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Survey Article

SEMILATTICES OF ARCHIMEDEAN SEMIGROUPS AND (COMPLETELY) π-REGULAR SEMIGROUPS I

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Dedicated to Professor Svetozar Milić on his 60th birthday

Contents

Introduction and preliminaries

Here we adapt the point of view, mostly propagated by T.Tamura, M.Petrich, M.S.Putcha, Л.Н.Шеврин and the authors, that a semigroup should be studied through its greatest semilattice decomposition. The idea consists of decomposing the given semigroup into subsemigroups (components), possibly of considerably

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simpler structure, studying these in detail, and finally studying their mutual relationships within the entire semigroup. Recall that semilattice decompositions of semigroups were first defined and studied by A.H.Clifford [56].

T.Tamura and N.Kimura [161] showed that every commutative semigroup is a semilattice of Archimedean semigroups. This well known result has since been generalized by many authors. The first characterization of semilattices of Archimedean semigroups is due to M.S.Putcha [113] (see also T.Tamura [158]). Some other characterizations of these semigroups are given by M.Ćirić and S.Bogdanović [44].

R.Arens and I.Kaplansky [2] and I.Kaplansky [81] investigated, as generalizations of algebraic algebras and rings with minimum conditions, following two types of rings: One are *π-regular* rings and the other are *right π-regular* rings (for definitions see below).

We present here a summary of the main results on the decompositions of (*completely*) *π-regular* semigroups into semilattices of Archimedean semigroups. These decompositions were first studied by M.S.Putcha [113]. This very important matter is after treated by Л.Н.Шеврин [146,147,148] (see also [130]) and intensively by J.L.Galbiati and M.L.Veronesi [69,70,71,72,73] and after all in a series of papers by the authors of this work.

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers and $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{H}$ will denote known Green's relations. By $Reg(S)$ ($Gr(S), E(S)$, $Intra(S)$) we denote the set of all *regular* (*completely regular, idempotent, intra-regular*) elements of a semigroup S. If e is an idempotent of a semigroup S, then by G_e we denote the maximal subgroup of S with e as its identity. It is known that $Gr(S) = \cup\{G_e \mid e \in E(S)\}$. A nonzero idempotent e of a semigroup S is *primitive* if for every nonzero $f \in E(S)$, $f = ef = fe \Rightarrow f = e$, i.e. if e is minimal in the set of all nonzero idempotents of S relative to the partial order on this set.

By a *radical* of the subset A of a semigroup S we mean the set $\sqrt{A}$ defined by $\sqrt{A} = \{a \in S \mid (\exists n \in \mathbf{Z}^+)\ a^n \in A\}$. By $S = S^0$ we denote that S is a semigroup with the zero 0 and in this case $S^* = S - \{0\}$. If $S = S^0$, then element from the set $Nil(S) = \sqrt{\{0\}}$ are *nilpotent elements* (*nilpotents*). A semigroup $S = S^0$ is a *nil-semigroup* if $S = Nil(S)$. A semigroup $S = S^0$ is *n-nilpotent* if $S^n = \{0\}$, $n \in \mathbf{Z}^+$. An ideal extension S of T is a *nil-extension* if S/T is a nil-semigroup (i.e. $S = \sqrt{T}$). An ideal extension S of a semigroup K is a *n-nilpotent extension* if S/K is a n-nilpotent semigroup.

Let $a, b \in S$. Then $a \mid b$ if $b \in S^1aS^1$, $a \underset{l}{\mid} b$ if $b \in S^1a$, $a \underset{r}{\mid} b$ if $b \in aS^1$, $a \underset{t}{\mid} b$ if $a \underset{l}{\mid} b$ and $a \underset{r}{\mid} b$, $a \longrightarrow b$ if $a \mid b^n$ for some $n \in \mathbf{Z}^+$, $a \stackrel{h}{\longrightarrow} b$ if $a \underset{h}{\mid} b^n$, for some $n \in \mathbf{Z}^+$, where h is l, r or t, $a \stackrel{p}{—} b$ if $a^m = b^n$ for some $m, n \in \mathbf{Z}^+$, $a — b$ if $a \longrightarrow b \longrightarrow a$, $a \stackrel{h}{—} b$ if $a \stackrel{h}{\longrightarrow} b \stackrel{h}{\longrightarrow} a$, where h is l, r or t.

A semigroup S is: *Archimedean* if $S = \sqrt{SaS}$, *left Archimedean* if $S = \sqrt{Sa}$, *right Archimedean* if $S = \sqrt{aS}$, *t-Archimedean* if it is left Archimedean and right Archimedean, *power joined* if $\stackrel{p}{—} = S \times S$, *completely Archimedean* if it is Archimedean and has a primitive idempotent, *intra-π-regular* if $(\forall a \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sa^{2n}S$, *left π-regular* if $(\forall a \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sa^{n+1}$, *$\pi$-regular* if $(\forall a \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^nSa^n$, *completely π-regular* if $(\forall a \in S)(\exists n \in \mathbf{Z}^+)(\exists x \in S)\ a^n = a^nxa^n,\ a^nx = xa^n$ (equivalently if $S = \sqrt{Gr(S)}$). On a completely π-regular semigroup can be introduced two unary operations $x \mapsto \overline{x}$ and $x \mapsto x^0$ by: $\overline{x} = (xe_x)^{-1}$, where $e_x \in E(S)$ such that $x^n \in G_{e_x}$ for some $n \in \mathbf{Z}^+$ and $^{-1}$ is the inversion in G_{e_x}, and $x^0 = x\overline{x}$, [143].

We will use the following notations for some classes of semigroups:

NOTATION	CLASS OF SEMIGROUPS	NOTATION	CLASS OF SEMIGROUPS
$\mathcal{A}$	*Archimedean*	$\mathcal{CA}$	*completely Archimedean*
$\mathcal{LA}$	*left Archimedean*	$\mathcal{LG}$	*left groups*
$\mathcal{TA}$	*t-Archimedean*	$\mathcal{G}$	*groups*
$\mathcal{B}$	*bands*	$\mathcal{N}$	*nil-semigroups*
$\mathcal{S}$	*semilattices*	$\mathcal{N}_k$	$(k+1)$-*nilpotent*
$\pi\mathcal{R}$	*π-regular*	$\mathcal{UG}$	*unions of groups*
$\mathcal{C}\pi\mathcal{R}$	*completely π-regular*	$\mathcal{CS}$	*completely simple*
$\mathcal{M}\times\mathcal{G}$	*rectangular groups*		

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be classes of semigroups. By $\mathcal{X}_1 \circ \mathcal{X}_2$ we denote the *Maljcev's product* of classes $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ρ such that S/ρ is in $\mathcal{X}_2$ and every ρ-class which is a subsemigroup is in $\mathcal{X}_1$ [90]. The related decomposition is an $\mathcal{X}_1 \circ \mathcal{X}_2$-*decomposition.* It is clear that $\mathcal{X} \circ \mathcal{B}$ ($\mathcal{X} \circ \mathcal{S}$) is the class of all bands (semilattices) of semigroups from the class $\mathcal{X}$. If $\mathcal{X}_2$ is a subclass of the class $\mathcal{N}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is a class of all semigroups which are ideal extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$. Also, in such a case, by $\mathcal{X}_1 \circledast \mathcal{X}_2$ we denote a class of all semigroups which are retract extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

In this paper we will use several semigroups given by the following presentations:

$$B_2 = \langle\ a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b\ \rangle\ ,$$
$$A_2 = \langle\ a, e \mid a^2 = 0,\ e^2 = e,\ aea = a,\ eae = e\ \rangle\ ,$$
$$N_m = \langle\ a \mid a^{m+1} = a^{m+2},\ a^m \neq a^{m+1}\ \rangle\ ,$$
$$L_{3,1} = \langle a, f \mid a^2 = a^3,\ f^2 = f,\ a^2f = a^2,\ fa = f \rangle\ .$$
$$LZ(n) = \langle\ a, e \mid a^{n+1} = a,\ e^2 = e,\ ea = a^ne = e\ \rangle\ ,$$
$$C_{1,1} = \langle\ a, e \mid a^2 = a^3,\ e^2 = e,\ ae = a,\ ea = a\ \rangle\ ,$$
$$C_{1,2} = \langle\ a, e \mid a^2 = a^3,\ e^2 = e,\ ae = a,\ ea = a^2\ \rangle\ ,$$
$$V = \langle\ e, f \mid e^2 = e,\ f^2 = f,\ fe = 0\ \rangle\ ,$$

$m, n \in \mathbf{Z}^+$, $n \geq 2$, and $R_{3,1}$ ($RZ(n)$, $C_{2,1}$) will be the dual semigroup of $L_{3,1}$ ($LZ(n)$, $C_{1,2}$). By L_2 (R_2) we denote the twoelement left zero (right zero) semigroup. Semigroups B_2 and A_2 are not semilattices of Archimedean semigroups. $L_{3,1}$ is a nil-extension of a union of groups, but it is not a retractive nil-extension of a union of groups. The semigroup $LZ(n)$ has $2n$ elements, it is a chain of the cyclic group $\langle a\rangle = \{a, a^2, \dots, a^n\}$ and the left zero band $\{e, ae, \dots, a^{n-1}e\}$, it is a union of groups and it is not a band of groups. Semigroups

$C_{1,1}$, $C_{1,2}$ and $C_{2,1}$ are examples for semigroups which are not nil-extensions of a union of groups, and the semigroup N_m is an example for a semigroup which is an $(m+1)$-nilpotent semigroup and it is not a m-nilpotent semigroup.

Let R be the ring $\mathbb{Z}$ of integers or the ring $\mathbb{Z}_p$ of residues of mod$p$, $p \geq 2$, and let $I = \{0, 1\} \subseteq R$. Define a multiplication on the set $R \times I \times I$ by

$$(m; i, \lambda)(n; j, \mu) = (m + n - (i - j)(\lambda - \mu); i, \mu),$$

$m, n \in R$, $i, j, \lambda, \mu \in I$. Then $R \times I \times I$ is a semigroup, and we will use notations: $E(\infty) = \mathbb{Z} \times I \times I$, $E(p) = \mathbb{Z}_p \times I \times I$. The semigroup $E(\infty)$ $(E(p))$ is isomorphic to the Rees matrix semigroup over the additive group of the ring $\mathbb{Z}$ $(\mathbb{Z}_p)$ with the sandwich matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and it is not a rectangular group.

Let S be a semigroup and let $a, b \in S$. By a *sequence between* a *and* b we mean a (possibly empty) finite sequence $(x_i)_{i=1}^n$ in S such that $a — x_1$, $x_i — x_{i+1} (i = 1, \dots, n-1)$, $x_n — b$. We call n the *length* of $(x_i)_{i=1}^n$. By $n = 0$ (or $(x_i)_{i=1}^n$ empty) we mean $a — b$. We say $(x_i)_{i=1}^n$ is *minimal* if it is nonempty and there is no sequence of smaller length (including the empty sequence) between a and b. By a *sequence from* a *to* b we mean a (possibly empty) finite sequence $(x_i)_{i=1}^n$ in S such that $a \longrightarrow x_1$, $x_i \longrightarrow x_{i+1} (i = 1, \dots, n-1)$, $x_n \longrightarrow b$. Again n is the *length* of $(x_i)_{i=1}^n$ and by $n = 0$ (or $(x_i)_{i=1}^n$ empty) we mean $a \longrightarrow b$. We say $(x_i)_{i=1}^n$ is *minimal* if it is nonempty and there is no sequence of smaller length (including the empty sequence) from a to b. The *rank* $\rho_1(S)$ of a semigroup S is a zero if there is no minimal sequence between any two points. Otherwise $\rho_1(S)$ is the supremum of the lengths of the minimal sequences between points in S. The *semirank* $\rho_2(S)$ of a semigroup S is a zero if there is no minimal sequence from a point to another in S. Otherwise $\rho_2(S)$ is the supremum of the lengths of the minimal sequences from one point to another in S, [116].

A subset A of a semigroup S is *consistent* if $xy \in A \Rightarrow x, y \in A$, $x, y \in S$. A subsemigroup A of a semigroup S is a *filter* if A is consistent. By $N(a)$ we denote the least filter of S containing an element a of S (i.e. the intersection of all filters of S containing a).

For undefined notions and notations we refer to [57], [58], [102] and [103].

Chapter 1. Semilattices of Archimedean semigroups

1.1. The general case

Theorem 1.1. *The following conditions on a semigroup* S *are equivalent:*

(i) $S \in \mathcal{A} \circ \mathcal{B}$;

(ii) $S \in \mathcal{A} \circ \mathcal{S}$;

(iii) $(\forall a, b \in S)\ a \mid b \ \Rightarrow \ a^2 \longrightarrow b$;

(iv) $(\forall a, b \in S)\ a^2 \longrightarrow ab$;

(v) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\ a^k \longrightarrow ab$;

(vi) $\sqrt{A}$ *is an ideal of* S*, for every ideal* A *of* S;

(vii) $\sqrt{SaS}$ *is an ideal of* S*, for every* $a \in S$;

(viii) in every homomorphic image with zero of S*, the set of all nilpotent elements is an ideal;*

(ix) $N(x) = \{y \in S \mid y \to x\}$, *for all* $x \in S$;
(x) $(\forall a, b, c \in S)\ a \longrightarrow b \wedge b \longrightarrow c \ \Rightarrow\ a \longrightarrow c$;
(xi) $(\forall a, b, c \in S)\ a \longrightarrow c \wedge b \longrightarrow c \ \Rightarrow\ ab \longrightarrow c$;
(xii) $\rho_1(S) = 0$;
$(xiii)$ $\rho_2(S) = 0$;
(xiv) $(\forall a, b \in S)\ \sqrt{SabS} = \sqrt{SaS} \cap \sqrt{SbS}$.

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are from M.S.Putcha [113], $(ii) \Leftrightarrow (xii) \Leftrightarrow (xiii)$ is, also, due by M.S.Putcha [116]. The equivalences $(ii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ are from M.Ćirić and S.Bogdanović [44], the conditions (vii) and $(viii)$ are from S.Bogdanović and M.Ćirić [24] and the conditions (x) and (xi) are from T.Tamura [158]. For some related results we refer to P.Protić [123]. For some more general results we refer to M.Ćirić and S.Bogdanović [48].

Corollary 1.1. [115] *Let* $S \in \mathcal{A} \circ \mathcal{S}$. *If* $S = S^0$, *then* $Nil(S)$ *is an ideal of* S.

Theorem 1.2. [24] *The following conditions on a semigroup* S *are equivalent:*
(i) $(\forall a, b \in S)\ a \underset{r}{\mid} b \ \Rightarrow\ a^2 \xrightarrow{r} b$;
(ii) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\ a^k \xrightarrow{r} ab$;
(iii) $(\forall a, b \in S)\ a^2 \xrightarrow{r} ab$;
(iv) $\sqrt{aS}$ *is a right ideal of* S, *for every* $a \in S$;
(v) $\sqrt{R}$ *is a right ideal of* S, *for every right ideal* R *of* S.

Theorem 1.3. *The following conditions on a semigroup* S *are equivalent:*
(i) $S \in \mathcal{LA} \circ \mathcal{S}$;
(ii) $(\forall a, b \in S)\ a \mid b \ \Rightarrow\ a \xrightarrow{l} b$;
(iii) $(\forall a, b \in S)\ a \xrightarrow{l} ab$;
(iv) $\sqrt{L}$ *is a (right) ideal of* S, *for every left ideal* L *of* S;
(v) $\sqrt{Sa}$ *is a (right) ideal of* S, *for all* $a \in S$;
(vi) $N(x) = \{y \in S \mid y \xrightarrow{l} x\}$, *for all* $x \in S$;
(vii) $(\forall a, b \in S)\ \sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb}$.

The equivalence $(i) \Leftrightarrow (ii)$ is from M.S.Putcha [115]. The conditions (iii) and (vi) are from S.Bogdanović [14] and (iv) and (v) are given by S.Bogdanović and M.Ćirić [24].

Theorem 1.4. *The following conditions on a semigroup* S *are equivalent:*
(i) $S \in \mathcal{TA} \circ \mathcal{S}$;
(ii) $(\forall a, b \in S)\ a \mid b \ \Rightarrow\ a \xrightarrow{t} b$;
(iii) $(\forall a, b \in S)\ b \xrightarrow{r} ab \wedge a \xrightarrow{l} ab$;
(iv) $\sqrt{B}$ *is an ideal of* S, *for every bi-ideal* B *of* S;
(v) $\sqrt{aSa}$ *is an ideal of* S, *for all* $a \in S$;
(vi) $N(x) = \{y \in S \mid y \xrightarrow{l} x \wedge y \xrightarrow{r} x\}$, *for all* $x \in S$.

The equivalence $(i) \Leftrightarrow (ii)$ is from M.S.Putcha [115], $(iii) \Rightarrow (vi)$ is from M.Petrich [101] and $(i) \Leftrightarrow (iii) \Leftrightarrow (vi)$ is from S.Bogdanović [14]. The condition (iv) is from S.Bogdanović and M.Ćirić [24]. *Weakly commutative* semigroups (semigroups satisfying the condition (iii)) are treated also by B.Pondeliček [110].

Theorem 1.5. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of simple semigroups;*
(ii) *$S \in \mathcal{A} \circ \mathcal{S}$ and S is intra π-regular;*
(iii) *S is intra π-regular and $(\forall a \in S)(\forall b \in Intra(S))\ a \mid b \ \Rightarrow \ a^2 \longrightarrow b$;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^{4n} \mid (ab)^n$;
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)(\forall k \in \mathbf{Z}^+)\ a^k \mid (ab)^n$.

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are given by M.S.Putcha [113].

Theorem 1.6. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of left simple semigroups;*
(ii) *S is intra π-regular and $(\forall a \in S)(\forall b \in Intra(S))\ a \underset{r}{\mid} b \ \Rightarrow \ a \xrightarrow{l} b$;*
(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^{2n+1} \xrightarrow{l} (ab)^n$;
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)(\forall k \in \mathbf{Z}^+)\ a^k \xrightarrow{l} (ab)^n$;
(v) *$S \in \mathcal{LA} \circ \mathcal{S}$ and S is left π-regular.*

The equivalence $(i) \Leftrightarrow (ii)$ is given by M.S.Putcha [113].

Theorem 1.7. [113] *A semigroup S is a semilattice of nil-extensions of bi-simple regular semigroups if and only if S is π-regular and for all $a \in S$, $e \in E(S)$, $a \mid e \ \Rightarrow \ a^2 \mid e$ and $a\mathcal{J}e \ \Rightarrow \ a\mathcal{D}e$.*

Theorem 1.8. *The following conditions on a semigroup S are equivalent:*
(i) *S is a nil-extension of a simple semigroup:*
(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sb^{2n}S$;
(iii) *S is an Archimedean intra π-regular semigroup;*
(iv) *S is an Archimedean semigroup with an intra-regular element.*

The equivalences $(i) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are from M.S.Putcha [113].

Theorem 1.9. [39] *A semigroup S is Archimedean and contains an idempotent if and only if S is a nil-extension of a simple semigroup with an idempotent.*

Theorem 1.10. *The following conditions on a semigroup S are equivalent:*
(i) *S is a nil-extension of a left simple semigroup;*
(ii) *S is left Archimedean and left π-regular;*
(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sb^{n+1}$.

A subset A of a semigroup S is *semiprimary* if

$$(\forall x, y \in S)(\exists n \in \mathbf{Z}^+)\ xy \in A \ \Rightarrow \ x^n \in A \vee y^n \in A.$$

A semigroup S is *semiprimary* if all of its ideals are semiprimary [9]. A subset A of a semigroup S is *completely prime* if $xy \in A \Rightarrow x \in A \vee y \in A$, $x, y \in S$. An ideal A of a semigroup S is a *completely prime ideal* of S if it is completely prime subset of S.

Theorem 1.11. [29] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a chain of Archimedean semigroups;*
(*ii*) *S is semiprimary;*
(*iii*) *$\sqrt{A}$ is a completely prime ideal of S, for every ideal A of S;*
(*iii*) *$\sqrt{A}$ is a completely prime subset of S, for every ideal A of S;*
(*iii*) *S is a semilattice of Archimedean semigroups and completely prime ideals of S are totally ordered.*

Proposition 1.1. [29] *Every right ideal of a semigroup S is semiprimary if and only if $(\forall a,b\in S)\ ab\xrightarrow{r}a\ \vee\ ab\xrightarrow{r}b$.*

Theorem 1.12. [29] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a chain of right Archimedean semigroups;*
(*ii*) *$\sqrt{R}$ is a completely prime ideal of S, for every right ideal R of S;*
(*iii*) *S is a semilattice of right Archimedean semigroups and every right ideal of S is semiprimary;*
(*iv*) *S is a semilattice of right Archimedean semigroups and*

$$(\forall a,b\in S)\ a\xrightarrow{r}b\ \vee\ b\xrightarrow{r}a.$$

Theorem 1.13. [29] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a chain of t-Archimedean semigroups;*
(*ii*) *$\sqrt{B}$ is a completely prime ideal of S, for every bi-ideal B of S;*
(*iii*) *S is a semilattice of t-Archimedean semigroups and every*

$$(\forall a,b\in S)\ a\xrightarrow{r}b\ \vee\ b\xrightarrow{r}a.$$

Theorem 1.14. [29] *The radical of every subsemigroup of a semigroup S is completely prime if and only if $(\forall a,b\in S)(\exists n\in\mathbf{Z}^+)\ a^n\in\langle ab\rangle\ \vee\ b^n\in\langle ab\rangle$.*

1.2. Bands of left Archimedean semigroups

By Theorem 1.1. we have that bands of Archimedean semigroups are semilattices of Archimedean semigroups, but the class of bands of left (or right or twosided) Archimedean semigroups is not equal to the class of semilattices of left (or right or twosided) Archimedean semigroups.

Theorem 1.15. [114] *$S\in\mathcal{LA}\circ\mathcal{B}$ if and only if $xay\overset{l}{—}xa^2y$, for all $a\in S$, $x,y\in S^1$.*

Theorem 1.16. [114] *$S\in\mathcal{TA}\circ\mathcal{B}$ if and only if $xay\overset{t}{—}xa^2y$, for all $a\in S$, $x,y\in S^1$.*

A band E is *normal* (*left normal*) if it satisfies the identity $axya=ayxa$ ($axy=ayx$).

Theorem 1.17. [26] *A semigroup S is a normal band of t-Archimedean semigroups if and only if $ac\xrightarrow{t}abc$, for all $a,b,c\in S$.*

Theorem 1.18. [26] *The following conditions on a semigroup S are equivalent:*
(*i*) S *is a left normal band on t-Archimedean semigroups;*
(*ii*) $(\forall a,b,c\in S)\ ac \xrightarrow{r} abc \wedge a \xrightarrow{l} abc$;
(*iii*) $(\forall a,b,c\in S)\ ac \xrightarrow{r} abc \wedge b \xrightarrow{l} abc$.

Theorem 1.19. [10] *The following conditions on a semigroup S are equivalent:*
(*i*) S *is a band of power joined semigroups;*
(*ii*) $(\forall a,b\in S)\ ab \overset{p}{—} a^2b \overset{p}{—} ab^2$;
(*iii*) $(\forall a,b\in S)(\forall m,n\in \mathbf{Z}^+)\ ab \overset{p}{—} a^mb^n$.

Bands of power joined semigroups are studied by T.Nordahl [98] in the medial case ($xaby = xbay$). For the related results in the periodic case see M.Yamada [175].

Theorem 1.20. [10] *The following conditions on a semigroup S are equivalent:*
(*i*) S *is a semilattice of power joined semigroups;*
(*ii*) $(\forall a,b\in S)\ ba \overset{p}{—} a^2b \overset{p}{—} ab^2$;
(*iii*) $(\forall a,b\in S)(\forall m,n\in \mathbf{Z}^+)\ ba \overset{p}{—} a^mb^n$.

Theorem 1.21. [10] S *is a rectangular band of power joined semigroups if and only if* $abc \overset{p}{—} ac$, *for all* $a,b,c\in S$.

Corollary 1.2. [10] *A semigroup S is a left zero band of power joined semigroups if and only if* $ab \overset{p}{—} a$, *for all* $a,b\in S$.

Theorem 1.22. [13] S *is a band of periodic power joined semigroups if and only if for every* $a,b\in S$ *and* $n\in \mathbf{Z}^+$ *there exists* $r\in \mathbf{Z}^+$ *such that* $(ab)^r = (a^nb^n)^r$.

Lemma 1.1. [34] S *is a union of nil-semigroups if and only if for every* $a\in S$ *there exists* $r\in \mathbf{Z}^+$ *such that* $a^r = a^{r+1}$.

Theorem 1.23. *The following conditions on a semigroup S are equivalent:*
(*i*) S *is a band of nil-semigroups;*
(*ii*) S *is a union of nil-semigroups and S is a band of power joined semigroups;*
(*iii*) $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^{3n+1} = (a^2b)^{2n+1} = (ab^2)^{2n+1}$.

For the related results see also D.W.Miller [93].

Theorem 1.24. *A semigroup S is a semilattice of nil-semigroups if and only if* $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ba)^{3n+1} = (a^2b)^{2n+1} = (ab^2)^{2n+1}$.

Chapter 2. Semilattices of Completely Archimedean semigroups

2.1. The general case

In a π-regular semigroup S we consider the equivalence relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{J}^*$ and $\mathcal{H}^*$ defined by:

$a\mathcal{L}^*b \Leftrightarrow Sa^p = Sb^q$, $a\mathcal{R}^*b \Leftrightarrow a^pS = b^qS$, $a\mathcal{J}^*b \Leftrightarrow Sa^pS = Sb^qS$, $\mathcal{H}^* = \mathcal{L}^*\cap\mathcal{R}^*$,

where p,q are the smallest positive integers such that $a^p, b^q\in Reg(S)$ (J.L.Galbiati and M.L.Veronesi [70]). If $e\in E(S)$, then by G_e we denote the maximal subgroup of S with e as its identity and $T_e = \sqrt{G_e}$. On a semigroup S we denote the relation τ by $a\ \tau\ b \ \Leftrightarrow\ (\exists e \in E(S))\ a, b \in T_e$. The relation τ ia an equivalence on S if and only if S is completely π-regular. A semigroup S is a *π-group* if S is a nil-extension of a group.

Every (Rees) factor semigroup of any subsemigroup of S is a *(Rees) factor* of S. If such a (Rees) factor is completely π-regular we call it *q-(Rees)-factor*, [147]. In [143] q-factors are called *epifactors*.

Theorem 2.1. *The following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{CA} \circ \mathcal{S}$;
(ii) $S \in \pi\mathcal{R}$ *and every* $\mathcal{H}^*$*-class of* S *contains an idempotent;*
(iii) $S \in \pi\mathcal{R}$ *and every* $\mathcal{H}^*$*-class of* S *is a* π*-group;*
(iv) $S \in \mathcal{C}\pi\mathcal{R}$ *and* $\tau = \mathcal{H}^*$;
(v) $S \in \pi\mathcal{R}$ *and* $Gr(S) = Reg(S)$;
(vi) $S \in \mathcal{A} \circ \mathcal{S} \cap \mathcal{C}\pi\mathcal{R}$;
(vii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n bS(ab)^n$;
(viii) $S \in \mathcal{C}\pi\mathcal{R}$ *and* $a \mid e \ \Rightarrow\ a^2 \mid e$, *for every* $a \in S$, $e \in E(S)$;
(ix) $S \in \mathcal{C}\pi\mathcal{R}$ *and every* $\mathcal{J}$*-class of* S *with an idempotent is a subsemigroup;*
(x) $S \in \mathcal{C}\pi\mathcal{R}$ *and in every Rees q-factor of* S *the set of all nilpotent elements is an ideal;*
(xi) $S \in \mathcal{C}\pi\mathcal{R}$ *and* S *has not q-factor which is* A_2 *or* B_2;
(xii) $S \in \mathcal{C}\pi\mathcal{R}$ *and* S *has not Rees q-factor which is* A_2 *or* B_2;
(xiii) $S \in \mathcal{C}\pi\mathcal{R}$ *and every regular* $\mathcal{D}$*-class of* S *is a subsemigroup of* S;
(xiv) $S \in \mathcal{C}\pi\mathcal{R}$ *and* $(\forall a \in S)(\forall b \in Intra(S))\ a \mid b \ \Rightarrow\ a^2 \to b$.

The equivalences $(i) \Leftrightarrow (vi) \Leftrightarrow (viii) \Leftrightarrow (ix) \Leftrightarrow (xiv)$ are from M.S.Putcha [113]. The equivalence $(i) \Leftrightarrow (v)$ is given by Л.Н.Шеврин [146], and independently this equivalence is proved by M.L.Veronesi [172]. The conditions $(ii), (iii), (iv)$ are from [172], $(x) - (xii)$ are from [148] and $(xiii)$ is from [146]. The condition (vii) is from [16].

A completely simple semigroup in which $E^2(S) = E(S)$ is a *rectangular group.*

Theorem 2.2. *The following conditions on a semigroup S are equivalent:*

(i) S *is a semilattice of nil-extensions of rectangular groups;*
(ii) $S \in \mathcal{CA} \circ \mathcal{S}$ *and* $(\forall e, f \in E(S))(\exists n \in \mathbf{Z}^+)\ (ef)^n = (ef)^{n+1}$;
(iii) $S \in \pi\mathcal{R}$ *and* $a = axa$ *implies* $a = ax^2a^2$;
(iv) $S \in \mathcal{CA} \circ \mathcal{S}$ *and every inverse of an idempotent is an idempotent;*
(v) $S \in \pi\mathcal{R}$ *and for all* $a, b \in S$, $ab, ba \in E(S)$ *implies* $ab = (ab)(ba)(ab)$;
(vi) $S \in \mathcal{CA} \circ \mathcal{S}$ *and there are no* $E(\infty)$ *and* $E(p)$, $p \geq 2$, *among subsemigroups of* S;
(vii) $S \in \mathcal{C}\pi\mathcal{R}$ *and there are no* A_2, B_2, $E(\infty)$ *and* $E(p)$, $p \geq 2$, *among subsemigroups of homomorphic images of* S;
(viii) $S \in \mathcal{C}\pi\mathcal{R}$ *and there are no* A_2, B_2 *and* $E(p)$, $p \geq 2$, *among q-factors of* S;
(ix) $S \in \mathcal{C}\pi\mathcal{R}$ *with* $(xy)^0 = (xy)^0(yx)^0(xy)^0$.

The equivalence $(i) \Leftrightarrow (v)$ is due to M.S.Putcha [113]. The conditions $(ii), (iii), (iv)$ are from S.Bogdanović [14] and the remaining cases are from S.Bogdanović and M.Ćirić [33].

A semigroup S is a *left* (*right*) *group* if for every $a, b \in S$ there exists only one $x \in S$ such that $xa = b$ $(xa = b)$, (A.H.Clifford and G.B.Preston [57]). S is an *LR-semigroup* if for every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in Sa \cup bS$, [20].

Theorem 2.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left and right groups;*

(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n S(ba)^n \cup (ba)^n S(ab)^n$;

(iii) *$S \in \pi\mathcal{R}$ and S is an LR-semigroup;*

(iv) $S \in \mathcal{CA} \circ \mathcal{S}$ *and* $(\forall e, f \in E(S))(\exists n \in \mathbf{Z}^+)\ (ef)^n = (efe)^n \vee (ef)^n = (fef)^n$;

(v) $S \in \pi\mathcal{R}$ *and* $a = axa$ *implies* $ax = ax^2a$ *or* $ax = xa^2x$;

(vi) $S \in \pi\mathcal{R}$ *(or* $S \in \mathcal{C}\pi\mathcal{R}$*) and* $e \in E(S)$ *implies* $E(S) \cap SeS \subseteq eS \cup Se$.

The equivalence $(i) \Leftrightarrow (vi)$ is from M.S.Putcha [113]. The conditions $(i)-(iv)$ are from S.Bogdanović and M.Ćirić [20]. An open problem is to describe LR-semigroups in the general case.

Theorem 2.4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left groups;*

(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n S(ba)^n$;

(iii) $S \in \pi\mathcal{R}$ *and* $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in Sa$;

(iv) $S \in \mathcal{CA} \circ \mathcal{S}$ *and* $(\forall e, f \in E(S))(\exists n \in \mathbf{Z}^+)\ (ef)^n = (efe)^n$;

(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n Sa^{2n}$;

(vi) $S \in \mathcal{CA} \circ \mathcal{S}$ *and every $\mathcal{R}^*$-class contains only one idempotent;*

(vii) $S \in \mathcal{CA} \circ \mathcal{S}$ *and for every* $e, f \in E(S)$ *there exists* $n \in \mathbf{Z}^+$ *such that* $(ef)^n \mathcal{L} (fe)^n$;

$(viii)$ $S \in \mathcal{CA} \circ \mathcal{S}$ *and for all* $a, x, y \in S$, $a = axa = aya$ *implies* $ax = ay$;

(ix) $S \in \pi\mathcal{R}$ *(or* $S \in \mathcal{C}\pi\mathcal{R}$*) and* $e \in E(S)$ *implies* $E(S) \cap SeS \subseteq Se$;

(x) $S \in \pi\mathcal{R}$ *and* $a = axa$ *implies* $ax = xa^2x$;

(xi) $S \in \mathcal{C}\pi\mathcal{R}$ *with* $(xy)^0 = (xy)^0(yx)^0$.

The equivalence $(i) \Leftrightarrow (ix)$ is from M.S.Putcha [113]. The conditions $(i), (iii), (iv), (x)$ are from S.Bogdanović [14], the conditions (i) and (v) are from [17], $(vi)-(viii)$ are from [15] and (xi) is from [33].

An anti commutative band is a *rectangular band*.

Theorem 2.5. [17] *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of rectangular bands;*

(ii) $S \in \pi\mathcal{R}$ *and* $E(S) = Reg(S)$;

(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^{2n+1} = (ab)^n ba^2 (ab)^n$.

A band S is *singular* either S is a left or a right zero band.

Theorem 2.6. [20] *The following conditions on a semigroup* S *are equivalent:*

(*i*) S *is a semilattice of nil-extensions of singular bands;*

(*ii*) $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^n=(ab)^n a\ \vee\ (ab)^n=b(ab)^n$*;*

(*iii*) $S\in\pi\mathcal{R}$*,* $E(S)=Reg(S)$ *and* S *is a* LR*-semigroup;*

(*iv*) $S\in\pi\mathcal{R}$ *and* $a=axa$ *implies* $a=ax$ *or* $a=xa$*.*

2.2. Bands of π-groups

A subsemigroup K of a semigroup S is a *retract* of S if there exists a homomorphism φ of S onto K such that $\varphi(a)=a$ for all $a\in K$. An ideal extension S of K is a *retract extension* (or *retractive extension*) of K if K is a retract of S.

Theorem 2.7. [26] *Let* S *be a* π*-regular semigroup and let* $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^n\in a^2Sb^2$*. Then* S *is a semilattice of retractive nil-extensions of completely simple semigroups.*

Theorem 2.8. [26] *Let* S *be a* π*-regular semigroup and let* $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^n\in a^2Sa$*. Then* S *is a semilattice of retractive nil-extensions of left groups.*

The converses of Theorems 2.7. and 2.8. are open problems.

Lemma 2.1. [19] *The following conditions on a semigroup* S *are equivalent:*

(*i*) S *is* π*-regular and* $Reg^2(S)=Reg(S)$*;*

(*ii*) S *is* π*-regular and* $\langle E(S)\rangle$ *is a regular subsemigroup of* S*;*

(*iii*) $(\forall a,b\in S)(\exists m,n\in \mathbf{Z}^+)\ a^mb^n\in a^mb^nSa^mb^n$*.*

Proposition 2.1. [26] *Let* S *be a band of* π*-groups and let* $Reg(S)$ *be a subsemigroup of* S*. Then* $Reg(S)$ *is a band of groups and it is a retract of* S*.*

Conversely, if S *contain a retract* K *which is a band of groups and if* $S=\sqrt{K}$*, then* S *is a band of* π*-groups.*

Theorem 2.9. *The following conditions on a semigroup* S *are equivalent:*

(*i*) S *is a band of* π*-groups;*

(*ii*) $S\in\mathcal{CA}\circ\mathcal{S}$ *and* $\mathcal{H}^*$ *is a congruence on* S*;*

(*iii*) S *is* π*-regular and* $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^n\in a^2bSab^2$*;*

(*iv*) S *is completely* π*-regular and* $ab\ \tau\ a^2b\ \tau\ ab^2$*;*

(*v*) $S\in\mathcal{TA}\circ\mathcal{B}\cap\mathcal{C}\pi\mathcal{R}$*;*

(*vi*) $S\in\mathcal{C}\pi\mathcal{R}$ *and* $xay\overset{t}{\text{---}}xa^2y$*, for all* $a\in S$*,* $x,y\in S^1$*;*

(*vii*) $S\in\mathcal{C}\pi\mathcal{R}$ *with* $(xy)^0=(x^2y)^0=(xy^2)^0$*.*

The equivalence $(i)\Leftrightarrow(ii)$ is from J.L.Galbiati and M.L.Veronesi [69]. The condition (iv) is given by B.Madison, T.K.Mukherjee and M.K.Sen [88], see also [89]. The conditions (v) and (vi) are from M.S.Putcha [113], for (vi) see also [114]. The condition (iii) is from S.Bogdanović and M.Ćirić [26].

A band S is *left regular* if $ax=axa$ for all $a,x\in S$.

Theorem 2.10. [26] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a left regular band of π-groups;*
(*ii*) *S is completely π-regular and for all $a,b\in S$, $ab\ \tau\ a^2b\ \tau\ aba$;*
(*iii*) *S is π-regular and $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^n\in a^2bSa$;*
(*iv*) *$S\in \mathcal{C}\pi\mathcal{R}$ with $(xy)^0=(x^2y)^0=(xyx)^0$.*

Theorem 2.11. [26] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a normal band of π-groups;*
(*ii*) *S is completely π-regular and for all $a,b,c,d\in S$, $abcd\ \tau\ acbd$;*
(*iii*) *S is π-regular and $(\forall a,b,c\in S)(\exists n\in \mathbf{Z}^+)\ (abc)^n\in acSac$;*
(*iv*) *$S\in \mathcal{C}\pi\mathcal{R}$ with $(xyzu)^0=(xzyu)^0$.*

Theorem 2.12. [26] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a left normal band of π-groups;*
(*ii*) *S is completely π-regular and for all $a,b,c\in S$, $abc\ \tau\ acb$;*
(*iii*) *S is π-regular and $(\forall a,b,c\in S)(\exists n\in \mathbf{Z}^+)\ (abc)^n\in acSa$;*
(*iv*) *$S\in \mathcal{C}\pi\mathcal{R}$ with $(xyz)^0=(xzy)^0$.*

A semigroup S is a *GV-inverse* semigroup if $S\in \mathcal{CA}\circ\mathcal{S}$ and every regular element of S possesses a unique inverse, [71].

Theorem 2.13. *The following conditions on a semigroup S are equivalent:*
(*i*) *S is GV-inverse;*
(*ii*) *S is a semilattice of π-groups;*
(*iii*) *$S\in \mathcal{CA}\circ\mathcal{S}$ and for every $e,f\in E(S)$ there exists $n\in \mathbf{Z}^+$ such that $(ef)^n=(fe)^n$;*
(*iv*) *$S\in \pi\mathcal{R}$ and $Reg(aS)=Reg(Sa)$, for all $a\in S$;*
(*v*) *$S\in \pi\mathcal{R}$ and $a=axa$ implies $ax=xa$;*
(*vi*) *$S\in \pi\mathcal{R}$ and $\mathcal{L}^*=\mathcal{R}^*$;*
(*vii*) *$S\in \pi\mathcal{R}$ and $(ab)^n,(ba)^n\in E(S)$ implies $(ab)^n=(ba)^n$, $a,b\in S$, $n\in \mathbf{Z}^+$;*
(*viii*) *$S\in \pi\mathcal{R}$ and $\mathcal{H}^*=\mathcal{J}^*$;*
(*ix*) *$S\in \pi\mathcal{R}\cap\mathcal{TA}\circ\mathcal{S}$;*
(*x*) *S is the disjoint union of π-groups and for every $e,f\in E(S)$ there exists $n\in \mathbf{Z}^+$ such that $(ef)^n=(fe)^n$;*
(*xi*) *S is completely π-regular and $ab\ \tau\ ba$ for all $a,b\in S$;*
(*xii*) *$(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^n\in b^{2n}Sa^{2n}$;*
(*xiii*) *$S\in \mathcal{C}\pi\mathcal{R}$ with $(xy)^0=(yx)^0$.*

The equivalences $(i)\Leftrightarrow(ii)\Leftrightarrow(viii)\Leftrightarrow(x)$ are from M.L.Veronesi [172], the conditions (iv) and (vi) are from [71], (vii) is from [113], (xii) is from [17] and (v) and (ix) are from [14].

Theorem 2.14. [33] *A semigroup S is a band of π-groups and $E^2(S)=E(S)$ if and only if $S\in \mathcal{C}\pi\mathcal{R}$ and $(xy)^0=x^0y^0$.*

Theorem 2.15. *The following conditions on a semigroup S are equivalent:*
(i) $S \in \mathcal{C}\pi\mathcal{R}$ *with* $\overline{xy} = \overline{y}\,\overline{x}$*;*
(ii) $S \in \mathcal{C}\pi\mathcal{R}$ *and there are no semigroups* A_2, B_2, L_2, R_2 *and* V *among epifactors of* S*;*
(iii) S *is a semilattice of* π*-groups and* $E^2(S) = E(S)$*;*
(iv) S *is a semilattice of* π*-groups and* $Gr(S)$ *is a subsemigroup of* S*;*
(v) $S \in \mathcal{C}\pi\mathcal{R}$ *with* $(xy)^0 = y^0x^0$*;*
(vi) S *is a semilattice of* π*-groups and* $ef = fe$ *for all* $e, f \in E(S)$*.*

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are from Л.Н.Шеврин [143].

Lemma 2.2. *Let* $S \in \mathcal{C}\pi\mathcal{R}$*. Then* $E^2(S) = E(S)$ *if and only if* $(x^0y^0)^0 = x^0y^0$ *in* S*.*

Theorem 2.16. *A semigroup* S *is a semilattice of nil-extensions of rectangular groups and* $E^2(S) = E(S)$ *if and only if* $S \in \mathcal{C}\pi\mathcal{R}$ *with* $(xy)^0 = (xy)^0(yx)^0(xy)^0$ *and* $(xy)^0 = x^0y^0$*.*

Theorem 2.17. [113] S *is commutative and* S *is a* GV*-inverse semigroup if and only if* S *is* π*-regular (completely* π*-regular) and for all* $a, b \in S$*,*

$$a — b \;\;\Rightarrow\;\; ab = ba.$$

Theorem 2.18. [79] *If* S *is completely* π*-regular and* $\mathcal{J} \subseteq \tau$*, then* S *is a semilattice of* π*-groups.*

2.3. Chains of completely Archimedean semigroups

Theorem 2.19. [14] S *is a chain of completely Archimedean semigroups if and only if* $S \in \mathcal{CA} \circ \mathcal{S}$ *and for every* $e, f \in E(S)$, $e \in efS$ *or* $f \in feS$*.*

Theorem 2.20. [14, 20] *The following conditions on a semigroup* S *are equivalent:*
(i) S *is a chain of nil-extensions of rectangular groups;*
(ii) S *is completely* π*-regular and* $E(S)$ *is a chain of rectangular bands;*
(iii) $S \in \mathcal{CA} \circ \mathcal{S}$ *and* $E(S)$ *is a chain of rectangular bands;*
(iv) $S \in \mathcal{CA} \circ \mathcal{S}$ *and* $e = efe$ *or* $f = fef$ *for all* $e, f \in E(S)$*;*
(v) $S \in \mathcal{C}\pi\mathcal{R}$ *with* $x^0 = x^0y^0x^0 \;\vee\; y^0 = y^0x^0y^0$*.*

Corollary 2.1. [14] S *is a chain of* π*-groups if and only if* S *is completely* π*-regular and* $E(S)$ *is a chain.*

Corollary 2.2. [20] S *is a chain of nil-extensions of periodic rectangular groups if and only if* S *is periodic and* $E(S)$ *is a chain of rectangular bands.*

Theorem 2.21. [20] *The following conditions on a semigroup* S *are equivalent:*
(i) S *is a chain of nil-extensions of rectangular bands;*
(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\; a^{2n} = a^nba^n \;\vee\; b^{2n} = b^nab^n$*;*
(iii) $S \in \pi\mathcal{R}$ *and* $Reg(S)$ *is a chain of rectangular bands.*

Theorem 2.22. [20] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a chain of nil-extensions of left and right groups;*
(*ii*) *for every $a,b\in S$ there exists $n\in \mathbf{Z}^+$ such that*
$$a^n\in a^{2n}S(ab)^n\cup(ba)^nSa^{2n} \quad \text{or} \quad b^n\in b^{2n}S(ba)^n\cup(ab)^nSb^{2n} \ ;$$
(*iii*) *S is completely π-regular and for every $e,f\in E(S)$, $ef\in\{e,f\}$ or $fe\in\{e,f\}$.*

Theorem 2.23. [14, 20] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a chain of nil-extensions of left groups;*
(*ii*) $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ a^n\in a^{2n}S(ab)^n\cup(ba)^nSa^{2n}$*;*
(*iii*) *S is completely π-regular and for every $e,f\in E(S)$, $ef=e$ or $fe=f$;*
(*iv*) *$S\in\mathcal{C}\pi\mathcal{R}$ with $x^0=x^0y^0 \ \vee\ y^0=y^0x^0$.*

Theorem 2.24. [20] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a chain of nil-extensions of singular bands;*
(*ii*) $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ a^n=a^nb \ \vee\ a^n=ba^n \ \vee\ b^n=b^na \ \vee\ b^n=ab^n$*;*
(*iii*) *$S\in\pi\mathcal{R}$, $Reg(S)=E(S)$ and $E(S)$ is a chain of singular bands.*

Corollary 2.3. [20] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a chain of nil-extensions of left zero bands;*
(*ii*) $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ a^n=a^nb \ \vee\ b^n=b^na$*;*
(*iii*) *$S\in\pi\mathcal{R}$, and for all $a,b\in Reg(S)$, $ab=a$ or $ba=b$.*

In connection with a study of a lattice of subsemigroups of some semigroup the important place is captured by $\mathcal{U}$-semigroups. A semigroup S is a *$\mathcal{U}$-semigroup* if the union of every two subsemigroups of S is a subsemigroup of S, which is equivalent with $xy\in\langle x\rangle\cup\langle y\rangle$ for all $x,y\in S$. A more detailed description can be find in M.Petrich [103]. These semigroups have been considered more recently, predominantly in special cases. Here we present some general results of Rédei's bands of π-groups. Several special cases of this the reader can find in Е.Г.Шутов [150], N.Kimura, T.Tamura and R.Merkel [82], Е.С.Ляпин и А.Е.Евсеев [87], А.Е.Евсеев [64], B.Trpenovski [141], S.Bogdanović, P.Kržovski, P.Protić and B.Trpenovski [34], J.Pelikán [100], B.Pondeliček [111], L.Rédei [125], S.Bogdanović and B.Stamenković [35], B.Trpenovski and N.Celakoski [142], S.Bogdanović and M.Ćirić [20] and M.Ćirić and S.Bogdanović [41].

A semigroup S is a *Rédei's band* if $xy=x$ or $xy=y$ for all $x,y\in S$, [125].

Theorem 2.25. [26] *The following conditions on a semigroup S are equivalent:*
(*ii*) *S is a Rédei's band of π-groups;*
(*ii*) *S has a retract K which is a Rédei's band of groups and $\sqrt{K}=S$;*
(*iii*) $(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ a^n\in(ab)^nS(ab)^n \ \vee\ b^n\in(ab)^nS(ab)^n$.

Corollary 2.4. [26] *A semigroup S is a Rédei's band of groups if and only if*
$$(\forall a,b\in S)\ a\in abSab \ \vee\ b\in abSab\ .$$

Let $n\in\mathbf{Z}^+$. A semigroup S is a *generalized $\mathcal{U}_{n+1}$-semigroup* or simply *$\mathcal{GU}_{n+1}$-semigroup* if S satisfies the following condition:

$$(\forall x_1, x_2, \dots, x_{n+1})(\exists m)\ (x_1x_2\cdots x_{n+1})^m \in \langle x_1\rangle \cup \langle x_2\rangle \cup \ldots \cup \langle x_{n+1}\rangle\ .$$

A $\mathcal{GU}_2$-semigroup we call $\mathcal{GU}$-semigroup. A chain Y of semigroups S_α, $\alpha \in Y$, is a *$\mathcal{GU}_{n+1}$-chain of semigroups* if for all $\alpha_1, \alpha_2, \ldots \alpha_{n+1} \in Y$ such that $\alpha_i \neq \alpha_j$ for some $i, j \in \{1, 2, \dots, n+1\}$, and for all $x_k \in S_{\alpha_k}$, $k \in \{1, 2, \dots, n+1\}$ there exists $m \in \mathbf{Z}^+$ such that $(x_1x_2\cdots x_{n+1})^m \in \langle x_1\rangle \cup \langle x_2\rangle \cup \ldots \cup \langle x_{n+1}\rangle$, [18,43].

Theorem 2.26. [43] *The following conditions on a semigroup S are equivalent:*
- (i) *S is a Rédei's band of periodic π-groups;*
- (ii) *S is a π-regular $\mathcal{GU}_{n+1}$-semigroup;*
- (iii) *S is a periodic $\mathcal{GU}_{n+1}$-semigroup;*
- (iv) *S is a $\mathcal{GU}_{n+1}$-chain of retractive nil-extensions of periodic left and right groups;*
- (v) *S is a π-regular $\mathcal{GU}$-semigroup;*
- (vi) *S is a periodic $\mathcal{GU}$-semigroup;*
- (vii) *S contains a retract K which is a regular $\mathcal{GU}$-semigroup and $\sqrt{K} = S$;*

Theorem 2.27. [43] *The following conditions on a semigroup S are equivalent:*
- (i) *S is a left zero band of periodic π-groups;*
- (ii) *S is a π-regular $\mathcal{GU}$-semigroup and $E(S)$ is a left zero band;*
- (iii) *S is a retractive nil-extension of a periodic left group.*

A semigroup S is a *$\mathcal{U}_{n+1}$-semigroup* if

$$(\forall x_1, x_2, \dots, x_{n+1} \in S)\ x_1x_2\cdots x_n \in \langle x_1\rangle \cup \langle x_2\rangle \cup \cdots \cup \langle x_{n+1}\rangle\,,$$

$n \in \mathbf{Z}^+$. A band I of semigroups S_i, $i \in I$, is a *$\mathcal{U}_{n+1}$-band of semigroups* if $x_1x_2\cdots x_{n+1} \in \langle x_1\rangle \cup \langle x_2\rangle \cup \cdots \cup \langle x_{n+1}\rangle$, for all $x_1 \in S_{i_1}$, $x_2 \in S_{i_2}, \dots, x_{n+1} \in S_{i_{n+1}}$, such that $i_k \neq i_l$ for some $k, l \in \{1, 2, \dots, n\}$. One defines analogously *$\mathcal{U}_{n+1}$-semilattice* and *$\mathcal{U}_{n+1}$-chain of semigroups*.

Theorem 2.28. [21] *The following conditions on a semigroup S are equivalent:*
- (i) *S is a $\mathcal{U}_{n+1}$-semigroup;*
- (ii) *S is a $\mathcal{U}_{n+1}$-chain of retract extensions of $\mathcal{U}$-groups and singular bands by $\mathcal{U}_{n+1}$-nil-semigroups;*
- (iii) *S is a $\mathcal{U}_{n+1}$-bands of ideal extensions of $\mathcal{U}$-groups by $\mathcal{U}_{n+1}$-nil-semigroups.*

Theorem 2.29. [21] *A semigroup S is a $\mathcal{U}_{n+1}$-semigroup and $Reg(S)$ is an ideal of S if and only if for every $x_1, x_2, \dots, x_{n+1} \in S$,*

$$x_1x_2\cdots x_{n+1} \in \bigcup_{i=1}^{n+1} \left\{x_i^k \mid k \geq 2\right\}. \tag{1}$$

Corollary 2.5. [21] *A semigroup S is a retract extension of a regular $\mathcal{U}$-semigroup by a $\mathcal{U}_{n+1}$-nil-semigroup if and only if S satisfies (1).*

Chapter 3. Nil-extensions of (completely) regular semigroups

3.1. The general case

Here we will consider π-regular semigroups S in which $Reg(S)$ is an ideal, predominantly π-regular semigroups in which $Reg(S) = Gr(S)$ is an ideal (see also Theorem 2.1.(v)).

Theorem 3.1. [23] *A semigroup S is a nil-extension of a regular semigroup if and only if for every $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xa^nySxa^ny$.*

Theorem 3.2. [23] *A semigroup S is a nil-extension of a union of groups if and only if for every $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xa^nyxSxa^ny$.*

Theorem 3.3. *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a nil-extension of a semilattice of left and right groups;*
(*ii*) $(\forall x, a, y \in S)(\exists n \in \mathbf{Z}^+)\ xa^ny \in xa^nySya^nx \cup ya^nxSxa^ny$;
(*iii*) *for all $a, b \in S$ and $x, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that*
$$x(ab)^ny \in x(ab)^nySy(ba)^nx \cup x(ba)^nySx(ab)^ny;$$
(*iv*) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xa^nySyx \cup yxSxa^ny$.*

The equivalence $(i) \Leftrightarrow (iii)$ is from S.Bogdanović and M.Ćirić [23].

Theorem 3.4. *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a nil-extension of a semilattice of left groups;*
(*ii*) $(\forall x, a, y \in S)(\exists n \in \mathbf{Z}^+)\ xa^ny \in xa^nySya^nx$;
(*iii*) $(\forall a, b \in S)(\forall x, y \in S^1)(\exists n \in \mathbf{Z}^+)\ x(ab)^ny \in x(ab)^nySy(ba)^nx$;
(*iv*) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xSx$.*

The equivalences $(i) \Leftrightarrow (iii)$ and $(i) \Leftrightarrow (iv)$ are from S.Bogdanović and M.Ćirić [23] and [28], respectively.

Theorem 3.5. [23] *The following conditions on a semigroup S are equivalent:*
(*i*) *S is a nil-extension of a semilattice of groups;*
(*ii*) *S is a retractive nil-extension of a semilattice of groups;*
(*iii*) $(\forall x, a, y \in S)(\exists n \in \mathbf{Z}^+)\ xa^ny \in xa^nySya^nx \cap ya^nxSxa^ny$;
(*iv*) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xSx \cap ySy$.*

Corollary 3.1. [23] *A semigroup S is a nil-extension of a group (π-group) if and only if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in b^nSb^n$.*

Several other characterizations for π-groups the reader can found in P.Chu, Y.Guo and X.Ren [41].

Corollary 3.2. [23] *A semigroup S is a nil-extension of a periodic group if and only if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = b^n$.*

Lemma 3.1. [37] *If S is a π-regular semigroup all of whose idempotents are primitive, then S is completely π-regular with the maximal subgroups given by*
$$G_e = eSe, \qquad e \in E(S).$$

A simple semigroup with a primitive idempotent is *completely simple*.

Theorem 3.6. [94] *A simple semigroup S is completely simple if and only if S is completely π-regular.*

A semigroup S is *weakly cancellative* if for any $a, b \in S$, $ax = bx$ and $xa = xb$ for some $x \in S$ imply $a = b$, [102].

Theorem 3.7. *The following conditions of a semigroup S are equivalent:*

(i) *S is completely Archimedean;*

(ii) *S is a nil-extension of a completely simple semigroup;*

(iii) *S is Archimedean and completely π-regular;*

(iv) *S is π-regular and all idempotents of S are primitive;*

(v) *S is Archimedean and contains at least one minimal left and at least one minimal right ideal;*

(vi) *S is Archimedean, π-regular and $Reg(S)$ is weakly cancellative subsemigroup of S;*

(vii) *S is π-regular and $Reg(S)$ is an ideal of S, in which $a = axa$ implies $x = xax$;*

$(viii)$ $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^n bSa^n$.

The equivalences $(i) \Leftrightarrow (iv) \Leftrightarrow (viii)$ are from S.Bogdanović and S.Milić [37] and $(i) \Leftrightarrow (iii) \Leftrightarrow (v)$ are from J.L.Galbiati and M.L.Veronesi [72].

Corollary 3.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is Archimedean and periodic;*

(ii) *S is π-regular and for every $x, y \in S$, $xy = yx$ implies $x^n = y^n$ for some $n \in \mathbf{Z}^+$;*

(iii) *S is a nil-extension of a completely simple periodic semigroup.*

Theorem 3.8. [113] *The following conditions of a semigroup S are equivalent:*

(i) *S is a nil-extension of a rectangular group;*

(ii) *S is a subdirect product of a group and a nil-extension of a rectangular band;*

(iii) *S is a subdirect product of a group, a nil-extension of a right zero semigroup and a nil-extension of a left zero semigroup.*

Theorem 3.9. [37] *A semigroup S is a nil-extension of a rectangular band if and only if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = a^n ba^n$.*

Theorem 3.10. [37] *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a left group;*

(ii) *S is left Archimedean and contains an idempotent;*

(iii) *S is π-regular and $E(S)$ is a left zero band;*

(iv) *for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^n Sa^n b$.*

Theorem 3.11. [37] *A semigroup S is a nil-extension of a left zero band if and only if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = a^n b$.*

Corollary 3.4. [37] *A semigroup S is a nil-semigroup if and only if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = ba^n = a^n b$.*

Theorem 3.12. [37] *Let S be a semigroup. If*

$$(\forall a \in S)(\exists_1 x \in S)(\exists n \in \mathbf{Z}^+)\ a^n = xa^{n+1}$$

then S is a nil-extension of a left group.

Theorem 3.13. [37] *Let S be a semigroup. If*
$$(\forall a \in S)(\exists_1 x \in S)(\exists n \in \mathbf{Z}^+)\ a^n = a^n x a^n$$
then S is a π-group.

The following theorem is proved by T.Tamura:

Theorem 3.14. *A semigroup S is a π-group if and only if S is an Archimedean semigroup with exactly one idempotent.*

Theorem 3.15. *A semigroup S is a nil-extension of a periodic left group if and only if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = a^n b^n$.*

3.2. Retractive nil-extensions of regular semigroups

A retract extensions can be more easily constructed than many other kinds of extensions, it is of interest to know whether a given extension is a retract extension. The purpose of this section is to present criterions for retractive nil-extensions, especially for the very important class of regular semigroups. Also, we summarize the results of retractive nilpotent extensions of regular semigroups. The constructions of retractive (nilpotent) extensions are given in [38].

Theorem 3.16. [27] *A semigroup S is a retractive nil-extension of a regular semigroup K if and only if S is a subdirect product of K and a nil-semigroup.*

Corollary 3.5. [113] *A semigroup S is a retractive nil-extension of a rectangular group if and only if S is a subdirect product of a group, a left zero semigroup, a right zero semigroup and a nil-semigroup.*

A semigroup S is an *n-inflation* of K if it is a retractive extension of K by a $(n+1)$-nilpotent semigroup. A 1-inflation is an *inflation*.

Corollary 3.6. [27] *A semigroup S is an n-inflation of a regular semigroup K if and only if S is a subdirect product of K and a $(n+1)$-nilpotent semigroup.*

Lemma 3.2. [23] *Let S be a nil-extension of a union of groups T. Then every retraction φ of S onto T has the representation $\varphi(a) = ea$, $a \in S$, where $e \in E(S)$ such that $a \in \sqrt{G_e}$.*

Theorem 3.17. [19, 28] *The following conditions on a semigroup S are equivalent:*

(*i*) *S is a retractive nil-extension of a union of groups;*

(*ii*) *for every $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*
$$c(ab)^n \in c(ab)^n S c(ab)^n \quad \text{and} \quad (ab)^n c \in e(ab)^n cbS(ab)^n cf\ ,$$
where $e, f \in E(S)$ such that $a \in \sqrt{G_e}$, $c \in \sqrt{G_f}$.

(*iii*) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^n y \in x^2 S y^2$;*

(*iv*) *S is a subdirect product of a union of groups and a nil-semigroup.*

Corollary 3.7. [28, 38] *The following conditions on a semigroup S are equivalent:*
(*i*) S *is an* n*-inflation of a union of groups;*
(*ii*) $xS^{n-1}y = x^2S^ny^2$, *for all* $x, y \in S$;
(*iii*) S *is a subdirect product of a union of groups and an* $(n+1)$*-nilpotent semigroup.*

Theorem 3.18. [30] *A semigroup S is a retractive nil-extension of a band of groups if and only if S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in x^2aSay^2$.*

Corollary 3.8. [30] *Let $n \in \mathbf{Z}^+$, $n \geq 3$. Then a semigroup S is an n-inflation of a band of groups if and only if $xaS^{n-3}by \subseteq x^2aS^nby^2$, for all $x, a, b, y \in S$.*

Corollary 3.9. [30] *A semigroup S is a 2-inflation of a band of groups if and only if $xay \in x^2aSay^2$, for all $x, a, y \in S$.*

Corollary 3.10. [30] *A semigroup S is an inflation of a band of groups if and only if $xy \in x^2ySxy^2$, for all $x, y \in S$.*

Theorem 3.19. [30] *A semigroup S is a retractive nil-extension of a left regular band of groups if and only if S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in x^2aSx$.*

Corollary 3.11. [30] *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then a semigroup S is an n-inflation of a left regular band of groups if and only if $xaS^{n-2}y \subseteq x^2aS^nx$, for all $x, a, y \in S$.*

Corollary 3.12. [30] *A semigroup S is an inflation of a left regular band of groups if and only if $xy \in x^2ySx$, for all $x, y \in S$.*

Theorem 3.20. [30] *The following conditions on a semigroup S are equivalent:*
(*i*) S *is a retractive nil-extension of a normal band of groups;*
(*ii*) S *is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xyaSaxy$;*
(*iii*) $S \in \mathcal{CA} \circ \mathcal{S}$ *and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xySxy$.*

Corollary 3.13. [30] *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then a semigroup S is an n-inflation of a normal band of groups if and only if $xS^{n-1}y \subseteq xyS^nxy$, for all $x, y \in S$.*

Theorem 3.21. [30] *A semigroup S is a retractive nil-extension of a left normal band of groups if and only if S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny \in xySx$.*

Corollary 3.14. [30] *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then a semigroup S is an n-inflation of a left normal band of groups if and only if $xS^{n-1}y \subseteq xyS^nx$, for all $x, y \in S$.*

Theorem 3.22. *The following conditions on a semigroup S are equivalent:*
(*i*) S *is a retractive nil-extension of a completely simple semigroup;*
(*ii*) S *is a rectangular band of π-groups;*
(*iii*) S *is completely Archimedean and for all $a \in S$, $x, y \in S^1$ there exists $p, q, r, s \in \mathbf{Z}^+$ such that*

$$(xay)^pS = (xa^2y)^pS \quad \textit{and} \quad S(xay)^r = S(xa^2y)^s;$$

(iv) *for all* $a, b, c \in S$ *there exists* $n \in \mathbf{Z}^+$ *such that*

$$a^nc \in ea^nbSa^ncf \quad \textit{and} \quad ca^n \in fca^nbSa^ne \ ,$$

where $a \in \sqrt{G_e}$, $c \in \sqrt{G_f}$;

(v) S *is an Archimedean semigroup with an idempotent and for every* $a, b \in S$ *there exists* $n \in \mathbf{Z}^+$ *such that* $(ab)^n \in a^2Sb^2$;

(vi) S *is a subdirect product of a completely simple semigroup and a nil-semigroup.*

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are proved by J.L.Galbiati and M.L.Veronesi [69]. The condition (vi) is given by Л.Н.Шеврин [146] and this is also a consequence of Theorem 3.16. The condition (iv) is from S.Bogdanović [19] and (v) is from S.Bogdanović and M.Ćirić [28].

Corollary 3.15. *A semigroup* S *is a retractive nil-extension of a periodic completely simple semigroup if and only if* S *is a rectangular band of nil-extensions of periodic groups.*

Corollary 3.16. *A semigroup* S *is a retractive nil-extension of a rectangular band if and only if* S *is a rectangular band of nil-semigroups.*

Theorem 3.23. [28] *A semigroup* S *is a retractive nil-extension of a semilattice of left and right groups if and only if* S *is* π*-regular and for all* $x, a, y \in S$ *there exists* $n \in \mathbf{Z}^+$ *such that* $xa^ny \in x^2Sy^2x \cup yx^2Sy^2$.

Corollary 3.17. [28] *A semigroup* S *is an* n*-inflation of a semilattice of left and right groups if and only if* $xS^{n-1}y \subseteq x^2S^ny^2x \cup yx^2S^ny^2$ $(xy \in x^2Sy^2x \cup yx^2Sy^2$, *if* $n = 1)$, *for all* $x, y \in S$.

Theorem 3.24. [28] *A semigroup* S *is a retractive nil-extension of a semilattice of left groups if and only if* S *is* π*-regular and for all* $x, a, y \in S$ *there exists* $n \in \mathbf{Z}^+$ *such that* $xa^ny \in x^2Sx$.

Corollary 3.18. [28] *A semigroup* S *is an* n*-inflation of a semilattice of left groups if and only if* $xS^{n-1}y \subseteq x^2S^nx$ $(xy \in x^2Sx$, *if* $n = 1)$, *for all* $x, y \in S$.

Theorem 3.25. *The following conditions on a semigroup* S *are equivalent:*

(i) S *is a retractive nil-extension of a left group;*

(ii) S *is a left zero band of* π*-groups;*

(iii) S *is left Archimedean with an idempotent and for all* $a \in S$, $x, y \in S^1$ *there exists* $p, q, r, s \in \mathbf{Z}^+$ *such that*

$$(xay)^pS = (xa^2y)^qS \quad \textit{and} \quad S(xay)^r = S(xa^2y)^s;$$

(iv) S *is completely* π*-regular and for all* $a, b, c \in S$ *there exists* $n \in \mathbf{Z}^+$ *such that* $ca^n \in gca^nSa^nf$, *where* $f, g \in E(S)$ *such that* $b \in \sqrt{G_f}$, $c \in \sqrt{G_g}$;

(v) S *is Archimedean* π*-regular and for all* $a, b \in S$ *there exists* $n \in \mathbf{Z}^+$ *such that* $(ab)^n \in a^2Sa$.

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are from J.L.Galbiati and M.L.Veronesi [69]. The condition (iv) is from S.Bogdanović [19] and (v) is from S.Bogdanović and M.Ćirić [28].

3.3. Nil-extensions of bands

Theorem 3.26. [23, 126] *The following conditions on a semigroup S are equivalent:*

(*i*) *S is a nil-extension of a band;*
(*ii*) $(\forall x, a, y \in S)(\exists n \in \mathbf{Z}^+)\ xa^ny = (xa^ny)^2$*;*
(*iii*) *S is periodic and $E(S)$ is an ideal of S;*
(*iv*) *S is a semilattice of nil-extensions of rectangular bands and $SeS = E(S)$ for all $e \in E(S)$.*

A construction for semigroups from Theorem 3.26. is given by X.M.Ren and Y.Q.Guo [126].

Theorem 3.27. [23] *A semigroup S is a nil-extension of a semilattice of singular bands if and only if for all $a, b \in S$ and $a, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that $x(ab)^n = x(ab)^nyxbay$ or $x(ab)^ny = xbayx(ab)^ny$.*

Corollary 3.19. [23] *A semigroup S is a nil-extension of a semilattice of left zero bands if and only if for all $a, b \in S$ and $a, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that $x(ab)^n = x(ab)^nyxbay$.*

Corollary 3.20. [126] *The following conditions on a semigroup S are equivalent:*

(*i*) *S is a nil-extension of a left regular band;*
(*ii*) *S is a semilattice of nil-extensions of left zero bands, $E(S)$ is a left ideal of S and $eae = ea = ea^2$ for all $a \in S$, $e \in E(S)$;*
(*iii*) *S is periodic, $E(S)$ is a left ideal of S and $eae = ea = ea^2$ for all $a \in S$, $e \in E(S)$.*

Theorem 3.28. [126] *The following conditions on a semigroup S are equivalent:*

(*i*) *S is a nil-extension of a semilattice;*
(*ii*) *S is a semilattice of nil-semigroups and $ea = ae = a^2e$ for all $a \in S$, $e \in E(S)$;*
(*iii*) *S is periodic and $ea = ae = a^2e$ for all $a \in S$, $e \in E(S)$.*

Theorem 3.29. [23] *A semigroup S is a nil-extension of a chain of rectangular bands if and only if for all $a, b \in S$ and $x, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that $xa^ny = (xa^nba^ny)^2$ or $xb^ny = (xb^nab^ny)^2$.*

Corollary 3.21. [23] *A semigroup S is a nil-extension of a chain of left zero bands if and only if for all $a, b \in S$ and $x, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that $xa^nb^ny = (xa^ny)^2$ or $xb^na^ny = (xb^ny)^2$.*

Corollary 3.22. [23] *A semigroup S is a nil-extension of a Rédei's band if and only if for all $a, b \in S$ and $x, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that $xa^nb^ny = (xa^ny)^2$ or $xa^nb^ny = (xb^ny)^2$.*

Theorem 3.30. [23] *A semigroup S is a retractive nil-extension of a band if and only if for all $a, b \in S$ and $x, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that $x(ab)^{n+k}y = (xa^{n+k}b^{n+k}y)^2$, for all $k \in \mathbf{Z}^+$.*

Theorem 3.31. [23] *A semigroup S is a retractive nil-extension of a rectangular band if and only if for all $a,b,c\in S$ there exists $n\in\mathbf{Z}^+$ such that $(ac)^{n+k}=a^{n+k}bc^{n+k}$, for all $k\in\mathbf{Z}^+$.*

Theorem 3.32. [23] *A semigroup S is a retractive nil-extension of a left zero band if and only if for all $a,b\in S$ there exists $n\in\mathbf{Z}^+$ such that $(ab)^n=a^{2n+1}$.*

Theorem 3.33. [23] *A semigroup S is a retractive nil-extension of a semilattice of singular bands if and only if for all $a,b,\in S$ and $x,y\in S^1$ there exists $n\in\mathbf{Z}^+$ such that $x(ab)^{n+k}y=xa^{n+k}b^{n+k}yxb^{n+k}a^{n+k}y$ or $x(ab)^{n+k}y=xb^{n+k}b^{n+k}yxa^{n+k}b^{n+k}y$, for all $k\in\mathbf{Z}^+$.*

Corollary 3.23. [23] *A semigroup S is a retractive nil-extension of a semilattice of left zero bands if and only if for all $a,b,\in S$ and $x,y\in S^1$ there exists $n\in\mathbf{Z}^+$ such that $x(ab)^{n+k}y=xa^{n+k}b^{n+k}yxb^{n+k}a^{n+k}y$, for all $k\in\mathbf{Z}^+$.*

Theorem 3.34. [23] *A semigroup S is a retractive nil-extension of a Rédei's band if and only if for all $a,b,\in S$ and $x,y\in S^1$ there exists $n\in\mathbf{Z}^+$ such that $x(ab)^ny=(xa^ny)^2$ or $x(ab)^ny=(xb^ny)^2$.*

3.4. Primitive π-regular semigroups

Various characterizations for primitive regular semigroups has been obtained by T.E.Hall [78], G.Lallement and M.Petrich [86], G.B.Preston [112], O.Steinfeld [135] and P.S.Venkatesan [170,171] (this appeared also in the book of A.H.Clifford and G.B.Preston [58]). J.Fountain [68] considered primitive abundant semigroups. In this section some characterizations of primitive π-regular semigroups are given, which generalize the previous results for primitive regular semigroups.

An ideal I of a semigroup $S=S^0$ is a *nil-ideal* of S if I is a nil-semigroup. By $R^*(S)$ we denote *Clifford's radical* of a semigroup $S=S^0$, i.e. the union of all nil-ideals of S (it is the greatest nil-ideal of S).

Lemma 3.3. [31] *Let $S=S^0$ be a semigroup. If eS (Se) is a 0-minimal right (left) ideal of S generated by a nonzero idempotent e, then e is primitive.*

A nonzero idempotent e of a semigroup $S=S^0$ which generates 0-minimal left (right) ideal is called *left (right) completely primitive*. An idempotent e is *completely primitive* if it is both left and right completely primitive. A semigroup S is (*left, right*) *completely primitive* if all of its nonzero idempotents are (left, right) completely primitive. In regular semigroups the notions "primitive" and "completely primitive" coincide.

Lemma 3.4. [31] *Let $S=S^0$ be a regular semigroup and let $e\in E(S^*)$. Then e is primitive if and only if eS (Se) is a 0-minimal left (right) ideal of S.* $\square$

Theorem 3.35. [31] *The following conditions on a semigroup $S=S^0$ are equivalent:*

(*i*) *S is a nil-extension of a primitive regular semigroup;*
(*ii*) *S is a completely primitive π-regular semigroup;*

(iii) S *is completely* π*-regular and* SeS *is a 0-minimal ideal of* S *for every* $e\in E(S^*)$*;*

(iv) S *is a primitive* π*-regular semigroup and* $R^*(SE(S)S)=\{0\}$.

Theorem 3.36. [31] *The following conditions on a semigroup* S *are equivalent:*

(i) S *is a primitive* π*-regular semigroup;*

(ii) S *is an ideal extension of a nil-semigroup by a completely primitive* π*-regular semigroup;*

(iii) S *is a nil-extension of a semigroup which is an ideal extension of a nil-semigroup by a primitive regular semigroup.*

Corollary 3.24. [31] *A semigroup* $S=S^0$ *is a completely primitive* π*-inverse semigroup if and only if* S *is a nil-extension of a primitive inverse semigroup.*

Corollary 3.25. [31] *The following conditions on a semigroup* S *are equivalent:*

(i) S *is a primitive* π*-inverse semigroup;*

(ii) S *is an ideal extension of a nil-semigroup by a completely primitive* π*-inverse semigroup;*

(iii) S *is a nil-extension of a semigroup which is an ideal extension of a nil-semigroup by a primitive inverse semigroup.*

Chapter 4. Decompositions induced by identities

4.1. Basic definitions

By A^+ we denote the *free semigroup* over an alphabet A and by A^* we denote the *free monoid* over an alphabet A. By $|u|$ we denote the *length* of a word $u\in A^+$ and by $|x|_u$ we denote the number of appearances of the letter x in u. A word $v\in A^+$ is a *subword* (*left cut, right cut*) of a word $u\in A^+$ if $v\mid u$ ($v\underset{l}{\mid} u$, $v\underset{r}{\mid} u$). If $u\in A^+$, $|u|\geq 2$, then by $h^{(2)}(u)$ ($t^{(2)}(u)$) we denote the left (right) cut of u of the length 2. By $h(u)$ ($t(u)$) we denote the *head* (*tail*), by $i(u)$ ($f(u)$) we denote the *initial* (*final*) *part*, by $l(u)$ ($r(u)$) we denote the *left* (*right*) *part*, by $\overline{u}$ we denote the *mirror image* and by $c(u)$ we denote the *content* of the word u [103]. For a word $u\in A^+$ by $\Pi(u)$ we denote the set $\Pi(u)=\{\,x\in A\mid |x|_u=1\,\}$, by $u=u(x_1,\dots,x_n)$ we denote that $c(u)=\{x_1,\dots,x_n\}$. If $u\in A^+$ and let $x\in A$, then by $x\underset{l}{\parallel} u$ ($x\underset{r}{\parallel} u$) we denote that $u=xu'$, $u'\in A^+$, $x\nmid u'$ ($u=u'x$, $u'\in A^+$, $x\nmid u'$), where "$\nmid$" is the complement of "$\mid$". Otherwise we write $x\underset{l}{\nparallel} u$ ($x\underset{r}{\nparallel} u$). Since in this chapter we consider free semigroups over finite alphabets, then we introduce the following notations for finite alphabets: For $n\in\mathbf{Z}^+$, $n\geq 3$, $A_n=\{x_1,x_2,\dots,x_n\}$, and $A_2=\{x,y\}$.

Let $u\in A_n^+$ and let S be a semigroup. By a *value* of the word u in the *valuation* $\overline{a}=(a_1,\dots,a_n)\in S^n$ we mean the element $u(\overline{a})=F(u)\in S$, where $F:A_n^+\to S$ is the homomorphism determined by $F(x_i)=a_i$, $i\in\{1,2,\dots,n\}$.

By $[u=v]$ we denote the *variety* determined by the identity $u=v$. Identities $u=v$ and $u'=v'$ over an alphabet A_n^+ are *p-equivalent* if $u'=v'$ can be obtained from $u=v$ by some permutation of letters. It is clear that p-equivalent identities determines the same variety. If $\mathcal{X}$ is a class of semigroups, then $u=v$ is an $\mathcal{X}$*-identity* if $[u=v]\subseteq\mathcal{X}$. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are classes of semigroups, then $u=v$ is a $\mathcal{X}_1\triangleright\mathcal{X}_2$*-identity* if $[u=v]\cap\mathcal{X}_1\subseteq\mathcal{X}_2$.

Throughout this chapter we will consider problems of recognition of identities which induce several types of semigroup decompositions, i.e. problems of the following two types:

$(P1)$ *if* $\mathcal{X}$ *is a class of semigroups, find all* $\mathcal{X}$*-identities;*

$(P2)$ *if* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are classes of semigroups, find all* $\mathcal{X}_1\triangleright\mathcal{X}_2$*-identities;*

4.2. Identities and semilattices of Archimedean semigroups

An identity $u=v$ is *homotype* if $c(u)=c(v)$ and it is *heterotype* if $c(u)\neq c(v)$. The homotype identities will be considered firstly, i.e. the identity of the form

$$u(x_1,x_2,\dots,x_n)=v(x_1,x_2,\dots,x_n). \tag{1}$$

Lemma 4.1. [44] *Let* ϑ *be an equivalence on* A_2^+ *determined by the partition*

$$C_a=\{(xy)^nx\mid n\in\mathbf{Z}^+\cup\{0\}\ \},\quad C_b=\{(yx)^ny\mid n\in\mathbf{Z}^+\cup\{0\}\ \},$$
$$C_{ab}=\{(xy)^n\mid n\in\mathbf{Z}^+\},\quad C_{ba}=\{(yx)^n\mid n\in\mathbf{Z}^+\},$$
$$C_0=A_2^+-(C_a\cup C_b\cup C_{ab}\cup C_{ba}).$$

Then ϑ *is a congruence and the factor* A_2^+/ϑ *is isomorphic to* B_2.

By the following theorem some characterizations of all identities which induce decompositions into a semilattice of Archimedean semigroups are given.

Theorem 4.1. [44] *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity;*

(ii) (1) *is not satisfied in the semigroup* B_2*;*

(iii) *there exists a homomorphism* $T:A_n^+\to A_2^+$ *such that* $(T(u),T(v))\notin\vartheta$*;*

(iv) *there exists a homomorphism* $T:A_n^+\to A_2^+$ *and a permutation* π *of a set* $\{u,v\}$ *such that one of the following conditions hold:*

$(A1)$ $T(\pi(u))\in C_{ab}$ *and* $T(\pi(v))\notin C_{ab}$*;*

$(A2)$ $T(\pi(u))\in C_a$ *and* $T(\pi(v))\notin C_a$*;*

(v) *there exists* $k\in\mathbf{Z}^+$ *and* $w\in C_0\subset A_2^+$ *such that* $[u=v]\subseteq[(xy)^k=w]$.

For one description of identities satisfied on B_2 we refer to the paper of Г.И.Машевицкий [176].

Several special types of $\mathcal{A}\circ\mathcal{S}$-identities are considered by T.Tamura and N.Kimura [161], J.L.Chrislock [39], T.Tamura and J.Shafer [164], T.Tamura and T.Nordahl [163], T.Nordahl [97], М.В.Сапир и Е.В.Суханов [130]. By Theorem 4.1. it follows that *permutation identities*, i.e. identities of the form $x_1x_2\dots x_n=x_{\pi(1)}x_{\pi(2)}\dots x_{\pi(n)}$, where π is a nonidentical permutation of a set $\{1,2,\dots,n\}$, and *quasi-permutation identities*, i.e. identities of the form $x_1\dots vx_{k-1}yx_{k+1}\dots x_n=x_{\pi(1)}\dots x_{\pi(l-1)}y^2x_{\pi(l)}\dots x_{\pi(n)}$, where π is a permutation of a set $\{1,2,\dots,n\}$, are $\mathcal{A}\circ\mathcal{S}$-identities. For connections with semigroup varieties see Л.Н.Шеврин и М.В.Волков [151].

Theorem 4.2. [44] *The identity* (1) *is a* $\pi\mathcal{R}\triangleright\mathcal{CA}\circ\mathcal{S}$*-identity if and only if* (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity.*

Theorem 4.3. [44] *Let* $\mathcal{X}$ *be any variety of semigroups. The following conditions are equivalent:*

(*i*) $\mathcal{X}\subseteq\mathcal{A}\circ\mathcal{S}$*;*
(*ii*) $\mathcal{X}$ *not contain the semigroup* B_2*;*
(*iii*) *every regular semigroup from* $\mathcal{X}$ *is completely regular;*
(*iv*) *every completely 0-simple semigroup from* $\mathcal{X}$ *have not zero divisors;*
(*v*) *in every semigroup with zero from* $\mathcal{X}$ *the set of all nilpotents is a subsemigroup;*
(*vi*) *in every semigroup with zero from* $\mathcal{X}$ *the set of all nilpotents is an ideal.*

For other connections of these results with semigroup varieties we refer to М.В. Сапир и Е.В.Суханов [130], M.Schutzenberger [131], Л.Н.Шеврин и М.В.Волков [151] and Л.Н.Шеврин и М.В.Суханов [152].

Theorem 4.4. [44] *The following conditions for an identity* (1) *are equivalent:*

(*i*) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{LG}\circ\mathcal{N})\circ\mathcal{S}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* B_2 *and* R_2*;*
(*iii*) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity and* $t(u)\neq t(v)$*.*

Corollary 4.1. [44] *The following conditions for an identity* (1) *are equivalent:*

(*i*) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{G}\circ\mathcal{N})\circ\mathcal{S}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* B_2*,* R_2 *and* L_2*;*
(*iii*) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity,* $h(u)\neq h(v)$ *and* $t(u)\neq t(v)$*.*

Note that the description of $\mathcal{LA}\circ\mathcal{S}$-identities and $\mathcal{TA}\circ\mathcal{S}$-identities in general case is an open problem. In the section 4.5. these identities over the twoelement alphabet will be described. Semigroups satisfying the identity $x_1\dots x_m x_{m+1}\dots x_{m+n} = x_{m+1}\dots x_{m+n}x_1\dots x_m$, $m,n\in\mathbf{Z}^+$, called (m,n)*-commutative semigroups* are considered by I.Babcsány [5], I.Babcsány and A.Nagy [6], S.Lajos [83,84,85] and A.Nagy [95,96]. S.Lajos in [83] proved that these identities are $\mathcal{TA}\circ\mathcal{S}$-identities.

Theorem 4.5. [44] *The following conditions for an identity* (1) *are equivalent:*

(*i*) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{CS}\circledast\mathcal{N})\circ\mathcal{S}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* B_2*,* $L_{3,1}$ *and* $R_{3,1}$*;*
(*iii*) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity,* $h^{(2)}(u)\neq h^{(2)}(v)$ *and* $t^{(2)}(u)\neq t^{(2)}(v)$*.*

Corollary 4.2. [44] *The following conditions for an identity* (1) *are equivalent:*

(*i*) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{LG}\circledast\mathcal{N})\circ\mathcal{S}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* B_2*,* $L_{3,1}$ *and* R_2*;*
(*iii*) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity,* $h^{(2)}(u)\neq h^{(2)}(v)$ *and* $t(u)\neq t(v)$*.*

4.3. Identities and bands of π-groups

For an identity (1), by p_i we denote the number

$$p_i = ||x_i|_u - |x_i|_v| \ ,$$

$i \in \{1, 2, \dots , n\}$. An identity (1) is *periodic* if $p_i \neq 0$ for some $i \in \{1, 2, \dots , n\}$. In such a case the number $p = \gcd(p_1, \dots , p_n)$ is the *period* of the identity (1). Otherwise we say that (1) is *aperiodic* and that it is of the period $p = 0$, [44].

Periodic identities have been considered with various names, but, following the sense of the Proposition 4.1, we will use the previous name.

Proposition 4.1. [44] *The following conditions on an identity* (1) *are equivalent:*

(i) $[u = v]$ *consists of* π*-regular semigroups;*

(ii) $[u = v]$ *consists of completely* π*-regular semigroups;*

(iii) $[u = v]$ *consists of periodic semigroups;*

(iv) (1) *is a periodic identity.*

Let (1) be an identity for which

$$i(u) = i(v) = x_{\pi(1)}x_{\pi(2)} \dots x_{\pi(n)}$$

for some permutation π of a set $\{1, 2, \dots , n\}$. For $k \in \{1, 2, \dots , n-1\}$ by u_k (v_k) we denote the left cut of u (v) of the greatest length which contains exactly k letters (i.e. which not contains the letter $x_{\pi(k+1)}$). It is clear that

$$u_k = u_k(x_{\pi(1)}, \dots , x_{\pi(k)}) \ , \quad v_k = v_k(x_{\pi(1)}, \dots , x_{\pi(k)}) \ .$$

For $k \in \{1, 2, \dots , n-1\}$ and $i \in \{1, \dots , k\}$ we will use the notation

$$l_{k,i} = ||x_{\pi(i)}|_{u_k} - |x_{\pi(i)}|_{v_k}| \ .$$

An identity (1) is an identity *with left distortion* if $i(u) \neq i(v)$. Otherwise, (1) is *without left distortion*. Similarly we define identities *with* (*without*) *right distortion*, [44].

We define the *left characteristic* l of an identity (1) in the following way:

(i) $l = 1$, if (1) is an identity with left distortion;

(ii) l is the greatest common divisor of integers

$$p, \ l_{k,i}, \ 1 \leq k \leq n-1, \ 1 \leq i \leq k,$$

if (1) is without left distortion and some of integers p and $l_{k,i}$ is different to 0;

(iii) $l = 0$, if (1) is without left distortion and all of numbers p and $l_{k,i}$ are equal to 0, [44].

By *right characteristic* of an identity (1) we mean the left characteristic of the identity $\overline{u} = \overline{v}$.

Theorem 4.6. [44] *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{G} \circ \mathcal{N}) \circ \mathcal{B}$*-identity;*

(ii) (1) *is not satisfied in semigroups* B_2, $L_{3,1}$, $R_{3,1}$, $LZ(d)$ *and* $RZ(d)$ *for* $d \in \mathbf{Z}^+$, $d \geq 2$*;*

(iii) (1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{CS} \circledast \mathcal{N}) \circ \mathcal{S}$*-identity of the left and the right characteristic equal to* 1.

Using existing classifications of bands (see, for example, [103]), descriptions of identities which induce decompositions of π-regular semigroups into some special types of bands of π-groups can be obtained.

Theorem 4.7. [44] *The following conditions for an identity* (1) *are equivalent:*
(*i*) (1) *is a* $\mathcal{UG} \triangleright \mathcal{G} \circ \mathcal{B}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* $LZ(d)$ *and* $RZ(d)$ *for* $d \in \mathbf{Z}^+$, $d \geq 2$*;*
(*iii*) (1) *is an identity of left and right characteristic equal to* 1.

The description of $\mathcal{TA} \circ \mathcal{B}$-identities in general case is an open problem. Identities of this type are considered only by M.S.Putcha [114], where M.S.Putcha proved that the identity $(xy)^2 = x^2y^2$ is $\mathcal{TA} \circ \mathcal{B}$-identity.

The identities of the form $(xy)^n = x^ny^n$, $n \in \mathbf{Z}^+$, $n \geq 2$, are considered many a times. By Theorem 4.6. we obtain that this identity is a $\pi\mathcal{R} \triangleright (\mathcal{G} \circ \mathcal{N}) \circ \mathcal{B}$- identity if and only if $n = 2$.

For connections with semigroup varieties we refer to V.V.Rasin [124] and Л.Н.Шеврин и Е.В.Суханов [152].

4.4. Identities and nil-extensions of unions of groups

Theorem 4.8. [45] *The following conditions are equivalent for the identity* (1)*:*
(*i*) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$ *and* $C_{2,1}$*;*
(*iii*) $\Pi(u) \neq \Pi(v)$ *and* (1) *is* p*-equivalent to some identity of one of the following forms:*

(*A*1)
$$x_1u'(x_2, \dots, x_n) = v'(x_1, \dots, x_{n-1})x_n \ ,$$
where $x_1 \underset{l}{\nparallel} v'$ *and* $x_n \underset{r}{\nparallel} u'$*;*

(*A*2)
$$x_1u'x_n = v' \ ,$$
where $x_1, x_n \nmid u'$, $x_1 \underset{l}{\nparallel} v'$ *and* $x_n \underset{r}{\nparallel} v'$*;*

(*A*3)
$$x_1u'(x_2, \dots, x_n) = v'(x_2, \dots, x_n)x_1 \ .$$

Corollary 4.3. [45] *The following conditions are equivalent for the identity* (1)*:*
(*i*) (1) *is a* $(\mathcal{LG} \circ \mathcal{S}) \circ \mathcal{N}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$, $C_{2,1}$ *and* R_2*;*
(*iii*) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity and* $t(u) \neq t(v)$.

Corollary 4.4. [45] *The following conditions are equivalent for the identity* (1)*:*
(*i*) (1) *is a* $(\mathcal{G} \circ \mathcal{S}) \circ \mathcal{N}$*-identity;*
(*ii*) (1) *is a* $(\mathcal{G} \circ \mathcal{S}) \circledast \mathcal{N}$*-identity;*
(*iii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$, $C_{2,1}$, R_2 *and* L_2*;*
(*iv*) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $t(u) \neq t(v)$ *and* $h(u) \neq h(v)$.

Corollary 4.5. [45] *The following conditions are equivalent for the identity* (1)*:*
(*i*) (1) *is a* $(\mathcal{G} \circ \mathcal{B}) \circ \mathcal{N}$*-identity;*
(*ii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$, $C_{2,1}$, $LZ(n)$ *and* $RZ(n)$*, for* $n \in \mathbf{Z}^+$, $n \geq 2$*;*
(*iii*) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity and* $\mathcal{UG} \triangleright \mathcal{G} \circ \mathcal{B}$*-identity.*

Theorem 4.9. [45] *The following conditions are equivalent for the identity* (1)*:*

(*i*) (1) *is a* $\mathcal{UG} \circledast \mathcal{N}$*-identity;*

(*ii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$, $C_{2,1}$, $L_{3,1}$ *and* $R_{3,1}$*;*

(*iii*) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity and*

$$h^{(2)}(u) \neq h^{(2)}(v) \qquad \text{and} \qquad t^{(2)}(u) \neq t^{(2)}(v) \ .$$

Corollary 4.6. [45] *The following conditions are equivalent for the identity* (1)*:*

(*i*) (1) *is a* $(\mathcal{LG} \circ \mathcal{S}) \circledast \mathcal{N}$*-identity;*

(*ii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$, $C_{2,1}$, $L_{3,1}$ *and* R_2*;*

(*iii*) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t(u) \neq t(v)$.

Lemma 4.2. [130] *Let* Q *be a nil-semigroup which satisfies the identity*

$$x_1x_2 \dots x_n = w \ ,$$

where $|w| \geq n+1$. *Then* $Q^n = \{0\}$.

Let A_N^+ be the free semigroup over an alphabet $A_N = \{\ x_k \mid k \in \mathbf{Z}^+\ \}$ and let

$$I = \{\ u \in A_N^+ \mid (\exists x_i \in A_N)\ |x_i|_u \geq 2\ \}.$$

Then I is an ideal of A_N^+. By D_N we will denote the factor semigroup $(A_N^+)/I$. It is clear that D_N is isomorphic to the semigroup

$$(\{\ u \in A_N^+ \mid \Pi(u) = c(u)\ \} \cup \{0\}, \cdot) \ ,$$

where the multiplication " $\cdot$ " is defined by

$$u \cdot v = \begin{cases} uv & \text{if } u, v \neq 0 \text{ and } c(u) \cap c(v) = \varnothing \\ 0 & \text{otherwise} \end{cases} .$$

D_N is a nil-semigroup and it is not nilpotent.

Theorem 4.10. [45] *Let* $k \in \mathbf{Z}^+$. *Then the following conditions are equivalent for the identity* (1)*:*

(*i*) (1) *is a* $\mathcal{UG} \circ \mathcal{N}_k$*-identity;*

(*ii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$, $C_{2,1}$, D_N *and* N_{k+1}*;*

(*iii*) $n \leq k+1$ *and* (1) *is p-equivalent to some identity of the form*

$$x_1x_2 \dots x_n = w \ , \tag{15}$$

where $|w| \geq n+1$, $x_1 \underset{l}{\nparallel} w$ *and* $x_n \underset{l}{\nparallel} w$.

Corollary 4.7. [45] *Let* $k \in \mathbf{Z}^+$. *Then the following conditions are equivalent for the identity* (1)*:*

(*i*) (1) *is a* $\mathcal{UG} \circledast \mathcal{N}_k$*-identity;*

(*ii*) (1) *is not satisfied in semigroups* $C_{1,1}$, $C_{1,2}$, $C_{2,1}$, $L_{3,1}$, $R_{3,1}$, D_N *and* N_{k+1}*;*

(*iii*) (1) *is p-equivalent to some identity of the form*

$$x_1x_2 \dots x_n = w \ ,$$

where $|w| \geq n+1$, $h^{(2)}(u) \neq x_1x_2$ *and and* $t^{(2)}(v) \neq x_{n-1}x_n$.

Corollary 4.8. [52] *A semigroup identity determines a variety of inflations of unions of groups if and only if this identity has one of the following forms:*
(*i*) $x = w$, *where* w *is a word other than* x;
(*ii*) $xy = w$, *where* w *is a word other that* yx *and which neither begins nor ends in* xy.

Structure of semigroups satisfying some of $\mathcal{UG} \circ \mathcal{N}$-identities are considered by E.Tully (see [157]), T.Tamura [157], Lee Sin-Min [132], M.Petrich [104], J.Gerhard [74], M.Ćirić and S.Bogdanović [42] and so on. Connections between some types of $\mathcal{UG} \circ \mathcal{N}$-identities and semigroup varieties are considered by А.Б.Тищенко in [169].

4.5. Identities over the twoelement alphabet

In the next the following identity will be considered

$$u(x, y) = v(x, y). \tag{2}$$

Theorem 4.11. [46] *The identity* (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity if and only if it is p-equivalent to one of the following identities:*
(*A*1) $xy = w$, *where* $w \in A_2^+ - \{xy\}$;
(*A*2) $(xy)^k = w$, *where* $k \in \mathbf{Z}^+$, $k \geq 2$ *and* $w \in A_2^+ - \{(xy)^m \mid m \in \mathbf{Z}^+\}$;
(*A*3) $(xy)^k x = w$, *where* $k \in \mathbf{Z}^+$ *and* $w \in A_2^+ - \{(xy)^m x \mid m \in \mathbf{Z}^+\}$;
(*A*4) $xy^k = w$, *where* $k \in \mathbf{Z}^+$, $k \geq 2$ *and* $w \in A_2^+ - \{xy^m \mid m \in \mathbf{Z}^+\}$;
(*A*5) $x^k y = w$, *where* $k \in \mathbf{Z}^+$, $k \geq 2$ *and* $w \in A_2^+ - \{x^m y \mid m \in \mathbf{Z}^+\}$.

Theorem 4.12. [46] *The following conditions for the identity* (2) *are equivalent:*
(*i*) (2) *is a* $\mathcal{LA} \circ \mathcal{S}$*-identity;*
(*ii*) (2) *is not satisfied in semigroups* B_2 *and* R_2;
(*iii*) (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity and* $t(u) \neq t(v)$.

Corollary 4.9. [46] *The following conditions for the identity* (2) *are equivalent:*
(*i*) (2) *is a* $\mathcal{TA} \circ \mathcal{S}$*-identity;*
(*ii*) (2) *is not satisfied in semigroups* B_2, R_2 *and* L_2;
(*iii*) (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity,* $t(u) \neq t(v)$ *and* $h(u) \neq h(v)$.

Theorem 4.13. [46] *The identity* (2) *is a* $\mathcal{CS} \triangleright \mathcal{M} \times \mathcal{G}$*-identity if and only if one of the following conditions holds:*
(*B*1) $h(u) \neq h(v)$ *or* $t(u) \neq t(v)$;
(*B*2) (1) *is p-equivalent to some identity of the form*

$$x^{m_1} y^{n_1} x^{m_2} y^{n_2} \cdots x^{m_h} y^{n_h} = x^{k_1} y^{l_1} x^{k_2} y^{l_2} \cdots x^{k_s} y^{l_s}$$

$m_i,\ n_i,\ k_j,\ l_j \in \mathbf{Z}^+$, *with* $\gcd(p_x, p_y, h - s) = 1$, *where* $p_x = \Sigma_{i=1}^{h} m_i - \Sigma_{j=1}^{s} k_j$ *and* $p_y = \Sigma_{i=1}^{h} n_i - \Sigma_{j=1}^{s} l_j$.
(*B*3) (1) *is p-equivalent to some identity of the form*

$$x^{m_1} y^{n_1} x^{m_2} y^{n_2} \cdots x^{m_h} y^{n_h} x^{m_{h+1}} = x^{k_1} y^{l_1} x^{k_2} y^{l_2} \cdots x^{k_s} y^{l_s} x^{k_{s+1}}$$

$m_i,\ n_i,\ k_j,\ l_j \in \mathbf{Z}^+$, *with* $\gcd(p_x, p_y, h - s) = 1$, *where* $p_x = \Sigma_{i=1}^{h+1} m_i - \Sigma_{j=1}^{s+1} k_j$ *and* $p_y = \Sigma_{i=1}^{h} n_i - \Sigma_{j=1}^{s} l_j$.

Corollary 4.10. [46] *The identity* (2) *is a* $\pi\mathcal{R}\triangleright(\mathcal{M}\times\mathcal{G}\circ\mathcal{N})\circ\mathcal{S}$*-identity if and only if* (2) *is a* $\mathcal{A}\circ\mathcal{S}$*-identity and a* $\mathcal{CS}\triangleright\mathcal{M}\times\mathcal{G}$*-identity.*

Proposition 4.2. [46] *The identity* (2) *is a* $\mathcal{UG}\circ\mathcal{N}$*-identity if and only if it is p-equivalent to an identity of one of the following forms:*
$(C1)$ $xy = w$, *where* $w\in A_2^+$ *and* $w\notin\{xy^m \mid m\in\mathbf{Z}^+\}\cup\{x^my \mid m\in\mathbf{Z}^+\}\cup\{yx\}$;
$(C2)$ $xy^m = x^ny$, *where* $m,n\in\mathbf{Z}^+$, $m,n\geq 2$.

Proposition 4.3. [46] *The identity* (2) *is a* $\mathcal{UG}\circledast\mathcal{N}$*-identity if and only if it is p-equivalent to an identity of one of the following forms:*
$(D1)$ $xy = w$, *where* $w\in A_2^+$, $|w|\geq 3$ *and* $h^{(2)}(w)\neq xy\neq t^{(2)}(w)$;
$(D2)$ $xy^m = x^ny$, *where* $m,n\in\mathbf{Z}^+$, $m,n\geq 2$.

4.6. Problems of Tamura's type

T.Tamura in [157] posed the problem of describing the structure of semigroups satisfying the identity of the form $xy = w(x,y)$, $|w|\geq 3$, which we call *Tamura's problems*. Firstly, solutions of some types of Tamura's problems will be presented.

A regular semigroup is *orthodox* if its idempotents form a subsemigroup. An orthodox union of groups we call an *orthogroup*, a band of groups we call a *cryptogroup* and an orthodox band of groups we call an *orthocryptogroup*. If S is a union of groups, then Green's $\mathcal{J}$-classes of S we call *completely simple components* of S.

Theorem 4.14. [46] *Let* S *be an orthogroup. Then* S *satisfies the identity* $u = v$ *over the alphabet* A_n *if and only if all of its subgroups satisfy* $u = v$, $l(u)(\overline{a})\ \mathcal{L}\ l(v)(\overline{a})$ *and* $r(u)(\overline{a})\ \mathcal{R}\ r(v)(\overline{a})$, *for all* $\overline{a}\in S^n$.

Corollary 4.11. [46] *Let* S *be an orthocryptogroup. Then* S *satisfies the identity* $u = v$ *if and only if all of its subgroups satisfy* $u = v$ *and* $S/\mathcal{H}$ *satisfies* $u = v$.

Theorem 4.15. [46] *Let* S *be an cryptogroup and let* $u = v$ *be an identity over the alphabet* A_2 *such that* $h(u) = h(v)$ *and* $t(u) = t(v)$. *Then* S *satisfies* $u = v$ *if and only if all of its completely simple components satisfy* $u = v$.

Let we consider the identity

$$xy = x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots x^{m_h}y^{n_h}\ , \tag{3}$$

with $h,\ m_i,\ n_i\in\mathbf{Z}^+$, $i\in\{1,2,\dots,h\}$ and $h = 1 \Rightarrow m_1,n_1\geq 2$. Also, $p_x = \Sigma_{i=1}^h m_i - 1$, $p_y = \Sigma_{i=1}^h n_i - 1$, $p = \gcd(p_x,p_y)$ (i.e. p is the period of the identity (3)).

Theorem 4.16. [46] *A semigroup* S *satisfies the identity* (2) *with* $\gcd(p_x,p_y, h-1) = 1$ *if and only if* S^2 *is an orthogroup whose subgroups satisfy* (2), $ab\ \mathcal{L}\ a^{m_1}b$ *and* $ab\ \mathcal{R}\ ab^{n_h}$, *for all* $a,b\in S$.

Corollary 4.12. [46] *A semigroup* S *satisfies the identity* (3) *with* $m_1,n_h = 1$ *if and only if* S^2 *is an orthogroup whose subgroups satisfy* (3).

Theorem 4.17. [46] *A semigroup* S *satisfies the identity* (3) *with* $m_1, n_h \geq 2$ *and* $\gcd(p_x, p_y, m_1 - 1) = \gcd(p_x, p_y, n_h - 1) = 1$ *if and only if* S *is an inflation of a cryptogroup whose completely simple components satisfy* (3)*.*

Corollary 4.13. [46] (*i*) *A semigroup* S *satisfies the identity* (3) *with* $p = m_1 = n_h = 1$ *if and only if* S^2 *is a band.*

(*ii*) *A semigroup* S *satisfies the identity* (3) *with* $p = 1$ *and* $m_1, n_h \geq 2$ *if and only if* S *is an inflation of a band if and only if* S *satisfies the system of identities* $xy = x^2y = xy^2$ *if and only if* S *satisfies the identity* $xy = x^2y^2$*.*

(*iii*) *A semigroup* S *satisfies the identity* (3) *with* $p = 1$ *and* $m_1 \geq 2$, $n_h = 1$ ($m_1 = 1$, $n_h \geq 2$) *if and only if* S^2 *is a band and* S *satisfies the identity* $xy = x^2y$ ($xy = xy^2$)*.*

Corollary 4.14. [46] *A semigroup* S *satisfies the identity* (3) *with* $m_1, n_h \geq 2$ *and* $\gcd(p_x, p_y, m_1 - 1) = \gcd(p_x, p_y, n_h - 1) = \gcd(p_x, p_y, h - 1) = 1$ *if and only if* S *is an inflation of an orthocryptogroup whose subgroups satisfy* (3)*.*

Let we consider the identity

$$xy = y^{n_0}x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots x^{m_h}y^{n_h}\ , \tag{4}$$

with $h,\ n_0,\ m_i,\ n_i \in \mathbf{Z}^+$, $i \in \{1, 2, \ldots, h\}$. Also, $p_x = \Sigma_{i=1}^{h} m_i - 1$, $p_y = \Sigma_{i=0}^{h} n_i - 1$, $p = \gcd(p_x, p_y)$ (i.e. p is the period of the identity (4)).

Theorem 4.18. [46] *A semigroup* S *satisfies the identity* (4) *if and only if* S^2 *is a semilattice of right groups whose subgroups satisfy* (4) *and* $ab\ \mathcal{R}\ ab^{n_h}$*, for all* $a, b \in S$*.*

Corollary 4.15. [17] *A semigroup* S *satisfies the identity* (4) *with* $n_h = 1$ *if and only if* S^2 *is a semilattice of right groups whose subgroups satisfy* (4)*.*

Corollary 4.16. [46] (*i*) *A semigroup* S *satisfies the identity* (4) *with* $p = n_h = 1$ *if and only if* S^2 *is a right regular band.*

(*ii*) *A semigroup* S *satisfies the identity* (4) *with* $p = 1$ *and* $n_h \geq 2$ *if and only if* S *is an inflation of a right regular band.*

Theorem 4.19. [46] *A semigroup* S *satisfy the identity* (4) *with* $n_h \geq 2$ *and* $\gcd(p_x, p_y, n_h - 1) = 1$ *if and only if* S *is an inflation of a right regular band of groups whose subgroups satisfy* (4)*.*

Let we consider the identity

$$xy = y^{n_1}x^{m_1}y^{n_2}x^{m_2}y^{n_2}\cdots y^{n_h}x^{m_h}\ , \tag{5}$$

with $h,\ m_i,\ n_i \in \mathbf{Z}^+$, $i \in \{1, 2, \ldots, h\}$, $\Sigma_{i=1}^{h} m_i + \Sigma_{i=0}^{h} n_i \geq 3$.

Theorem 4.20. [157] *A semigroup* S *satisfies the identity* (5) *if and only if* S *is an inflation of a semilattice of groups satisfying* (5)*.*

Except Tamura's problems, some other problems of this type will be also quoted. Let we consider the identity

$$xy^m = x^ny\ , \tag{6}$$

with $m, n \in \mathbf{Z}^+$, $m, n \geq 2$. By $p = \gcd(m - 1, n - 1)$ we will denote the period of the identity (6).

Theorem 4.21. [46] *A semigroup* S *satisfies the identity* (6) *if and only if* S *is a retractive extension of a semigroup which satisfies* $x = x^{p+1}$ *by a nil-semigroup which satisfies* (6).

Let $n \in \mathbf{Z}^+$, $n \geq 2$. A semigroup S is *left* (*right*) *n-distributive* if it satisfies the identity $a(x_1x_2\dots x_n) = (ax_1)(ax_2)\dots(ax_n)$ $((x_1x_2\dots x_n)a = (x_1a)(x_2a)\dots(x_na))$. A semigroup S is *n-distributive* if it is both left and right n-distributive. A 2-distributive (left 2-distributive, right 2-distributive) semigroup is *distributive* (*left distributive, right distributive*).

Theorem 4.22. [42] *A semigroup* S *is* n*-distributive if and only if* S *is an* n*-inflation of an orthodox semigroup which is a normal band of commutative groups satisfying* $x^n = x$.

Corollary 4.17. [104] *A semigroup* S *is distributive if and only if* S *is a* 2*-inflation of a normal band.*

Theorem 4.23. [92] *The following conditions on a semigroup* S *are equivalent:*

(i) S *satisfies the identities* $xy^m = yx^m = (xy^m)^n$, $n > 1$;

(ii) S *contains a commutative Clifford's subsemigroup* M *and satisfies*

($A1$) $x^{k+1} = x$ *for all* $x \in M$, *where* $k = \gcd(m-1, n-1)$;

($A2$) $xy^m \in M$ *for all* $x, y \in S$;

(i) S *is a semilattice* Y *of semigroups* S_α, $\alpha \in Y$, *such that each* S_α *is an ideal extension of a group* G_α *by* Q_α *and the following conditions are satisfied:*

($B1$) *each* G_α *is commutative and satisfies* $x^k = e$ *for all* $x \in G_\alpha$, *where* e *is the identity element of* G_α *and* k *being defined in* ($A1$);

($B2$) Q_α *satisfies* $xy^m = 0$ *for all* $x, y \in Q_\alpha$, *where* 0 *is the zero element of* Q_α;

($B3$) *if* $x \in S_\alpha$, $y \in S_\beta$, $\alpha \neq \beta$, *then* $xy^m \in G_{\alpha\beta}$.

Finally, we quote a result of J.Chrislock [40] which describe semigroups satisfying a heterotype identity.

Theorem 4.24. *The following conditions on a semigroup* S *are equivalent:*

(i) S *satisfies a heterotype identity;*

(ii) S *satisfies an identity of the form* $(x^ky^kx^k)^k = x^k$, $k \in \mathbf{Z}^+$;

(i) *there exists* $r \in \mathbf{Z}^+$ *such that* S *is an ideal extension of a completely simple semigroup whose structure group satisfies* $x^r = 1$ *by a semigroup that satisfies* $y^r = 0$.

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