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DIRECT SUM DECOMPOSITIONS OF QUASI-ORDERED SETS

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Dedicated to Professor B. M. Schein on the occasion of his 60th birthday

Abstract. The main purpose of this paper is to advance a general theory of direct sum decompositions of quasi-ordered sets and to show that certain significant decompositions of semigroups, automata and graphs can be carried out as direct sum decompositions of certain quasi-ordered sets associated to them. In this way we generalize certain results given by M. Petrich in [16] and [17], S. Bogdanović and M. Ćirić in [3], and [4], and M. Ćirić and S. Bogdanović in [5] and [6].

1. Introduction and preliminaries

This paper is inspred by certain results in Semigroup Theory and Automata Theory given by M. Petrich and the first two authors of this paper. M. Petrich investigated in [16], 1966, some general properties of left zero band, right zero band and matrix decompositions of semigroups, S. Bogdanović and M. Ćirić studied in [3] and [4], 1995, orthogonal, left, right and matrix sum decompositions of semigroups with zero, and in [6] they were occupied with direct sum decompositions of automata. Analyzing these papers we observe that a similar methodology was used and the results of similar form were obtained. So we ask a natural question: Where do these similarities go from?

In this paper we give one answer to this question. The main purpose of the paper is to advance a general theory of direct sum decompositions of quasi-ordered sets and to show that all the above mentioned decompositions of semigroups and automata can be carried out as direct sum decompositions of certain quasi-ordered sets associated to them.

In Section 2 we study double ideals of quasi-ordered sets. We prove the set $\boldsymbol{D}(Q)$ of all double ideals of a quasi-ordered set Q is a complete atomic

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Boolean algebra and we give some algorithms for finding the atoms of $\boldsymbol{D}(Q)$. In Section 3 we show that direct sum decompositions of a quasi-ordered set form a complete lattice that is isomorphic to the lattice of complete Boolean subalgebras of $\boldsymbol{D}(Q)$, and that every quasi-ordered set can be represented as a direct sum of direct sum indecomposable quasi-ordered sets. Section 4 is devoted to the lattice $\boldsymbol{I}(Q)$ of ideals of a quasi-ordered set Q. We establish a connection between direct sum decompositions of Q and decompositions of $\boldsymbol{I}(Q)$ into a direct product, so prove that $\boldsymbol{I}(Q)$ can be represented as a direct product of directly indecomposable lattices, and we apply the obtained results distributive, algebraic and dually algebraic lattices. Finally, in Section 5 we apply the obtained results to semigroups, automata and graphs. We show that some well-known and vary important decompositions of semigroups, automata and graphs can be carried out as direct sum decompositions of certain quasi-ordered sets associated to them. This approach makes possible to obtain as special cases certain results of M. Petrich [16] and [17], S. Bogdanović and M. Ćirić [1], [2], [3] and [4], M. Ćirić and S. Bogdanović [5] and [6], and M. Ćirić, S. Bogdanović and T. Petković [7] and [8].

By a quasi-order on a non-empty set Q we mean a reflexive and transitive binary relation on Q. Throughout the paper, quasi-orders will be usually denoted by $\preccurlyeq$. A non-empty set equipped with a quasi-oorder will be called a *quasi-ordered*. It will be usually denoted by Q. As known, a anti-symmetric quasi-order is called a *partial order*, and related quasi-ordered set is called a *partially ordered set*, or shortly a *poset*. Partial orders will be usually denoted by $\leq$. Let P and P' be two posets. A mapping φ of P into P' is called *isotone* if for $a, b \in P$, $a \leq b$ implies $a\varphi \leq b\varphi$, and it is called *antitone* if for $a, b \in P$, $a \leq b$ implies $b\varphi \leq a\varphi$. If φ is isotone, one-to-one and onto, then it is called an *order isomorphism* of P onto P'. If φ is antitone, one-to-one and onto, then it is a *dual order isomorphism* of P onto P', and then P' is called a *dual* of P, and vice versa.

When a quasi-ordered set Q with a quasi-order $\preccurlyeq$ is given, any non-empty subset H of Q will be treated as a quasi-ordered set, with respect to the restriction of $\preccurlyeq$ to H. A quasi-ordered set Q is called *directed* if for all $a, b \in Q$ there exists $x \in Q$ such that $a \preccurlyeq x$ and $b \preccurlyeq x$, and is called *dually directed* if for all $a, b \in Q$ there exists $x \in Q$ such that $x \preccurlyeq a$ and $x \preccurlyeq b$. A non-empty subset H of Q is called a *directed subset* of Q if it is a directed quasi-ordered set, with respect to the restriction of $\preccurlyeq$ to H. For $a, b \in Q$, we say that they are *incomparable*, and we write $a \parallel b$, if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$. For non-empty subsets $H, G \subseteq Q$, we write $H \parallel G$ if $a \parallel b$, for each $a \in H$ and each $b \in G$.

Throughout the paper, L will denote the complete lattice with the zero 0 and the unity 1. A sublattice K of L is called a $\{0,1\}$-*sublattice* if $0,1 \in K$. It is called a *complete sublattice* if it contains the meet and the join of any its non-empty subset. A mapping φ of L into some complete lattice L' is called a *lattice isomorphism* if it is one-to-one, onto and $(a \wedge b)\varphi = a\varphi \wedge b\varphi$ and $(a \vee b)\varphi = a\varphi \vee b\varphi$, for all $a, b \in L$. If, in addition, $\big(\bigwedge_{i\in I} a_i\big)\varphi = \bigwedge_{i\in I}(a_i\varphi)$ and $\big(\bigvee_{i\in I} a_i\big)\varphi = \bigvee_{i\in I}(a_i\varphi)$, for each non-empty subset $\{a_i\}_{i\in I}$ of L, then φ is called a *complete lattice isomorphism*. Note that every order isomorphism of L onto L' is a complete lattice isomorphism.

A mapping φ of L into itself is called *extensive* if $a \leq a\varphi$, for every $a \in L$, and *idempotent* if $\varphi^2 = \varphi$, that is $(a\varphi)\varphi = a\varphi$, for every $a \in L$. If φ is extensive, isotone and idempotent, then it is called a *closure operator* on L. An element $a \in L$ is called *closed* with respect to φ, or φ-*closed*, if $a\varphi = a$. The set of all closure operators on L will be treated as a poset with respect to the partial order defined by $\varphi \leq \psi$ if and only if $a\varphi \leq a\psi$, for every $a \in L$.

A subset C of L is called a *closure system* in L if $1 \in C$ and $\bigwedge_{i\in I} a_i \in C$, for each non-empty subset $\{a_i\}_{i\in I}$ of C. The intersection of an arbitrary non-empty family of closure systems in L is also a closure system in L, so closure systems in L form a complete lattice. As known, there is a natural correspondence between closure operators on L and closure systems in L. Namely, for a closure operator φ on L, the set C_φ of all φ-closed elements of L is a closure system on L. Conversely, if C is a closure system in L, then the mapping φ of L into itself defined by $a\varphi = \bigwedge\{x \in C \mid a \leq x\}$, $a \in L$, is a closure operator on L. Furthermore, the mappings $\varphi \mapsto C_\varphi$ and $C \mapsto \varphi_C$ are mutually inverse dual order isomorphisms of the poset of closure operators on L onto the complete lattice of closure systems in L. Therefore, the poset of closure operators on L is a complete lattice.

A closure operator φ is called an *algebraic closure operator* if $\big(\bigvee_{i\in I} a_i\big)\varphi = \bigvee_{i\in I}(a_i\varphi)$, for every non-empty directed subset $\{a_i\}_{i\in I}$ of L. The corresponding closure system is called an *algebraic closure system*. In other words, a closure system C is algebraic if and only if $\bigvee H \in C$, for every non-empty directed subset H of C. If $\big(\bigvee_{i\in I} a_i\big)\varphi = \bigvee_{i\in I}(a_i\varphi)$ for every non-empty subset $\{a_i\}_{i\in I}$ (not necessarily directed), then φ is called a *complete closure operator* and related closure system is called a *complete closure system* in L. This means that C is a complete closure system in L if and only if it is a complete sublattice of L.

An element $a \in L$ is called an *atom* of L, if $0 < a$ and there exists no $x \in L$ such that $0 < x < a$. We say that L is *atomic* if for each $a \in L$, $a \neq 0$, there exists an atom p of L such that $p \leq a$, and that L is *atomistic* if every non-zero element of L can be represented as the join of some family of atoms of L. Clearly, any atomistic lattice is atomic. A complete Boolean algebra is atomic if and only if it is atomistic.

An element $a \in L$ is called *neutral* if $(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$, for all $x, y \in L$. The set of all neutral complemented elements of L is called a *center* of L, and if L is distributive, then it consists simply of all complemented elements of L.

For a non-empty set X, $\mathcal{P}(X)$ will denote the *Boolean algebra of subsets* of X, $\mathfrak{B}(X)$ will denote the *Boolean algebra of binary relations* on X, and $\mathcal{E}(X)$ will denote the *lattice of equivalence relations* on X. The dual lattice of $\mathcal{E}(X)$ is called the *lattice of partitions* of X and is denoted by $\mathrm{Part}(X)$. The relation $\between$ on $\mathcal{P}(X)$ is defined by: for $H, G \subseteq X$, $H \between G$ if and only if $H \cap G \neq \varnothing$. For $\xi \in \mathfrak{B}(X)$, ξ^{-1} will denote the relation on X defined by: $(a, b) \in \xi^{-1}$ if and only if $(b, a) \in \xi$. The *equality relation* and the *universal relation* on X will be denoted by Δ_X and ∇_X, respectively. We say that an equivalence relation ϱ on X *saturates* a subset H of X, or that H is *saturated by* ϱ, if H is the union of some family of ϱ-classes.

Let C be a closure system in $\mathcal{P}(X)$. Then for every $a \in X$ there exists the smallest element of C containing a (this is exactly the intersection of all elements of C containing a), that will be called the *principal element* of C generated by a.

The operators $R : \xi \mapsto \xi R$, $S : \xi \mapsto \xi S$ and $T : \xi \mapsto \xi T$ on $\mathfrak{B}(X)$ defined by $\xi R = \xi \cup \Delta_X$, $\xi S = \xi \cup \xi^{-1}$ and $\xi T = \bigcup_{n \in \mathbb{N}} \xi^n$ are closure operators on $\mathfrak{B}(X)$, where $\mathbb{N}$ is the set of positive integers. They are called a *reflexive*, *symmetric* and *transitive closure* operators on $\mathfrak{B}(X)$, respectively. The operator $E : \xi \mapsto \xi E$ defined by $E = RST$ is also a closure operator and for each $\xi \in \mathfrak{B}(X)$, ξE is the smallest equivalence relation on X containing ξ. We say that E is a *equivalence closure operator* on $\mathfrak{B}(X)$.

For undefined notions and notations concerning partial orders (quasi-orders) and lattices we refer to the books of P. Crawley and R. P. Dilworth [9], B. A. Davey and H. A. Priestley [10] and G. Grätzer [12], and for that concerning semigroups and automata we refer to the books of S. Bogdanović and M. Ćirić [1], M. Petrich [17] and F. Gécseg and I. Peák [11].

2. Double ideals

A subset H of a quasi-ordered set Q is called an *ideal* of Q if for $a, x \in Q$, $a \in H$ and $x \preccurlyeq a$ implies $x \in H$, and it is a *filter* or a *dual ideal* of Q if for $a, x \in Q$, $a \in H$ and $a \preccurlyeq x$ implies $x \in H$. If H is both an ideal and a filter of Q, it will be called a *double ideal* of Q. The empty subset of Q is defined to be a double ideal of Q. Every ideal (resp. filter, double ideal) of Q different than $\varnothing$ and Q is called a *proper ideal* (resp. *proper filter*, *proper double ideal*) of Q.

As known, the set $\boldsymbol{I}(Q)$ of all ideals of Q and the set $\boldsymbol{F}(Q)$ of all filters of Q are complete $\{0, 1\}$-sublattices of the lattice (Boolean algebra) $\mathcal{P}(Q)$ of all subsets of Q. They will be called the *lattice of ideals* and the *lattice of filters* of Q, respectively. For the set $\boldsymbol{D}(Q)$ of all double ideals of Q we have the following:

Theorem 2.1. *Let Q be a quasi-ordered set. Then $\boldsymbol{D}(Q)$ is the center both of $\boldsymbol{I}(Q)$ and $\boldsymbol{F}(Q)$, it is a complete $\{0,1\}$-sublattice of $\boldsymbol{I}(Q)$ and $\boldsymbol{F}(Q)$ and it is a complete atomic Boolean algebra.*

Furthermore, each complete atomic Boolean algebra can be represented as the Boolean algebra of double ideals of some quasi-ordered set.

Proof. Since $\boldsymbol{I}(Q)$ and $\boldsymbol{F}(Q)$ are complete $\{0,1\}$-sublattices of $\mathcal{P}(Q)$, then $\boldsymbol{D}(Q)$ is also a complete $\{0,1\}$-sublattice of Q. On the other hand, a subset of Q is an ideal if and only if its complement in $\mathcal{P}(Q)$ is a filter, so a subset of Q is a double ideal if and only if its complement in $\mathcal{P}(Q)$ is a double ideal. Therefore, $\boldsymbol{D}(Q)$ is a Boolean algebra. Since it is a complete Boolean subalgebra of $\mathcal{P}(Q)$, then it is completely distributive and atomic, by Theorem 4.6 of [9].

Further, let B be an arbitrary complete atomic Boolean algebra and let A be the set of all atoms of B. Associate to each $a \in A$ a quasi-ordered set Q_a without proper double ideals. For example, we can assume an arbitrary directed quasi-ordered set or a lattice. Let $Q = \bigcup_{a\in A} Q_a$ and define a relation $\preccurlyeq$ on Q by: $x \preccurlyeq y$ in Q if and only if $x, y \in Q_a$ for some $a \in A$ and $x \preccurlyeq y$ in Q_a. Then $\preccurlyeq$ is a quasi-order on Q and for so defined quasi-ordered set Q we have that $\boldsymbol{D}(Q)$ is a complete atomic Boolean algebra whose atoms are exactly Q_a, $a \in A$. Therefore, B and $\boldsymbol{D}(Q)$ are complete atomic Boolean algebras whose sets of atoms have the same cardinality, so they are isomorphic. This completes the proof of the theorem. $\square$

Let Q be a quasi-ordered set. As we have seen, $\boldsymbol{I}(Q)$, $\boldsymbol{F}(Q)$ and $\boldsymbol{D}(Q)$ are complete closure systems on $\mathcal{P}(Q)$, and we can consider the closure operators $I : H \mapsto I(H)$, $F : H \mapsto F(H)$ and $D : H \mapsto D(H)$ ($H \subseteq Q$) associated to them. In other words, $I(H)$, $F(H)$ and $D(H)$ are respectively the smallest elements of $\boldsymbol{I}(Q)$, $\boldsymbol{F}(Q)$ and $\boldsymbol{D}(Q)$ containing H, that is, the intersections of all elements of $\boldsymbol{I}(Q)$, $\boldsymbol{F}(Q)$ and $\boldsymbol{D}(Q)$ containing H. For $a \in Q$, $I(a)$, $F(a)$ and $D(a)$ will denote the principal elements of $\boldsymbol{I}(Q)$, $\boldsymbol{F}(Q)$ and $\boldsymbol{D}(Q)$ generated by a, called the *principal ideal*, *principal filter* and *principal double ideal* of Q generated by a, respectively. The following theorem characterizes the atoms in $\boldsymbol{D}(Q)$ as its principal elements:

Theorem 2.2. *Let Q be a quasi-ordered set. Then the atoms in $\boldsymbol{D}(Q)$ are exactly the principal double ideals of Q.*

Proof. Let D be an arbitrary atom of $\boldsymbol{D}(Q)$ and let $a \in D$. Then $D(a) \subseteq D$, and since D is an atom and $D(a)$ is non-empty, then $D(a) = D$. Therefore, every atom of $\boldsymbol{D}(Q)$ is a principal duoble ideal.

Conversely, let $D(a)$ be the principal double ideal of Q generated by $a \in Q$. Assume a double ideal D of Q such that $D \subseteq D(a)$. If $a \in D$, then $D(a) \subseteq D$ and we have $D = D(a)$. Otherwise, if $a \notin D$, that is $a \in D(a) \setminus D$, then $D(a) \setminus D = D(a) \cap D'$ (where D' denotes the complement of D in $\mathcal{P}(Q)$) is a double ideal of Q containing a, whence we have $D(a) = D(a) \setminus D$, and hence $D = \varnothing$. Therefore, we have proved that $D(a)$ is an atom of $\boldsymbol{D}(Q)$. This completes the proof of the theorem. $\square$

There are considerably simple characterizations of principal ideals and principal filters of quasi-ordered sets. Namely, for a quasi-ordered set Q and $a \in Q$ we have $I(a) = \{x \in Q \mid x \preccurlyeq a\}$ and $F(a) = \{x \in Q \mid x \preccurlyeq a\}$.* Using these characterizations we give the following characterization for principal double ideals of quasi-ordered sets:

Theorem 2.3. *Let Q be a quasi-ordered set and let $a \in Q$. Define sequences $\{I_n(a)\}_{n\in\mathbb{N}}$ and $\{F_n(a)\}_{n\in\mathbb{N}}$ of subsets of Q by:*

$$I_1(a) = I(F(a)), \quad I_{n+1}(a) = I(F(I_n(a))), \ \textit{for } n \in \mathbb{N},$$
$$F_1(a) = F(I(a)), \quad F_{n+1}(a) = F(I(F_n(a))), \ \textit{for } n \in \mathbb{N}.$$

Then $\{I_n(a)\}_{n\in\mathbb{N}}$ and $\{F_n(a)\}_{n\in\mathbb{N}}$ are increasing sequences of sets and

$$D(a) = \bigcup_{n\in\mathbb{N}} I_n(a) = \bigcup_{n\in\mathbb{N}} F_n(a).$$

Proof. Seeing that D is the join of I and F in the lattice of closure operators on $\mathcal{P}(Q)$, the proof of the theorem can be derived from some more general results of T. Tamura given in [18], which concern the joins of algebraic closure operators on complete lattices (he called them join-conservative closure operators). But, such a proof requires many new notions, notations and auxiliary results, so we have decided to give an immediate, simpler proof.

Since I, F and D are closure operators on $\mathcal{P}(Q)$, we have the following: First, $I_n(a) \subseteq F(I_n(a)) \subseteq I(F(I_n(a))) = I_{n+1}(a)$, for every $n \in \mathbb{N}$, so the sequence $\{I_n(a)\}_{n\in\mathbb{N}}$ is increasing. Further, if $I_n(a) \subseteq D(a)$, for some $n \in \mathbb{N}$, then $F(I_n(a)) \subseteq F(D(a)) = D(a)$ and $I_{n+1}(a) = I(F(I_n(a))) \subseteq I(D(a)) = D(a)$. Therefore, by induction we obtain $I_n(a) \subseteq D(a)$, for each $a \in Q$, whence $\bigcup_{n\in\mathbb{N}} I_n(a) \subseteq D(a)$. Set $D = \bigcup_{n\in\mathbb{N}} I_n(a)$. To prove $D = D(a)$, it remains to prove that D is a double ideal of Q. Indeed, D is an ideal of

*Note that the notations $(a]$, instead of $I(a)$, and $[a)$, instead of $F(a)$, are used more frequently.

Q, since it is the union of ideals $I_n(a)$, $n \in \mathbb{N}$. On the other hand, assume $d \in D$ and $x \in Q$ such that $d \preccurlyeq x$. Then $d \in I_n(a)$, for some $n \in \mathbb{N}$, and $x \in F(d) \subseteq F(I_n(a)) \subseteq I(F(I_n(a))) = I_{n+1}(a) \subseteq D$, so D is also a filter of Q. Therefore, we have proved that $D = D(a)$. Similarly we prove the remaining assertions. □

The above theorem gives two nice procedures for finding the atoms of the Boolean algebra of double ideals of a quasi-ordered. In the case when this quasi-ordered set is finite, we have two finite algorithms for finding these atoms, as follows by the following:

Corollary 2.1. *Let Q be a finite quasi-ordered set. Then there exist $n = \min\{k \in \mathbb{N} \,|\, (\forall a \in Q)\, I_k(a) = I_{k+1}(a)\}$ and $m = \min\{k \in \mathbb{N} \,|\, (\forall a \in Q)\, F_k(a) = F_{k+1}(a)\}$ for which also holds $n, m \leq |Q|$ and $D(a) = I_n(a) = F_m(a)$, for each $a \in Q$.*

3. Direct sum decompositions

Let Q be a quasi-ordered set. We say that Q is a *direct sum* of its subsets Q_α, $\alpha \in Y$, in notation $Q = \sum_{\alpha \in Y} Q_\alpha$, if $Q = \bigcup_{\alpha \in Y} Q_\alpha$ and $Q_\alpha \parallel Q_\beta$, for $\alpha \neq \beta$. Clearly, the condition $Q_\alpha \parallel Q_\beta$ implies $Q_\alpha \cap Q_\beta = \varnothing$. Each of the sets Q_α, $\alpha \in Y$, will be called a *direct summand* of Q, the corresponding partition of Q will be called a *direct sum decomposition* of Q, and the corresponding equivalence relation on Q will be called a *direct sum equivalence* on Q. A quasi-ordered set Q will be called *direct sum indecomposable* if the universal relation ∇_Q is the unique direct sum equivalence on Q.

The main goal of this section is to give some general properties of direct sum decompositions of a quasi-ordered set. First we prove the following interesting lemma concerning direct sum equivalences on Q:

Lemma 3.1. *The set of all direct sum equivalences on a quasi-ordered set Q is a principal dual ideal of the lattice $\mathcal{E}(Q)$ of equivalence relations on Q.*

Proof. It is easy to see that an equivalence relation θ on Q is a direct sum equivalence on Q if and only if the quasi-order $\preccurlyeq$ is contained in θ. Therefore, the set of all direct sum equivalences on Q is the principal dual ideal of $\mathcal{E}(Q)$ generated by the smallest equivalence relation on Q containing $\preccurlyeq$, that is by the equivalence closure of $\preccurlyeq$. □

In terms of direct sum decompositions, the above lemma can be stated as follows:

Lemma 3.2. *The set of all direct sum decompositions of a quasi-ordered set Q is a principal ideal of the partition lattice* $\mathrm{Part}(Q)$.

The smallest direct sum equivalence on a quasi-ordered set Q will be denoted by σ_Q, or simply by σ, if we know on which quasi-ordered set it is considered. As we have seen in Lemma 3.1, we can construct σ as the equivalence closure of the quasi-order $\preccurlyeq$. But, it is often more convenient to use another method for construction of σ developed by M. Ćirić, S. Bogdanović and T. Petković in [7]. This method, that we use in the next theorem, involves a usage of principal ideals and filters of a quasi-ordered set.

Theorem 3.1. *Let Q be a quasi-ordered set and $a, b \in Q$. Then the following conditions are equivelent:*

(i) $(a, b) \in \sigma$;

(ii) *there exists a finite sequence $\{x_i\}_{i=1}^n \subseteq Q$, $n \in \mathbb{N} \cup \{0\}$ such that:*

$$I(a) \between I(x_1) \between I(x_2) \between \cdots \between I(x_n) \between I(b); \tag{1}$$

(iii) *there exists a finite sequence $\{x_i\}_{i=1}^n \subseteq Q$, $n \in \mathbb{N} \cup \{0\}$ such that:*

$$F(a) \between F(x_1) \between F(x_2) \between \cdots \between F(x_n) \between F(b). \tag{2}$$

Remark 3.1. If $n = 0$ in (ii) or (iii) of the above theorem, this means that the sequence $\{x_i\}_{i=1}^n$ is empty, and then (1) becomes $I(a) \between I(b)$ and (2) becomes $F(a) \between F(b)$.

Note also that $I(a) \between I(b)$ if and only if a and b have a common lower bound, and $F(a) \between F(b)$ if and only if a and b have a common upper bound. By this and the above theorem it follows that directed quasi-ordered sets and lattices are direct sum indecomposable, that was used in the proof of Theorem 2.1.

Proof of Theorem 3.1. Note first that M. Ćirić, S. Bogdanović and T. Petković defined in [7] two operators $U : \xi \mapsto \xi U$ and $L : \xi \mapsto \xi L$ on the Boolean algebra $\mathfrak{B}(Q)$ of all binary relations on Q by: $\xi U = \xi\xi^{-1}$ and $\xi L = \xi^{-1}\xi$, $\xi \in \mathfrak{B}(Q)$. Equivalently, for $\xi \in \mathfrak{B}(Q)$ we have

$$\begin{aligned} (a, b) \in \xi U \quad &\Leftrightarrow \quad (\exists c \in Q)\ a\,\xi\, c\ \ \&\ \ b\,\xi\, c; \\ (a, b) \in \xi L \quad &\Leftrightarrow \quad (\exists c \in Q)\ c\,\xi\, a\ \ \&\ \ c\,\xi\, b. \end{aligned}$$

In the same paper it was proved that $\xi E = \xi RUT = \xi RLT$, for each $\xi \in \mathfrak{B}(Q)$, which yields $\xi E = \xi UT = \xi LT$, when ξ is reflexive.

Set now $\xi = \preccurlyeq$. Then $\sigma = \xi E = \xi UT = \xi LT$, $(a, b) \in \xi L \Leftrightarrow I(a) \between I(b)$ and $(a, b) \in \xi U \Leftrightarrow F(a) \between F(b)$, whence it follows that the conditions (i), (ii) and (iii) are equivalent. $\square$

Now we give a characterization of the lattice of direct sum decompositions of a quasi-ordered set in terms of double ideals:

Theorem 3.2. *The lattice of direct sum decompositions of a quasi-ordered set Q is isomorphic to the lattice of complete Boolean subalgebras of the lattice $\boldsymbol{D}(Q)$ of double ideals of Q.*

Proof. Let $\boldsymbol{B}$ be an arbitrary complete Boolean subalgebra of $\boldsymbol{D}(Q)$. Then it is atomic, by Theorem 4.6 of [9]. We will first prove that the atoms of $\boldsymbol{B}$ are exactly the principal elements of $\boldsymbol{B}$. For $a \in Q$, let $B(a)$ denote the principal element of $\boldsymbol{B}$ generated by a. Assume $B \in \boldsymbol{B}$ such that $B \subseteq B(a)$. If $a \in B$, then $B(a) \subseteq B$, and hence $B = B(a)$. Otherwise, if $a \notin B$, then $a \in B(a) \setminus B = B(a) \cap B'$, where B' denotes the complement of B in $\mathcal{P}(Q)$. But, $B(a)$ and B' belong to $\boldsymbol{B}$, so $B(a) \setminus B$ also belongs to $\boldsymbol{B}$ and it contains a, whence $B(a) \setminus B = B(a)$, that is $B = \varnothing$. Thus, we have proved that each principal element of $\boldsymbol{B}$ is an atom in $\boldsymbol{B}$. Conversely, let B be an arbitrary atom in $\boldsymbol{B}$. Assume an arbitrary $a \in B$. Then $B(a) \subseteq B$ and $B(a) \neq \varnothing$, so we have $B(a) = B$. Hence, each atom of $\boldsymbol{B}$ is a principal element of $\boldsymbol{B}$.

We also have that the set of all atoms of $\boldsymbol{B}$ is the set of all summands of some direct sum decomposition of Q. Let $\mathcal{D}(\boldsymbol{B})$ denote this decomposition. The corresponding direct sum congruence $\sigma_{\boldsymbol{B}}$ on Q is then defined by: $(a,b) \in \sigma_{\boldsymbol{B}} \Leftrightarrow B(a) = B(b)$, for $a, b \in Q$.

Let $\mathcal{D}$ be an arbitrary direct sum decomposition of Q. Then its summands are double ideals of Q and they are exactly the atoms of a complete Boolean subalgebra $\boldsymbol{B}(\mathcal{D})$ of $\boldsymbol{D}(Q)$ defined in the following way: $\boldsymbol{B}(\mathcal{D})$ consists of the empty set and all subsets of Q that are unions (finite or infinite) of some family of summands of $\mathcal{D}$. In other words, $\boldsymbol{B}(\mathcal{D})$ consists of the empty set and all subsets of H that are saturated by the direct sum equivalence on Q which corresponds to $\mathcal{D}$.

It is easy to see that $\boldsymbol{B}(\mathcal{D}(\boldsymbol{B})) = \boldsymbol{B}$, for each complete Boolean subalgebra $\boldsymbol{B}$ of $\boldsymbol{D}(Q)$, and $\mathcal{D}(\boldsymbol{B}(\mathcal{D})) = \mathcal{D}$, for each direct sum decomposition $\mathcal{D}$ of Q. Therefore, the mappings $\boldsymbol{B} \mapsto \mathcal{D}(\boldsymbol{B})$ and $\mathcal{D} \mapsto \boldsymbol{B}(\mathcal{D})$ are mutually inverse bijections of the lattice of complete Boolean subalgebras of $\boldsymbol{D}(Q)$ onto the lattice of direct sum decompositions of Q, and vice versa. Therefore, it remains to prove that they are order isomorphisms, i.e. that both of these mappings are isotone.

Let $\boldsymbol{B}$ and $\boldsymbol{B}'$ be two complete Boolean subalgebras of $\boldsymbol{D}(Q)$ such that $\boldsymbol{B} \subseteq \boldsymbol{B}'$. Assume an arbitrary summand D of $\mathcal{D}(\boldsymbol{B}')$. Then $D = B'(a)$, i.e. it is the principal element of $\boldsymbol{B}'$ generated by some element $a \in Q$. But, $\boldsymbol{B} \subseteq \boldsymbol{B}'$ implies $D = B'(a) \subseteq B(a)$, where $B(a)$ is the principal element of $\boldsymbol{B}$ generated by a, and hence a summand in $\mathcal{D}(\boldsymbol{B})$. Therefore, we have proved $\mathcal{D}(\boldsymbol{B}) \leq \mathcal{D}(\boldsymbol{B}')$.

Assume now two direct sum decompositions $\mathcal{D}$ and $\mathcal{D}'$ of Q such that $\mathcal{D} \leq \mathcal{D}'$. Then every summand of $\mathcal{D}'$ is contained in some summand of $\mathcal{D}$, i.e. every atom of $\boldsymbol{B}(\mathcal{D}')$ is contained in some atom of $\boldsymbol{B}(\mathcal{D})$, so clearly

$\boldsymbol{B}(\mathcal{D}) \subseteq \boldsymbol{B}(\mathcal{D}')$, which was to be proved. This completes the proof of the theorem. $\square$

Finally, we prove the following representation theorem:

Theorem 3.3. *Every quasi-ordered set Q can be represented as a direct sum of direct sum indecomposable quasi-ordered sets.*

This decomposition is the greatest direct sum decomposition of Q and its summands are the principal double ideals of Q.

Proof. Let Q be an arbitrary quasi-ordered set. By Theorems 3.2 and 2.2 we have that there exists a greatest direct sum decomposition of Q and its summands are exactly the principal double ideals of Q, i.e. the atoms in $\boldsymbol{D}(Q)$. To prove the assertion of the theorem it remains to prove that each principal double ideal D of Q is direct sum indecomposable, or equivalently, that D does not have a proper double ideal. Let E be an arbitrary double ideal of D. By the hypothesis, $D = D(a)$, for some $a \in Q$, and as in the proofs of Theorems 2.2 and 3.2 we have the following: If $a \in E$, then $D(a) \subseteq E$, so $D = E$. Otherwise, if $a \notin E$, then $D \setminus E$ is a double ideal of Q containing a, whence we have $D(a) \subseteq D \setminus E$ and hence $D \setminus E = D$, i.e. $E = \varnothing$. Therefore, we have proved that D does not have a proper double ideal. This completes the proof of the theorem. $\square$

4. The lattice of ideals

There are numerous papers in which direct product decompositions of some lattices have been studied through the properties of their centers. For example, we refer to the recent papers of S. Bogdanović and M. Ćirić [3] and [4], M. Ćirić and S. Bogdanović [5] and [6], M. Ćirić, S. Bogdanović and T. Petković [7] and [8], L. Libkin [13] and [14], and L. Libkin and I. Muchnik [15]. Here we use a similar approach in investigation of the lattice of ideal of a quasi-ordered set. Namely, we connect direct product decompositions of the lattice of ideals of a quasi-ordered set Q with direct sum decompositions of Q.

Theorem 4.1. *The lattice $\boldsymbol{I}(Q)$ of ideals of a quasi-ordered set Q is a direct product of lattices L_α, $\alpha \in Y$, if and only if Q is a direct sum of quasi-ordered sets Q_α, $\alpha \in Y$, and $L_\alpha \cong \boldsymbol{I}(Q_\alpha)$, for each $\alpha \in Y$.*

Proof. Let $\boldsymbol{I}(Q)$ be a direct product of lattices L_α, $\alpha \in Y$. For each $\alpha \in Y$, L_α is a homomorphic image of $\boldsymbol{I}(Q)$, with respect to the projection homomorphism π_α of $\boldsymbol{I}(Q)$ onto L_α, so L_α has a zero 0_α and a unity 1_α. Let $Q_\alpha \in \boldsymbol{I}(Q)$ be an element satisfying the following condition:

$$Q_\alpha \pi_\beta = \begin{cases} 1_\alpha & \text{for } \beta = \alpha \\ 0_\alpha & \text{for } \beta \neq \alpha \end{cases}, \qquad (\beta \in Y).$$

Then L_α is isomorphic to the principal ideal of $\boldsymbol{I}(Q)$ generated by Q_α. On the other hand, the principal ideal of $\boldsymbol{I}(Q)$ generated by Q_α is isomorphic to $\boldsymbol{I}(Q_\alpha)$, since for every subset I of Q_α, I is an ideal of Q_α if an only if it is an ideal of Q. Hence, $L_\alpha \cong \boldsymbol{I}(Q_\alpha)$, for each $\alpha \in Y$. We obtain immediately that Q is a direct sum of its subsets Q_α, $\alpha \in Y$.

To prove the converse, assume that Q is a direct sum of its subsets Q_α, $\alpha \in Y$. For $\alpha \in Y$ let L_α denote the principal ideal of $\boldsymbol{I}(Q)$ generated by Q_α, let $L = \prod_{\alpha \in Y} L_\alpha$, and for $\alpha \in Y$, let π_α denote the projection homomorphism of L onto L_α. Then the mapping $\phi : \boldsymbol{I}(Q) \to L$ defined by

$$(I\phi)\pi_\alpha = I \cap Q_\alpha \qquad\qquad (I \in \boldsymbol{I}(Q),\ \alpha \in Y),$$

is an isomorphism, seeing that $\boldsymbol{I}(Q)$ is infinitely distributive for meets. Finally, $L_\alpha = \boldsymbol{I}(Q_\alpha)$, for each $\alpha \in Y$. This completes the proof of the theorem. $\square$

Using the above theorem and Theorem 3.3 we obtain the following:

Corollary 4.1. *Let Q be an arbitrary quasi-ordered set. Then the lattice $\boldsymbol{I}(Q)$ can be represented as a direct product of directly indecomposable lattices, $\boldsymbol{I}(Q) \cong \prod_{\alpha \in Y} \boldsymbol{I}(Q_\alpha)$, where $Q = \sum_{\alpha \in Y} Q_\alpha$ is a representation of Q as the direct sum of direct sum indecomposable quasi-ordered sets.*

Finally, we give the following characterizations of direct sum indecomposable quasi-ordered sets:

Theorem 4.2. *The following conditions on a quasi-ordered set Q are equivalent:*

(i) *$\boldsymbol{I}(Q)$ is a direct product indecomposable lattice;*
(ii) *Q is a direct sum indecomposable quasi-ordered set;*
(iii) *Q has no proper double ideals;*
(iv) *$\boldsymbol{D}(Q)$ is a two-element Boolean algebra;*
(v) *for all $a, b \in Q$ there exists a sequence $c_1, c_2, \dots, c_n \in A$ such that*

$$I(a) \between I(c_1) \between I(c_2) \between \cdots \between I(c_n) \between I(b);$$

(vi) *for all $a, b \in Q$ there exists a sequence $c_1, c_2, \dots, c_n \in A$ such that*

$$F(a) \between F(c_1) \between F(c_2) \between \cdots \between F(c_n) \between F(b).$$

Proof. (i) $\Leftrightarrow$ (ii). This follows by Theorem 4.1.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). This is a consequence of Theorems 3.2 and 2.1.

(ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). This follows by Theorem 3.1. $\square$

As we noted earlier, all the results proved here for quasi-ordered sets, can be easily translated to the case of partially ordered sets. Therefore, Theorem 4.1 holds also for the lattice of ideals of a poset. On the other hand, it is known which lattices can be represented as the lattices of ideals of some poset. In addition, such lattices were studied in Chapter 10 of the book of P. Crawley and R. P. Dilworth [9]. We intend to apply the above results to these lattices.

Recall first some definitions and notations. An element a of a complete lattice L is called *compact* if for arbitrary subset H of L, $c \leq \bigvee H$ implies $c \leq \bigvee H_0$, for some finite subset H_0 of H. A lattice L is defined to be *compactly generated* or *algebraic* if L is complete and every non-zero element of L is the join of some family of compact elements of L. A lattice L is called *dually algebraic* if its dual lattice is algebraic. An element $a \in L$ is called *completely join-irreducible* if $a \neq 0$ and for every subset K of L, $a = \bigvee K$ implies $a \in K$. The set of all completely join-irreducible elements of L will be denoted by $J(L)$. Let us observe that if L is an algebraic lattice, then any completely join-irreducible element is compact.

The following theorem, taken from the book of P. Crawley and R. P. Dilworth [9], characterizes the lattices that can be represented as the lattice of ideals of a poset:

Theorem 4.3. *The following conditions on a lattice L are equivalent:*

(i) *L is distributive, algebraic and dually algebraic;*
(ii) *L is distributive, algebraic and every non-zero element of L is the join of some family of completely join-irreducible elements of L;*
(iii) *L is isomorphic to the lattice $\boldsymbol{I}(P)$ of ideals of some poset P;*
(iv) *L is a complete sublattice of some complete atomic Boolean algebra.*

The poset P from the above theorem is in fact the poset $J(L)$ of all completely join-irreducible elements of L. This and Theorem 4.1 motivate us to establish the following connection between the direct product decompositions of L and direct sum decompositions of the poset $J(L)$.

Theorem 4.4. *Let L be a distributive algebraic lattice in which every element is the join of some family of completely join-irreducible elements and let $P = J(L)$. Then P is a direct sum of posets P_α, $\alpha \in Y$, if and only if L is a direct product of lattices L_α, $\alpha \in Y$, and $J(L_\alpha) \cong P_\alpha$, for every $\alpha \in Y$.*

Proof. Let P be a direct sum of posets P_α, $\alpha \in Y$. First we prove that $\alpha, \beta \in Y$, $\alpha \neq \beta$, $p_\alpha \in P_\alpha$ and $p_\beta \in P_\beta$ implies $p_\alpha \wedge p_\beta = 0$. Indeed, if $p_\alpha \wedge p_\beta \neq 0$, then $p_\alpha \wedge p_\beta = \bigvee H$, for some $H \subseteq P$. If $h \in H$, then $h \leq p_\alpha \wedge p_\beta$, whence $h \leq p_\alpha$ and $h \leq p_\beta$. But, this yields $h \in P_\alpha$ and $h \in P_\beta$, which is not possible. Therefore, we have proved that $p_\alpha \wedge p_\beta = 0$.

For $\alpha \in Y$ let L_α be the set of all elements of L which can be represented as the join of some subset of $P_\alpha \cup \{0\}$. By Theorem 4.3, L is completely distributive so L_α is a complete sublattice of L. Clearly, $P_\alpha \subseteq J(L_\alpha)$. To prove the opposite inclusion assume an arbitrary $a \in J(L_\alpha)$ and suppose that $a = \bigvee H$, for some $H \subseteq L$. By the assumptions of the theorem, without loss of generality we can assume that $H \subseteq P$. But now $a \leq b$, for every $b \in H$, whence $H \subseteq P_\alpha \subseteq L_\alpha$. Since $a \in J(L_\alpha)$, then $a \in H$, which was to be proved. Therefore, $J(L_\alpha) \cong P_\alpha$.

Further, for $\alpha, \beta \in Y$, $\alpha \neq \beta$, $x_\alpha \in L_\alpha$ and $x_\beta \in L_\beta$ we have $x_\alpha \wedge x_\beta = 0$. Indeed, if $x_\alpha \neq 0$ and $x_\beta \neq 0$, then $x_\alpha = \bigvee H$ and $x_\beta = \bigvee G$, for some $H \subseteq P_\alpha$ and $G \subseteq P_\beta$. Since L is infinitely distributive (by Theorems 4.6 of [9] and Theorem 4.3), then $x_\alpha \wedge x_\beta = (\bigvee H) \wedge (\bigvee G) = \bigvee_{h \in H} \bigvee_{g \in G} (h \wedge g) = 0$.

For $x \in L$, $x \neq 0$, and $\alpha \in Y$ let $P_\alpha^x = \{a \in P_\alpha \,|\, a \leq x\}$ and $x_\alpha = \bigvee P_\alpha^x$, and for $x = 0$ let $x_\alpha = 0$, for every $\alpha \in Y$. If $x \neq 0$, then $x = \bigvee H$, for some $H \subseteq P$. If we set $H_\alpha = H \cap P_\alpha$, for $\alpha \in Y$, then $H_\alpha \subseteq P_\alpha^x$, for each $\alpha \in Y$, whence $x = \bigvee H = \bigvee_{\alpha \in Y} \big(\bigvee H_\alpha\big) \leq \bigvee_{\alpha \in Y} \big(\bigvee P_\alpha^x\big) = \bigvee_{\alpha \in Y} x_\alpha \leq x$. Thus, $x = \bigvee_{\alpha \in Y} x_\alpha$.

Define a mapping φ of L into the direct product of L_α, $\alpha \in Y$, by: $x\varphi = (x_\alpha)_{\alpha \in Y}$. To prove that φ is onto assume $(y_\alpha)_{\alpha \in Y}$ with $y_\alpha \in L_\alpha$, for each $\alpha \in Y$. Let $x = \bigvee_{\alpha \in Y} y_\alpha$. Since $x_\alpha \wedge y_\beta = 0$, whenever $\alpha \neq \beta$, for an arbitrary $\alpha \in Y$ we have that $y_\alpha = y_\alpha \wedge x = y_\alpha \wedge \big(\bigvee_{\beta \in Y} x_\beta\big) = \bigvee_{\beta \in Y} (y_\alpha \wedge x_\beta) = y_\alpha \wedge x_\alpha = \bigvee_{\beta \in Y} (y_\beta \wedge x_\alpha) = \big(\bigvee_{\beta \in Y} y_\beta\big) \wedge x_\alpha = x \wedge x_\alpha = x_\alpha$. Hence, $y_\alpha = x_\alpha$, for each $\alpha \in Y$, so $x\varphi = (y_\alpha)_{\alpha \in Y}$, which was to be proved.

Assume arbitrary $x, y \in L$. If $x \leq y$, then $P_\alpha^x \subseteq P_\beta^x$ and $x_\alpha \leq y_\alpha$, for every $\alpha \in Y$, so $x\varphi \leq y\varphi$. On the other hand, if $x\varphi \leq y\varphi$, i.e. $x_\alpha \leq y_\alpha$, for each $\alpha \in Y$, then $x = \bigvee_{\alpha \in Y} x_\alpha \leq \bigvee_{\alpha \in Y} y_\alpha = y$. Therefore, φ is an order isomorphism, so it is a complete lattice isomorphism of L onto the direct product of lattices L_α, $\alpha \in Y$.

Conversely, let L be the direct product of lattices L_α, $\alpha \in Y$. For each $\alpha \in Y$, L_α is isomophic to the complete sublattice of L consisting of all elements $x \in L$ for which $x\pi_\beta = 0$, whenever $\beta \neq \alpha$, $\beta \in Y$ (here π_β denotes the projection homomorphism of L onto L_β. We will identify these two lattices, and we then have that $\bigcap_{\alpha \in Y} L_\alpha = \{0\}$, $a \wedge b = 0$, whenever $a \in L_\alpha$, $b \in L_\beta$ and $\alpha \neq \beta$, $\alpha, \beta \in Y$, and every $x \in L$ has a unique representation $x = \bigvee_{\alpha \in Y} x_\alpha$, with $x_\alpha \in L_\alpha$, for each $\alpha \in Y$.

For $\alpha \in Y$ let $P_\alpha = J(L_\alpha)$. Assume first an arbitrary $a \in P$. Then $a = \bigvee_{\alpha \in Y} a_\alpha$, with $a_\alpha \in L_\alpha$, for each $\alpha \in Y$, and since $a \in P = J(L)$, then $a = a_\alpha$, so $a \in L_\alpha$, for some $\alpha \in Y$. Furthermore, $a \in J(L)$ yields $a \in J(L_\alpha) = P_\alpha$. Therefore, we have proved $P \subseteq \bigcup_{\alpha \in Y} P_\alpha$. To prove the opposite inclusion, assume arbitrary $\alpha \in Y$ and $a \in P_\alpha = J(L_\alpha)$. Suppose that $a = \bigvee H$, for some $H \subseteq L$. For $\beta \in Y$ let $H_\beta = \big(H \cup \{0\}\big) \cap L_\beta$ and

$a_\beta = \bigvee H_\beta$. Now $a = \bigvee H = \bigvee_{\beta\in Y}\big(\bigvee H_\beta\big) = \bigvee_{\beta\in Y} a_\beta$, with $a_\beta \in L_\beta$, for every $\beta\in Y$. But, the uniqueness of the representation of this form yields $a_\alpha = a$ and $a_\beta = 0$, whenever $\beta \neq \alpha$, $\beta \in Y$. Therefore, $a = \bigvee H_\alpha$ and $H_\alpha \subseteq L_\alpha$, so $a \in J(L_\alpha)$ implies $a \in H_\alpha \subseteq H$, which was to be proved. Thus $a \in P$, and we have proved that $P = \bigcup_{\alpha\in Y} P_\alpha$.

It remains to prove that $P_\alpha \parallel P_\beta$, whenever $\alpha \neq \beta$, $\alpha,\beta \in Y$. Indeed, assume $\alpha,\beta\in Y$ such that $\alpha\neq\beta$, and assume arbitrary $a \in P_\alpha$ and $b \in P_\beta$. Then $a\wedge b = 0$, so $a \leq b$ implies $a = a\wedge b = 0$, and $b \leq a$ implies $b = b\wedge a = 0$, which is not possible, whence we conclude that $a \parallel b$. Therefore, we have proved that P is a direct sum of posets P_α, $\alpha\in Y$. This completes the proof of the theorem. $\square$

Theorem 4.4 and Corollary 4.1 give the following corollary:

Corollary 4.2. *Let L be a distributive algebraic lattice in which every element is the join of some family of completely join-irreducible elements and let $P = J(L)$. Then L can be represented as a direct product $L \cong \prod_{\alpha\in Y} L_\alpha$ of directly indecomposable lattices $L_\alpha \cong \boldsymbol{I}(P_\alpha)$, where where $P = \sum_{\alpha\in Y} P_\alpha$ is a representation of the poset P as the direct sum of direct sum indecomposable posets.*

5. Some applications

In the last section of the paper we talk about some applications of the results obtained in the previous sections. First we talk about certain applications in Semigroup Theory.

Let S be a semigroup and let S^1 denote a semigroup obtained from S by adjoining a new element to be an identity in S^1. Define the following relations on S:

$$a \mid b \;\Leftrightarrow\; b \in S^1aS^1; \qquad a \underset{l}{\mid} b \;\Leftrightarrow\; b \in S^1a;$$
$$a \underset{r}{\mid} b \;\Leftrightarrow\; b \in aS^1; \qquad a \underset{t}{\mid} b \;\Leftrightarrow\; b \in aS^1 \cap S^1a.$$

These relations are called the *division*, *left division*, *right division* and *two-sided division relations* on S, respectively. All of these relations are quasi-orders on S. Furthermore, the division relation is the join, and the two-sided division relation is the meet of the left division and the right division relation on S in the lattice of quasi-orders on S.

A congruence ϱ on a semigroup S is called a *right zero band* (resp. *left zero band*, *matrix*) congruence on S if S/ϱ is a right zero band (resp. left zero band, rectangular band), that is if $xy\,\varrho\, y$ (resp. $xy\,\varrho\, x$, $xyx\,\varrho\, x\,\varrho\, x^2$), for all

$x, y \in S$. The corresponding partition of S is called a *right zero band* (resp. *left zero band*, *matrix*) *decomposition* of S. The following theorem gives a connection between these decompositions and direct sum decompositions of the above defined quasi-ordered sets.

Theorem 5.1. *Let S be a semigroup and let we also consider S as a quasi-ordered set with respect to the left division* (*resp. right division, two-sided division*) *relation on S. Then the lattice of direct sum decompositions of the quasi-ordered set S is isomorphic to the lattice of right zero band* (*resp. left zero band, matrix*) *decompositions of the semigroup S.*

Proof. We will prove only the assertion concerning the left division relation. The remaining assertions can be proved similarly.

Let the quasi-ordered set S be a direct sum of its subsets S_α, $\alpha \in Y$. If $\alpha, \beta \in Y$, $a \in S_\alpha$ and $b \in S_\beta$, then $b \underset{l}{|} ab$ implies $ab \in S_\beta$, so $S_\alpha S_\beta \subseteq S_\beta$. Therefore, the semigroup S is a right zero band of semigroups S_α, $\alpha \in Y$.

Conversely, let the semigroup S be a right zero band of semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \underset{l}{|} b$, that is $b = xa$, for some $x \in S^1$. If $a \in S_\alpha$, for some $\alpha \in Y$, then we have that $b = xa \in S_\alpha$, too. Hence, the quasi-ordered set S is a direct sum of its subsets S_α, $\alpha \in Y$.

Therefore, we have proved that there exists a correspondence between the direct sum decompositions of the quasi-ordered set S and right zero band decompositions of the semigroup S. It is easy to verify that this correspondence is an order isomorphism, and hence, it is a complete lattice isomorphism. $\square$

Using Theorem 5.1 and the results from the precedding sections we can obtain the results proved by M. Petrich in [16], concerning right zero band, left zero band and matrix decompositions. More information about these decompositions can be also found in the Petrich's book [17], and the survey paper of M. Ćirić and S. Bogdanović [5].

Let us observe that the division relation on S does not appear in Theorem 5.1. The reason is the following: the quasi-ordered set which corresponds to the division relation on a semigroup is directed, so it is direct sum indecomposable. A similar situation arises when a semigroup S has a zero. In this case the quasi-ordered sets corresponding to all division quasi-orders on S are direct sum indecomposable, seeing that they have top elements.

As in the case of posets, an element a of a quasi-ordered set Q is called a *greatest element* of Q if $x \preccurlyeq a$, for every $x \in Q$. But, a greatest element of a quasi-ordered set is not necessary unique. If a quasi-ordered set Q has a unique greatest element, it will be called a *top element* and it will be denoted by $\top$. For $H \subseteq Q$ we write $H^* = H \setminus \{\top\}$. Clearly, a quasi-ordered set having a top element is direct sum indecomposable. On the other hand, if Q is a direct sum of quasi-ordered sets Q_α, $\alpha \in Y$, each of whose has a top element, then Q does not have a top element. As was notes by B. A. Davey and H. A. Priestley in [10], there are two general ways to modify this construction and stay within the class of quasi-ordered sets with a top element. The first one is to add a new element to Q to be a top element. The quasi-ordered set obtained in this manner is called a *separated sum* of Q_α, $\alpha \in Y$. Another way is to identify the top elements of all Q_α, $\alpha \in Y$. So constructed quasi-ordered set is called a *coalesced sum* of Q_α, $\alpha \in Y$.

In other words, a quasi-ordered set Q with the top element $\top$ is called a *coalesced sum* of its subsets Q_α, $\alpha \in Y$, if $Q = \bigcup_{\alpha\in Y} Q_\alpha$, and $Q_\alpha \cap Q_\beta = \varnothing$ and $Q^*_\alpha \parallel Q^*_\beta$, whenever $\alpha \neq \beta$, $\alpha, \beta \in Y$. The partition of Q whose components are $\{\top\}$ and Q^*_α, $\alpha \in Y$, is called a *coalesced sum decomposition* of Q. It is not hard to prove the following proposition:

Proposition 5.1. *Let Q be a quasi-ordered set with a top element. Then coalesced sum decompositions of Q form a complete lattice which is isomorphic to the lattice of direct sum decompositions of the quasi-ordered set Q^*.*

Now we are ready to consider certain decompositions of semigroups with zero.Let S be a semigroup with zero 0. We say that S is a 0-*sum* of semigroups S_α, $\alpha \in Y$, if $S = \bigcup_{\alpha\in Y} S_\alpha$, and $S_\alpha \cap S_\beta = \{0\}$, whenever $\alpha \neq \beta$, $\alpha, \beta \in Y$. The partition $\mathcal{D}$ of S whose components are $\{0\}$ and $S_\alpha \setminus \{0\}$, $\alpha \in Y$, is called a 0-*sum decomposition* of S. We distinguish several special types of 0-sum decompositions. We say that $\mathcal{D}$ is an *orthogonal sum decomposition* if $S_\alpha S_\beta = S_\beta S_\alpha = \{0\}$, whenever $\alpha \neq \beta$, $\alpha, \beta \in Y$. If $S_\alpha S_\beta \subseteq S_\alpha$, then $\mathcal{D}$ is a *left sum decomposition*, and if $S_\alpha S_\beta \subseteq S_\beta$, then it is called a *right sum decomposition* of S. Finally, if $Y \subseteq I \times \Lambda$, for some non-empty sets I and Λ, and if $S_{(i,\lambda)} S_{(j,\mu)} \subseteq S_{(i,\mu)}$, whenever $(i, \mu) \in Y$, and $S_{(i,\lambda)} S_{(j,\mu)} = \{0\}$, otherwise, then we say that $\mathcal{D}$ is a *matrix sum decomposition* of S.

Similarly as Theorem 5.1 we prove the following theorem:

Theorem 5.2. *Let S be a semigroup with zero and let we consider S as a quasi-ordered set with respect to the division* (*resp. left division, right division, two-sided division*) *relation on S. Then the lattice of direct sum decompositions of the quasi-ordered set S is isomorphic to the lattice of orthogonal* (*resp. right, left, matrix*) *sum decompositions of the semigroup S.*

From the above theorem and the results from the previous sections we obtain the results of S. Bogdanović and M. Ćirić from [3] and [4] concerning certain general properties of orthogonal, left, right and matrix sum decompositions of semigroups with zero. More information about these decompositions can be found in the book of S. Bogdanović and M. Ćirić [1], and their survey papers [2] and [5].

Further we consider direct sum decompositions of automata. A general theory of these decompositions was developed M. Ćirić and S. Bogdanović in [6], and also by M. Ćirić, S. Bogdanović and T. Petković in [7] and [8].

By an automaton we mean an automaton without outputs (in terms from the book of F. Gécseg and I. Péak [11]). Equivalently, we consider automata as algebras whose all fundamental operations are unary. We say that an automaton A is a *direct sum* of its subautomata A_α, $\alpha \in Y$, if $A = \bigcup_{\alpha\in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \varnothing$, whenever $\alpha \neq \beta$, $\alpha, \beta \in Y$. The corresponding partition of A is called a *direct sum decomposition* of A.

Direct sum decompositions of an automaton A can be studied through the quasi-order on A defined as follows: for two states $a, b \in A$ we write $a \mid b$ if there exists an input word u that induces a transition from the state a into the state b. Such defined relation is a quasi-order on A, and analogously to the related relation in Semigroup Theory, it is called the *division relation* on A. Without proof we give the following theorem:

Theorem 5.3. *Let A be an automaton and let us also consider A as a quasi-ordered set with respect to the division relation on A. Then the lattice of direct sum decompositions of the quasi-ordered set A is isomorphic to the lattice of direct sum decompositions of the automaton A.*

Similar results can be also given for graphs. Namely, we define a graph G to be a *direct sum* of its subsets G_α, $\alpha \in Y$, if $G = \bigcup_{\alpha\in Y} G_\alpha$, and arbitrary vertices $a \in G_\alpha$ and $b \in G_\beta$ are not incident whenever $\alpha \neq \beta$, $\alpha, \beta \in Y$. The corresponding partition of G is called a *direct sum decomposition* of G. A relation π on G will be defined in the following way: for vertices $a, b \in G$, $(a, b) \in \pi$ if there exists a path from a to b. Then this relation is a quasi-order on G and the following theorem holds:

Theorem 5.4. *Let G be a graph and let us also consider G as a quasi-ordered set, with respect to the relation π on G. Then the lattice of direct sum decompositions of the quasi-ordered set G is isomorphic to the lattice of direct sum decompositions of the graph G.*

Note that the summands in the greatest direct sum decomposition of a graph G are exactly the components of connectedeness of G.

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