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POWER SEMIGROUPS THAT ARE ARCHIMEDEAN

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Abstract. *Power semigroups of various semigroups were studied by a number of authors. Here we give structural characterizations for semigroups whose power semigroups are Archimedean and we generalize some results from* [1], [8], [10] *and* [11].

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. For an element a of a semigroup S, $\langle a\rangle$ wil denote the *cyclic subsemigroup* of S generated by a. For a semigroup S, let $\mathbf{P}(S) = \{A \mid \emptyset \neq A \subseteq S\}$. If the multiplication on $\mathbf{P}(S)$ is defined by $AB = \{ab \mid a \in A,\ b \in B\}$, then $\mathbf{P}(S)$ is a semigroup which will be called the *power semigroup* of S, [11].

A semigroup S is *intra-π-regular* if for each $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in Sa^{2n}S$. A semigroup S is *left π-regular* if for each $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in Sa^{n+1}$, and it is *left regular* if for any $a \in S$, $a \in Sa^2$. *Right π-regular* and *right regular* semigroups are defined dually.

A semigroup S is *Archimedean* if for any $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$. A semigroup S is *left Archimedean* (*weakly left Archimedean*) if for any $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in Sb$ ($a^n \in Sba$), [4]. *Right Archimedean* and *weakly right Archimedean* semigroups are defined dually. A semigroup S is *t-Archimedean* (*weakly t-Archimedean*) if it is both left and right Archimedean (weakly left and weakly right Archimedean). A semigroup S is *power joined* if for any $a, b \in S$ there exists $m, n \in \mathbf{Z}^+$ such that $a^m = b^n$. A semigroup S is *left completely Archimedean* if it is Archimedean and left π-regular. *Right completely Archimedean* semigroups are defined dually. A semigroup S is *completely Archimedean* if it is both left and right completely Archimedean. A semigroup S is *left completely simple* if it is simple and left regular. *Right completely simple* semigroups

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are defined dually. A semigroup S is *completely simple* if it is both left and right completely simple.

Further, $S = S^0$ will means that S is a semigroup with zero 0. A semigroup $S = S^0$ is a *nil-semigroup* if for any $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. For $n \in \mathbf{Z}^+$, a semigroup $S = S^0$ is *n-nilpotent* if $S^n = \{0\}$, and $S = S^0$ is *nilpotent* if it is n-nilpotent, for some $n \in \mathbf{Z}^+$. An ideal extension S of a semigroup T will be called a *nil-extension* (*nilpotent extension, n-nilpotent extension*) if S/T is a nil-semigroup (nilpotent semigroup, n-nilpotent semigroup).

Let T be a subsemigroup of a semigroup S. A mapping φ of S onto T will be called a *right retraction* if $a\varphi = a$, for each $a \in S$, and $(ab)\varphi = a(b\varphi)$, for all $a, b \in S$. *Left retractions* are defined dually. A mapping φ of S onto T is a *retraction* if it is a homomorphism and $a\varphi = a$, for each $a \in T$. If T is an ideal of S, then φ is a retraction of S onto T if and only if it is both left and right retraction of S onto T. An ideal extension S of a semigroup T is a (*left, right*) *retractive* extension of T if there exists a (*left, right*) *retraction* of S onto T. A (left, right) retractive extension by an n-nilpotent semigroup will be called a (*left, right*) *n-inflation*, 2-inflations will be called simply *inflations*, and (left, right) retractive extensions by nilpotent semigroups will be called (*left, right*) *inflationary* extensions.

A semigroup S is a *singular band* if it is either a left zero band or a right zero band.

For undefined notions and notations we refer to [2], [3] and [7].

Theorem 1. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is Archimedean;*
(ii) $\mathbf{P}(S)$ *is a nil-extension of a simple semigroup;*
(iii) $\mathbf{P}(S)$ *is Archimedean with an idempotent.*

Proof. (i) $\Rightarrow$ (iii). Asumme $a \in S$. For $\{a\}$, $\langle a\rangle \in \mathbf{P}(S)$ there exists $B,\ C \in \mathbf{P}(S)$ and $n \in \mathbf{Z}^+$ such that $\{a\}^n = B\langle a\rangle C$, so for $b \in B,\ c \in C$ and $a^{2n} \in \langle a\rangle$ we have

$$a^n = ba^{2n}c \in Sa^{2n}S.$$

Therefore, S is intra-π-regular semigroup. Since S is also Archimedean, then by Theorem VI 1.1 [2], S is a nil-extension of a simple semigroup K. Thus, $\mathbf{P}(S)$ is an Archimedean semigroup with an idempotent K.

(iii) $\Rightarrow$ (ii). This follows by Theorem 3.2 [6].

(ii) $\Rightarrow$ (i). This follows by Theorem VI 1.1 [2]. $\square$

Corollary 1. *If $\mathbf{P}(S)$ is Archimedean, then S is a nilpotent extension of a simple semigroup.*

Proof. By the proof of (i) $\Rightarrow$ (iii) in Theorem 1, S is a nil-extension of a simple semigroup K. Since $\mathbf{P}(S)$ is Archimedean, there exists $n \in \mathbf{Z}^+$, $A,\ B \in \mathbf{P}(S)$ such that $S^n = AKB$, whence $S^n = AKB \subseteq K = K^n \subseteq S^n$. Therefore, $S^n = K$, so S is a nilpotent extension of a simple semigroup. □

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is left completely Archimedean;*
(ii) $\mathbf{P}(S)$ *is completely Archimedean;*
(iii) $\mathbf{P}(S)$ *is a nil-extension of a rectangular band;*
(iv) S *is a nilpotent extension of a rectangular band.*

Proof. (i) $\Rightarrow$ (ii). By Theorem 1, $\mathbf{P}(S)$ has an idempotent, so by Corollary 4 [4], $\mathbf{P}(S)$ is completely Archimedean.

(ii) $\Rightarrow$ (iv). Let $a \in S$. By Theorem 1, $S^n = K$ is a simple semigroup, for some $n \in \mathbf{Z}^+$. Also, by Theorem VI 2.2.1 [2], there exists $m \in \mathbf{Z}^+,\ C \in \mathbf{P}(S)$ such that $\{a\}^m = \{a\}^m\langle a\rangle C\{a\}^m$. Now, for any $c \in C$ we have

$$a^m = a^m aca^m = a^m a^2 ca^m = aa^m aca^m = aa^m = a^{m+1},$$

and by this it follows that K is a rectangular band.

(iv) $\Rightarrow$ (iii). Let $S^n = K$ be a rectangular band, for some $n \in \mathbf{Z}^+$. By Lemma 4 [8], $\mathbf{P}(K)$ is an ideal of $\mathbf{P}(S)$, and by Theorem 4 [10], $\mathbf{P}(K)$ is an inflation of a rectangular band T. Since $T^2 = T$, T is an ideal of $\mathbf{P}(K)$ and $\mathbf{P}(K)$ is an ideal of $\mathbf{P}(S)$, then T is an ideal of $\mathbf{P}(S)$. Also, for $A \in \mathbf{P}(S)$, $A^n \subseteq S^n = K$, so $A^n \in \mathbf{P}(K)$, whence $A^{2n} \in T$. Thus, $\mathbf{P}(S)$ is a nil-extension of a rectangular band T.

(iii) $\Rightarrow$ (i). This follows immediately. □

Corollary 1. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is an inflation of a rectangular band;*
(ii) S *is an inflation of a rectangular band;*
(iii) $(\forall x, y, z \in S)\ xz = xyz$.

Proof. (ii) $\Leftrightarrow$ (iii). This follows by Corollary 3.5 [5].

(iii) $\Rightarrow$ (i). For $A, B, C \in \mathbf{P}(S)$, by (iii) we obtain that $AC = ABC$, so by (ii) $\Leftrightarrow$ (iii) we obtain (i).

(i) $\Rightarrow$ (ii). This follows immediately. □

Theorem 3. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is weakly left Archimedean;*
(ii) $\mathbf{P}(S)$ *is a right zero band of nil-extensions of left zero bands;*
(iii) S *is a right inflationary extension of a rectangular band.*

Proof. (i) ⇒ (ii). By Theorem 1, $\mathbf{P}(S)$ has an idempotent, so by Theorem 7 [4] we obtain (ii).

(ii) ⇒ (i). This follows immediately.

(i) ⇒ (iii). By Theorem 2, S is a nilpotent extension of a rectangular band K. On the other hand, it is not hard to check that S is weakly left Archimedean, so by Theorem 7 [4], S is a right retractive nil-extension of a rectangular band T. Clearly, $K = T$, so (iii) holds.

(iii) ⇒ (i). Let S be a right inflationary extension of a rectangular band K and let φ be a right retraction of S onto K. By the proof of Theorem 2, $\mathbf{P}(S)$ is a nil-extension of $\mathbf{P}(K)$ and $\mathbf{P}(K)$ is an inflation of a rectangular band T. Further, T is a right zero band Y of left zero bands T_α, $\alpha \in Y$, so $\mathbf{P}(K)$ is a right zero band Y of semigroups P_α, $\alpha \in Y$, where for each $\alpha \in Y$, P_α is an inflation of T_α. Assume $A, B \in \mathbf{P}(S)$. Then $A^n, B^n \in T$, for some $n \in \mathbf{Z}^+$, and $A^n \in T_\alpha$, $B^n \in T_\beta$, for some $\alpha, \beta \in Y$. Now, $A\varphi \in \mathbf{P}(K)$, i.e. $A\varphi \in P_\gamma$, for some $\gamma \in Y$, so

$$A^n = A^{n+1} = A^{n+1}\varphi = (A^n A)\varphi = A^n(A\varphi) \in P_\alpha P_\gamma \subseteq P_\gamma,$$

and by $A^n \in T_\alpha$ we obtain $\gamma = \alpha$, i.e. $A\varphi \in P_\alpha$, whence

$$B^n A = (B^n A)\varphi = B^n(A\varphi) \in T_\beta P_\alpha \subseteq T \cap P_\alpha = T_\alpha.$$

Therefore, $A^n, B^nA \in T$, whence $A^n = A^n B^n A$, since T_α is a left zero band. Hence, $\mathbf{P}(S)$ is weakly left Archimedean. □

Corollary 3. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is weakly t-Archimedean;*
(ii) $\mathbf{P}(S)$ *is a matrix of nil-semigroups;*
(iii) S *is an inflationary extension of a rectangular band.*

Proof. This follows by Theorems 1 and 3 and Corollary 5 [4]. □

Theorem 4. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is left Archimedean;*
(ii) $\mathbf{P}(S)$ *is a nil-extension of a left zero band;*
(iii) S *is a nilpotent extension of a left zero band.*

Proof. (i) ⇒ (ii). By Theorem 1, $\mathbf{P}(S)$ has an idempotent, so by Theorem VI 3.2.1 [2], $\mathbf{P}(S)$ is a nil-extension of a left group. On the other hand, by Theorem 2, $\mathbf{P}(S)$ is a nil-extension of a rectangular band, and so $\mathbf{P}(S)$ is a nil-extension of a left zero band.

(ii) ⇒ (iii). Let $\mathbf{P}(S)$ be a nil-extension of a left zero band T. By Theorem 2, S is an n-nilpotent extension of a rectangular band K, for some $n \in \mathbf{Z}^+$.

For $a, b \in K$, $\{a\}, \{b\} \in T$, whence $\{a\} \cdot \{b\} = \{a\}$, i.e. $ab = a$. Thus, K is a left zero band.

(iii) $\Rightarrow$ (ii). Let S be an n-nilpotent extension of a left zero band K, for some $n \in \mathbf{Z}^+$. By Theorem 2, $\mathbf{P}(S)$ is a nil-extension of a rectangular band T. Let $A, B \in T$. Then $A = A^n \subseteq S^n = K$ and also $B \subseteq K$, whence $AB = A$. Therefore, T is a left zero band.

(ii) $\Rightarrow$ (i). This follows immediately. □

Corollary 4. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is left completely simple;*
(ii) $\mathbf{P}(S)$ *is completely simple;*
(iii) $\mathbf{P}(S)$ *is a rectangular band;*
(iv) $\mathbf{P}(S)$ *is a singular band;*
(v) S *is a singular band.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). This follows by Theorem 2.

(iii) $\Rightarrow$ (v). By (iii), each subset of S is its subsemigroup, so by the well-known result of L. Rédei [9], S is an ordinal sum of singular bands (for the definition of an ordinal sum see [7]). By Theorem 2, S is semilattice indecomposable, whence S is a singular band.

(v) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i). This follows immediately. □

Corollary 5. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is t-Archimedean;*
(ii) $\mathbf{P}(S)$ *is power joined;*
(iii) $\mathbf{P}(S)$ *is a nil-extension of a group;*
(iv) $\mathbf{P}(S)$ *is a nil-semigroup;*
(v) $\mathbf{P}(S)$ *is nilpotent;*
(vi) S *is nilpotent.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) was proved by S. Bogdanović [1], and in the commutative case, (i) $\Leftrightarrow$ (vi) was proved by M. S. Putcha [8]. □

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