol{V}$ *does not contain* $\mathbb{C}_2$ *and* $\mathbb{G}_p$, *for any prime* p;
(iii) $\mathbb{C}_2 \notin \boldsymbol{P}(\boldsymbol{V})$.

The equivalence of the conditions (i) and (iii) is a well-known result of the theory of "power pseudovarieties" which has the following significant consequence: In the lattice of pseudovarieties, for a pseudovariety $\boldsymbol{V}$, the pseudovarieties $\boldsymbol{P}(\boldsymbol{V})$ and $\overline{\boldsymbol{P}}(\boldsymbol{V})$ are related by: $\overline{\boldsymbol{P}}(\boldsymbol{V}) = \boldsymbol{P}(\boldsymbol{V}) \vee \boldsymbol{Sl}$, where $\boldsymbol{Sl}$ denotes the pseudovariety of all finite semilattices. Since $\boldsymbol{Sl}$ is the pseudovariety generated by $\mathbb{C}_2$, then $\overline{\boldsymbol{P}}(\boldsymbol{V}) = \boldsymbol{P}(\boldsymbol{V})$, for any pseudovariety $\boldsymbol{V} \nsubseteq \boldsymbol{LI}$.

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