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INVITED LECTURE

THEORY OF GREATEST DECOMPOSITIONS OF SEMIGROUPS (A SURVEY)

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Dedicated to Professor L. N. Shevrin on his 60th birthday

Contents

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Introduction

As known, one of the best methods used in studying of structure of semigroups, as well as other algebras, is the *decomposition method*. The main idea of this method is to decompose a semigroup into components, possibly of simpler structure, to study the components in details and to establish mutual relationships between the components within the entire semigroup. We differentiate two general kinds of decompositions: *external decompositions*, where we include decompositions into a direct product and related concepts, and *internal decompositions*, by which we mean decompositions by equivalence relations. In this paper our attention will be aimed only to internal decompositions, which will be here called simply *decompositions*.

By a *kind of decompositions* we will mean a mapping $\mathfrak{T} : S \mapsto \mathfrak{T}_S$ by which to any semigroup S we associate a subset $\mathfrak{T}_S$, possibly empty, of the partition lattice Part (S) of S. But it is often of interest to consider such kinds of decompositions which can be applied on any semigroup, i.e. such that $\mathfrak{T}$ is nonempty subset of Part (S), for any semigroup S. For example, many kinds of decompositions have the property that for any semigroup S, $\mathfrak{T}_S$ contains the zero of Part (S), i.e. the one-component partition $\{S\}$. For that reason we define a *type of decompositions*, or a *decomposition type*, as a mapping $\mathfrak{T} : S \mapsto \mathfrak{T}_S$ by which to any semigroup S we associate a subset $\mathfrak{T}_S$ of the partition lattice Part (S) of S, containing its zero. In other words, a decomposition type $\mathfrak{T}$ is a collection of sets $\mathfrak{T}_S$ indexed by the set of all semigroups, and it is defined if for any semigroup S we define what are the elements of $\mathfrak{T}_S$. Of course, any type $\mathfrak{T}$ of decompositions induces a mapping $\mathfrak{T}' : S \mapsto \mathfrak{T}'_S$ by which to any semigroup S we associate a subset $\mathfrak{T}'_S$ of the lattice $\mathcal{E}(S)$ of equivalence relations on S, containing the universal relation on S, called a *type of equivalences*, and vice versa. For a given type $\mathfrak{T}$ of decompositions and a semigroup S, the elements of $\mathfrak{T}_S$ will be called $\mathfrak{T}$*-decompositions* of S, and related equivalence relations will be called $\mathfrak{T}$-equivalences on S, and S will be called $\mathfrak{T}$*-indecomposable* if the one-component partition $\{S\}$ is the unique $\mathfrak{T}$-decomposition of S, i.e. if the universal relation is the unique $\mathfrak{T}$-equivalence on S.

Consider a decomposition type $\mathfrak{T}$ and a semigroup S. Since $\mathfrak{T}_S$ is a subset of the lattice Part (S), then $\mathfrak{T}_S$ is a poset with respect to usual ordering of partitions, from where several very important questions follow:

(1) Does $\mathcal{T}_S$ have a greatest element?
(2) Is $\mathcal{T}_S$ a complete lattice?
(3) Does $\mathcal{T}_S$ a complete sublattice of the partition lattice on S?

Such problems have been treated first by T. Tamura and N. Kimura [112], 1954, and [113], 1955. After that, they have been considered by many authors. The aim of this paper is to make a survey of main ideas, concepts and results concerning greatest decompositions of semigroups of various types. We will talk about the mostly important decomposition types and the results concerning these.

We know that one of the most important algebraic theorems is the famous *Birkhoff's representation theorem*, proved by G. Birkhoff in [3], 1944, which says that any algebra can be decomposed into a subdirect product of subdirectly irreducible algebras. Of course, in Theory of semigroups similar theorems are also very important. A decomposition type $\mathfrak{T}$ will be called *atomic* if there exists the greatest $\mathfrak{T}$-decomposition and their components are $\mathfrak{T}$-indecomposable. But only four atomic types of decompositions of semigroups are known: *semilattice decompositions*, whose atomicity has been proved by T. Tamura [110], 1956, *ordinal decompositions*, whose atomicity has been proved by E. S. Lyapin [62], 1960, $\cup$-*decompositions*, whose atomicity has been proved by L. N. Shevrin [96], 1965, and *orthogonal decompositions*, whose atomicity has been established by S. Bogdanović and M. Ćirić in [10], 1995. In this paper these decomposition types will take an outstanding place.

This paper is divided into five chapters.

In the first chapter we introduce notions and notations that will be used in the further text, we give a classification of decomposition types and define the types that will considered in this paper, and we also present several general results concerning decompositions by congruences.

Because of the great importance and enormous quantity of the results concerning semilattice decompositions of semigroups, these results will be separated from the ones concerning band decompositions and they will be presented in Chapter 2.

Chapter 3 is devoted to the remaining significant types of band decompositions. Namely, in this chapter we make a survey of the results on matrix and normal band decompositions of semigroups.

In Chapter 4 we consider decompositions of semigroups with zero: orthogonal decompositions, decompositions into a left, right and matrix sum of semigroups, and quasi-semilattice decompositions.

Finally, in Chapter 5 we talk about yet other types of decompositions: $\cup$-decompositions, ordinal decompositions, I-matrix decompositions and semilattice-matrix decompositions.

1. Preliminaries

This chapter is divided into three sections. In Section 1.1 we introduce notions and notations that will be used in the further text. In Section 1.2 we make a classification of decompositions and we single out the most important decomposition types, which will be treated later. Finally, in Section 1.3 we consider decompositions by congruence relations and we present several general results concerning these decompositions.

1.1. Basic notions and notations

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. Further, $S = S^0$ means that S is a semigroup with zero 0, and $S \neq S^0$ means that S is a semigroup without zero. If $S = S^0$, we will write 0 instead $\{0\}$, and if A is a subset of S, then $A^\bullet = A - 0$, $A^0 = A \cup 0$ and $A' = (S - A)^0$. If A is a subset of a semigroup S, then $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbf{Z}^+)\, x^n \in A\}$.

For a binary relation ξ on a set A, ξ^∞ will denote the transitive closure of ξ, ξ^{-1} will denote the relation defined by $a\,\xi^{-1}\,b \Leftrightarrow b\,\xi\,a$, and for $a \in A$, $a\xi = \{x \in A \mid a\,\xi\,x\}$ and $\xi a = \{x \in A \mid x\,\xi\,a\}$. By a *quasi-order* we mean a reflexive and transitive binary relation. If ξ is a quasi-order on a set A, then the relation $\widetilde{\xi}$ defined by $\widetilde{\xi} = \xi \cap \xi^{-1}$ is an equivalence relation called the *natural equivalence* of ξ. A relation ξ on a semigroup $S = S^0$ is called *left 0-restricted* if $0\xi = 0$. A *right 0-restricted* relation on S is defined dually, and a relation ξ on $S = S^0$ will be called *0-restricted* if it is both left and right 0-restricted, i.e. if $0\xi = \xi 0 = 0$. We say that a relation ξ on a semigroup S satisfies the *common multiple property*, briefly the *cm-property*, if for all $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$. Similarly, for a relation ξ on a semigroup $S = S^0$ we say that ξ satisfies the *0-common multiple property*, briefly the *0-cm-property*, if for all $a, b, c \in S$, $ab \neq 0$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$.

Let K be a subset of a lattice L (not necessary complete). If K contains the meet of any its nonempty subset having the meet in L, then K is called a *complete meet-subsemilattice* of L. A *complete join-subsemilattice* is defined dually. If K is both complete meet-subsemilattice and complete join-subsemilattice of L, then it is called a *complete sublattice* of L. If L is a lattice with unity, then any sublattice of L containing its unity is called a *1-sublattice* of L. Dually we define a *0-sublattice* of a lattice with zero, and we define a sublattice of a lattice L with zero and unity to be a *0,1-sublattice* if it is both 0-sublattice and 1-sublattice of L. If any element of L is the meet of some nonempty subset of K, then K is called *meet-dense* in L.

An element a of a lattice L with the zero 0 is an *atom* of L if $a > 0$ and there exists no $x \in L$ such that $a > x > 0$. A complete Boolean algebra B is *atomic* if every element of B is the join of some set of atoms of B. If L is a distributive lattice with zero and unity, then the set $\mathfrak{B}(L)$ of all elements of L having a complement in L is a Boolean algebra and it is called the *greatest Boolean subalgebra* of L.

For a nonempty set A, $\mathcal{P}(A)$ will denote the *lattice of subsets* of A. Let A be a nonempty set and let L be a sublattice of $\mathcal{P}(A)$ containing its unity and having the property that any nonempty intersection of elements of L is also in A. Then for any $a \in A$ there exists the smallest element of L containing a (it is the intersection of all elements of L containing a), which will be called the *principal element* of L generated by a.

A subset A of a semigroup S is called *completely semiprime* if for $x \in S$, $x^2 \in A$ implies $x \in A$, *completely prime* if for $x, y \in S$, $xy \in A$ implies the either $x \in A$ or $y \in A$, *left consistent* if for $x, y \in S$, $xy \in A$ implies $x \in A$, *right consistent* if for $x, y \in S$, $xy \in A$ implies $y \in A$, and it is *consistent* if it is both left and right consistent. Clearly, the empty set has any of these properties and the sets of completely semiprime, completely prime, left consistent, right consistent and consistent subsets are complete sublattices of $\mathcal{P}(S)$. A consistent subsemigroup of a semigroup S will be called a *filter* of S. The empty set will be also defined to be a filter. By $\mathcal{F}(S)$ we denote the *lattice of filters* of S, which is a complete meet-subsemilattice of $\mathcal{P}(S)$, and therefore a complete lattice, but it is not necessary a sublattice of $\mathcal{P}(S)$. It is well known that a subset A of a semigroup S is a filter of S if and only its complement is either empty or a completely prime ideal of S. The principal element $\mathcal{F}(S)$, called the *principal filter*, generated by $a \in S$ will be denoted by $N(a)$.

In studying of semigroups with zero we use some similar notions. Namely, a subset A of a semigroup $S = S^0$ is called *left 0-consistent* if $A^\bullet$ is left consistent, *right 0-consistent* if $A^\bullet$ is right consistent, and it is *0-consistent* if $A^\bullet$ is consistent. Similarly, an equivalence relation θ on $S = S^0$ will be called *left 0-consistent* if for $x, y \in S$, $xy \neq 0$ implies $xy\,\theta\,x$, *right 0-consistent* if for $x, y \in S$, $xy \neq 0$ implies $xy\,\theta\,y$, and *0-consistent* if it is both left and right 0-consistent.

Let S be a semigroup. By $\mathcal{I}d(S)$ we denote the *lattice of ideals* of S. This lattice is a sublattice of $\mathcal{P}(S)$, it is also a complete join-subsemilattice of $\mathcal{P}(S)$, but it is not necessary a complete meet-subsemilattice, since the empty set is not included in $\mathcal{I}d(S)$. The principal element of $\mathcal{I}d(S)$, called the *principal ideal*, generated by $a \in S$ will be denoted by $J(a)$.Further, $\mathcal{LI}d(S)$ will denote the *lattice of left ideals* of a semigroup S defined in the following way: if $S = S^0$, then $\mathcal{LI}d(S)$ consists of all left ideals of S, and if S has no zero, then $\mathcal{LI}d(S)$ consists of the empty set and all left ideals of S. The *lattice of right ideals* of S, in notation $\mathcal{RI}d(S)$, is defined dually. Lattices $\mathcal{LI}d(S)$ and $\mathcal{RI}d(S)$ are complete sublattices of $\mathcal{P}(S)$. The principal element of $\mathcal{LI}d(S)$, called the *principal left ideal*, generated by $a \in S$ will be denoted by $L(a)$. The *principal right ideal* generated by $a \in S$, defined dually, will be denoted by $R(a)$. By $\mathcal{I}d^{\mathbf{cs}}(S)$ we denote the *lattice of completely semprime ideals* of S, which is a complete 1-subsemilattice of $\mathcal{I}d(S)$. The principal element of $\mathcal{I}d^{\mathbf{cs}}(S)$, called the *principal radical*, generated by $a \in S$ will be denoted by $\Sigma(a)$. By $\mathcal{RI}d^{\mathbf{lc}}(S)$ and $\mathcal{LI}d^{\mathbf{rc}}(S)$ we denote the lattice of left consistent right ideals and the lattice of right consistent left ideals of S, which are complete sublattices of $\mathcal{RI}d(S)$ and $\mathcal{LI}d(S)$, respectively.

For a nonempty subset A of a semigroup S define the relations P_A, R_A and L_A by:

$$\begin{aligned} a\,P_A\,b \quad &\Leftrightarrow \quad (\forall x, y \in S)(xay \in A \;\Leftrightarrow\; xby \in A),\\ a\,R_A\,b \quad &\Leftrightarrow \quad (\forall y \in S)(ay \in A \;\Leftrightarrow\; by \in A),\\ a\,L_A\,b \quad &\Leftrightarrow \quad (\forall x \in S)(xa \in A \;\Leftrightarrow\; xb \in A). \end{aligned}$$

Then P_A is a congruence on S called the *principal congruence* on S defined by A, R_A is a right congruence called the *principal right congruence* on S defined by A, and L_A is a left congruence called the *principal left congruence* on S defined by A. If $\mathcal{A}$ is a nonempty family of subsets of S, then $P(\mathcal{A})$ will denote the congruence which is the intersection of all principal congruence on S defined by elements from $\mathcal{A}$.

Let A be a nonempty set and let $X \in \mathcal{P}(A)$. The relation Θ_X on A defined by

$$a\,\Theta_X\,b \quad \Leftrightarrow \quad a, b \in X \;\text{ or }\; a, b \in A - X \qquad (a, b \in A),$$

is an equivalence relation on A whose classes are precisely the nonempty sets among the sets X and $A - X$. Clearly, when $X = \varnothing$ or $X = A$, then Θ_X is the universal relation on A. Also, for any $X \in \mathcal{P}(A)$, $\Theta_X = \Theta_{A_X}$. Further, for a nonempty subset $\mathcal{A}$ of $\mathcal{P}(A)$, $\Theta(\mathcal{A})$ will denote the equivalence relation on A defined by:

$$\Theta(\mathcal{A}) = \bigcap_{X \in \mathcal{A}} \Theta_X .$$

If $\mathcal{A}$ is a complete meet-subsemilattice of $\mathcal{P}(A)$, and it contains the unity of $\mathcal{P}(A)$, then $\Theta(\mathcal{A})$ can be alternatively defined by:

$$a\,\Theta(\mathcal{A})\,b \;\Leftrightarrow\; \mathcal{A}(a) = \mathcal{A}(b) \qquad (a, b \in S),$$

where for $x \in A$, $\mathcal{A}(x)$ denotes the principal element of $\mathcal{A}$ generated by x.

For a semigroup S, $\mathcal{Q}(S)$ will denote the lattice of quasi-orders on S, $\mathcal{E}(S)$ will denote the lattice of equivalence relations on S and $\mathrm{Con}\,(S)$ will denote the lattice of congruence relations on S. It is well-known that $\mathrm{Con}\,(S)$ is a complete sublattice of $\mathcal{E}(S)$ and $\mathcal{E}(S)$ is a complete sublattice of $\mathcal{Q}(S)$. By $\mathcal{E}^{\bullet}(S)$ we denote the lattice of 0-restricted equivalence relation on a semigroup $S = S^0$, which is the principal ideal of $\mathcal{E}(S)$ generated by the equivalence relation χ determined by the partition $\{S^{\bullet}, 0\}$.

An ideal A of a semigroup S is a *prime ideal* if for $x, y \in S$, $xSy \subseteq A$ implies that either $x \in A$ or $y \in A$, or, equivalently, if for all ideals M and N of S, $MN \subseteq A$ implies that either $M \subseteq A$ or $N \subseteq A$. A completely 0-simple semigroup with the property that the structure group of its Rees-matrix representation is the one-element group, is called a *rectangular 0-band*. Equivalently, a rectangular 0-band can be defined as a semigroup $S = S^0$ in which 0 is a prime ideal and for all $a, b \in S$, either $aba = a$ or $aba = 0$.

For undefined notions and notations we refer to the following books: G. Birkhoff [2], S. Bogdanović [4], S. Bogdanović and M. Ćirić [7], S. Burris and H. P. Sankappanavar [17], A. H. Clifford and G. B. Preston [35], [36], G. Grätzer [45], J. M. Howie [48], E. S. Lyapin [62], M. Petrich [72], [73], L. N. Shevrin [98], L. N. Shevrin and A. Ya. Ovsyanikov [102], [103], O. Steinfeld [105] and G. Szász [109].

1.2. A classification of decompositions

In this section we classify decompositions of semigroups into few classes and we single out the most important types of decompositions.

Let us say again that by a decompositions of a semigroup S we mean a family $\mathcal{D} = \{S_\alpha\}_{\alpha \in Y}$ of subsets of S satisfying the condition

$$S = \bigcup_{\alpha \in Y} S_\alpha, \qquad \text{where} \;\; S_\alpha \cap S_\beta = \varnothing, \;\text{for}\;\; \alpha, \beta \in Y,\, \alpha \neq \beta.$$

Various special kinds of decompositions we obtain in two general ways: imposing some requirements on the structure of the components S_α, and imposing some requirements on products of elements from different classes.

The first general type of decompositions that we single out are *decompositions S onto subsemigroups*, determined by the property that any S_α is a subsemigroup of S. Clearly, decompositions onto subsemigroups correspond to equivalence relations satisfying the *cm*-property, so the following theorem can be easily proved:

Theorem 1.1. *The poset of decompositions of a semigroup S onto subsemigroups is a complete lattice which is dually isomorphic to the lattice of equivalence relations on S satisfying the cm-property.*

If to a decomposition of a semigroup S onto subsemigroups we impose an additional condition

$$ab \in \langle a \rangle \cup \langle b \rangle \,,$$

for all elements $a, b \in S$ belonging to the different components, then we obtain so called $\cup$-*decompositions*. Decompositions of this type will be considered in Section 5.1.

The second general class of decompositions that we single out form decompositions whose related equivalence relations are congruences. Decompositions of this type are called *decompositions by congruences*. When the decomposition $\mathcal{D}$ is a decomposition by a congruence relation, then the index set Y is a factor semigroup of S and many properties of S are determined by structure of the semigroup Y. Special types of decompositions by congruences we obtain imposing some requirements on the structure of the related factor semigroup. If a class $\mathfrak{C}$ of semigroups and a semigroup S are given, then a congruence relation θ on S is called a $\mathfrak{C}$-*congruence* on S if the related factor S/θ is in $\mathfrak{C}$, the related decomposition is given a $\mathfrak{C}$-*decomposition*, and the related factor semigroup is called a $\mathfrak{C}$-*homomorphic image* of S. When there exists the greatest $\mathfrak{C}$-decomposition of S, i.e. the smallest $\mathfrak{C}$-congruence on S, then we say that the factor semigroup corresponding to this congruence is the *greatest* $\mathfrak{C}$-*homomorphic image* of S. A semigroup S is called $\mathfrak{C}$-*indecomposable* if the universal relation is the unique $\mathfrak{C}$-congruence on S. Of course, when the class $\mathfrak{C}$ contains the trivial (one-element) semigroup, then the $\mathfrak{C}$-decompositions determine a decomposition type.

If the decomposition $\mathcal{D}$ is both a decomposition by a congruence relation and a decomposition onto subsemigroups, then it is called a *band decomposition* of S and the related congruence relation is called a *band congruence* on S. Equivalently, the type of band decompositions is defined as the type of $\mathfrak{C}$-decompositions, where $\mathfrak{C}$ equals the variety $[x^2 = x]$ of bands. Moreover, by some subvarieties of the variety of bands we define the following very important special types of band decompositions and band congruences:

- *semilattice decompositions* and *congruences*, determined by the variety $[x^2 = x, xy = yx]$ of *semilattices*;
- *matrix decompositions* and *congruences*, determined by the variety $[x^2 = x, xyx = x] = [x^2 = x, xyz = xz]$ of *rectangular bands*;
- *left* (*right*) *zero band decompositions* and *congruences*, determined by the variety $[x^2 = x, xy = x]$ ($[x^2 = x, xy = y]$) of *left* (*right*) *zero bands*;

– *normal band decompositions* and *congruences*, determined by the variety $[x^2 = x, xyzx = xzyx] = [x^2 = x, xyzu = xzyu]$ of *normal bands*;
– *left* (*right*) *normal band decompositions* and *congruences*, determined by the variety $[x^2 = x, xyz = xzy]$ ($[x^2 = x, xyz = yxz]$) of *left* (*right*) *normal bands*.

Also, *chain decompositions* and *congruences* are determined by the class of *chains* (linearly ordered semilattices). The decomposition $\mathcal{D}$ is called an *ordinal decomposition* if it is a chain decomposition, i.e. Y is a chain, and for all $a, b \in S$,

$$a \in S_\alpha,\ b \in S_\beta,\ \alpha < \beta \quad \Rightarrow \quad ab = ba = a.$$

These decompositions will be considered in Section 5.2. In the last chapter of this paper we will also consider I-matrix decompositions and semilattice-matrix decompositions, which will be precisely defined in Sections 5.3 and 5.4, respectively.

Semigroups with zero have a specific structure and in studying of such semigroups it is often convenient to represent a semigroup $S = S^0$ in the form:

$$S = \bigcup_{\alpha \in Y} S_\alpha, \qquad \text{where } \ S_\alpha \cap S_\beta = 0, \ \text{for } \ \alpha, \beta \in Y,\, \alpha \neq \beta.$$

In this case, the partition $\mathcal{D}$ of S, whose components are 0 and $S_\alpha^\bullet$, $\alpha \in Y$, is called a *0-decomposition* of S. If, moreover, any S_α is a subsemigroup of S, we say that $\mathcal{D}$ is a *0-decomposition* of S *onto subsemigroups* and that S is a *0-sum* of semigroups S_α, $\alpha \in Y$, and the semigroups S_α will be called the *summands* of this decomposition. Equivalence relations corresponding to such decompositions are exactly the ones which satisfy the 0-*cm*-property, so the following theorem follows:

Theorem 1.2. *The poset of 0-decompositions of a semigroup $S = S^0$ onto subsemigroups is a complete lattice which is dually isomorphic to the lattice of equivalence relations on S satisfying the 0-cm-property.*

Special decompositions of this type may be determined by some properties of the index set Y. Namely, it is often convenient to assume that Y is a partial groupoid whose all elements are idempotents, and to require that the multiplication on S is carried by Y, by the following condition:

$$\begin{cases} S_\alpha S_\beta \subseteq S_{\alpha\beta} & \text{if } \alpha\beta \text{ is defined in } Y \\ S_\alpha S_\beta = 0 & \text{otherwise} \end{cases}, \qquad \text{for all } \alpha, \beta \in Y.$$

For example, if Y is a semigroup, i.e. a band, we obtain so called *0-band decompositions*. If the product $\alpha\beta$ is undefined, whenever $\alpha \neq \beta$, then $S_\alpha S_\beta = 0$, whenever $\alpha \neq \beta$, and such decompositions are called *orthogonal decompositions*. If Y is a left (right) zero band, then the corresponding decomposition is called a *decomposition into a left* (*right*) *sum* of semigroups. If Y is a nonempty subset of $I \times \Lambda$, where I and Λ are nonempty sets, and the partial multiplication on Y is defined by: for $(i,\lambda),(j,\mu) \in Y$, the product $(i,\lambda)(j,\mu)$ equals (i,μ), if $(i,\mu) \in Y$, and it is undefined, otherwise, then the decomposition $\mathcal{D}$ carried by Y is called a *decomposition into a matrix sum* of semigroups S_α, $\alpha \in Y$. Finally, if Y is an arbitrary poset and for $\alpha,\beta \in Y$, the product $\alpha\beta$ is defined as the meet of α and β, if it exists, then the related decomposition is called a *quasi-semilattice decomposition* of S.

1.3. Decompositions by congruences

Given a nonempty class $\mathfrak{C}$ of semigroups. Let $\mathrm{Con}_{\mathfrak{C}}(S)$ denotes the set of all $\mathfrak{C}$-congruences on S. Of course, $\mathrm{Con}_{\mathfrak{C}}(S)$ is a subset of $\mathrm{Con}(S)$ and it can be treated as a poset with respect to the usual ordering of congruences. Properties of posets of $\mathfrak{C}$-congruences inside the lattice $\mathrm{Con}(S)$ have been first investigated by T. Tamura and N. Kimura in [123], 1955, where they proved the following theorem:

Theorem 1.3. (T. Tamura and N. Kimura [123]) *If* $\mathfrak{C}$ *is a variety of semigroups, then* $\mathrm{Con}_{\mathfrak{C}}(S)$ *is a complete lattice, for any semigroup* S.

For the variety of semilattices, the previous theorem has been proved also by T. Tamura and N. Kimura [122], 1954 (see Theorem 2.1).

The problem of existence of the greatest decomposition of a given type has been solved in a special case, for so-called μ-decompositions, by T. Tamura [110], 1956. The solution of this problem in the general case has been given by N. Kimura [54], 1958, by the next theorem. Note that by an *algebraic class of* of semigroups we mean a class of semigroups closed under isomorphisms.

Theorem 1.4. (N. Kimura [54]) *Let* $\mathfrak{C}$ *be a nonempty algebraic class of semigroups. Then* $\mathfrak{C}$ *is closed under subdirect products if and only if* $\mathrm{Con}_{\mathfrak{C}}(S)$ *has the smallest element, for any semigroup* S *for which* $\mathrm{Con}_{\mathfrak{C}}(S) \neq \varnothing$.

As N. Kimura [54] noted, this theorem has been also found by E. J. Tully. Note that if $\mathrm{Con}_{\mathfrak{C}}(S)$ has the smallest elements, then it is a complete meet-subsemilattice of $\mathrm{Con}(S)$.

The converse of Theorem 1.3 has been proved in a recent paper of M. Ćirić and S. Bogdanović [24]. Namely, they proved the following theorem:

Theorem 1.5. (M. Ćirić and S. Bogdanović [24]) *Let $\mathfrak{C}$ be a nonempty algebraic class of semigroups. Then $\mathfrak{C}$ is a variety if and only if* $\mathrm{Con}_{\mathfrak{C}}(S)$ *is a complete sublattice of* $\mathrm{Con}(S)$*, for any semigroup S.*

By the proof of the previous theorem, given in [24], the next theorem also follows:

Theorem 1.6. (T. Tamura and N. Kimura [123]) *If $\mathfrak{C}$ is a variety of semigroups, then* $\mathrm{Con}_{\mathfrak{C}}(S)$ *is a principal dual ideal of* $\mathrm{Con}(S)$*, for any semigroup S.*

Note that Theorems 1.4, 1.5 and 1.6 holds also for any algebra.

The following theorem, proved by M. Petrich in [72], 1973, has been very useful in his investigations of some greatest decompositions of semigroups.

Theorem 1.7. (M. Petrich [72]) *Let $\mathfrak{C}$ be a variety of semigroups, $\mathfrak{D}$ the class of subdirectly irreducible semigroups from $\mathfrak{C}$ and S any semigroup. Then a congruence θ on a semigroup S, different from the universal congruence, is a $\mathfrak{C}$-congruence if and only if it is the intersection of some family of $\mathfrak{D}$-congruences.*

If we define the trivial semigroup to be subdirectly irreducible, then Theorem 1.7 says that $\mathrm{Con}_{\mathfrak{D}}(S)$ is meet-dense in $\mathrm{Con}_{\mathfrak{C}}(S)$.

2. Semilattice decompositions

Semilattice decompositions of semigroups have been first defined and studied by A. H. Clifford [29], 1941. After that they have been investigated by many authors and they have been systematically studied in several monographs: by E. S. Lyapin [62], 1960, A. H. Clifford and G. B. Preston [35], 1961, M. Petrich [72], 1973, and [73], 1977, S. Bogdanović [4], 1985, S. Bogdanović and M. Ćirić [7], 1993, and other.

First general results concerning semilattice decompositions of semigroups have been the results of T. Tamura and N. Kimura from [122], 1954. There they proved a theorem, given below as Theorem 2.1, by which it follows the existence of the greatest semilattice decomposition on any semigroup. This theorem initiated intensive studying of the greatest semilattice decompositions of semigroups and Section 2.1 is devoted to the results from this area. We present various characterizations of the greatest semilattice decomposition of a semigroup, the smallest semilattice congruence on a semigroup and the greatest semilattice homomorphic image of a semigroup, given by M. Yamada [132], 1955, T. Tamura [110], 1956, [112], 1964, and [117], 1972, M.

Petrich [69], 1964, and [72], 1973, M. S. Putcha [79], 1973, and [80], 1975, and M. Ćirić and S. Bogdanović [21]. We also quote the famous theorem of T. Tamura [110], 1956, on atomicity of semilattice decompositions, which is probably the most important result of the theory of semilattice decompositions of semigroups, and we give several characterizations of semilattice indecomposable semigroups given by M. Petrich [69], 1964, and [72], 1973, and T. Tamura [117], 1972. For the related results concerning decompositions of groupoids we refer to G. Thierrin [127], 1956.

Section 2.2 is devoted to lattices of semilattice decompositions of a semigroup, i.e. to lattices of semilattice congruence on a semigroup. We present characterizations of these lattices of T. Tamura [120], 1975, M. Ćirić and S. Bogdanović [23], and S. Bogdanović and M. Ćirić [12].

2.1. The greatest semilattice decomposition

As we noted above, the first general result concerning semilattice decompositions of semigroups is the one of T. Tamura and N. Kimura [122], 1954, which is given by the following theorem:

Theorem 2.1. (T. Tamura and N. Kimura [122]) *The poset of semilattice decompositions of any semigroup is a complete lattice.*

By the previous theorem it follows that any semigroup has a greatest semilattice decomposition. The first characterization of the greatest semilattice decomposition has been given by M. Yamada [132], 1955, in terms of P-subsemigroups. A subsemigroup T of a semigroup S is called a *P-semigroup* of S if for all $a_1, \dots, a_n \in S$,

$$a_1 \cdots a_n \in T \quad \Rightarrow \quad C(a_1, \dots, a_n) \subseteq T,$$

where $C(a_1, \dots, a_n)$ denotes the subsemigroup of S consisting of all products of elements $a_1, \dots, a_n \in S$ with each a_i appearing at least once [132]. Recall that $P(\mathcal{A})$ denotes the intersection of all principal congruences defined by elements of a nonempty family $\mathcal{A}$ of subsets of a semigroup.

Theorem 2.2. (M. Yamada [132]) *A relation θ on a semigroup S is a semilattice congruence if and only if $\theta = P(\mathcal{A})$, for some nonempty family $\mathcal{A}$ of P-subsemigroups of S.*

As a consequence of the previous theorem it can be deduced the following theorem:

Theorem 2.3. (M. Yamada [132]) *The smallest semilattice congruence on a semigroup S equals the congruence $P(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all P-subsemigroups of S.*

Another approach to the greatest decompositions of semigroups, through completely prime ideals and filters, has been developed by M. Petrich [69], 1964. He proved the following four theorems:

Theorem 2.4. (M. Petrich [69]) *A relation θ on a semigroup S is a semilattice congruence if and only if $\theta = \Theta(\mathcal{A})$, for some nonempty family $\mathcal{A}$ of completely prime ideals of S.*

Theorem 2.5. (M. Petrich [69]) *The smallest semilattice congruence on a semigroup S equals the congruence $\Theta(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all completely prime ideals of S.*

Theorem 2.6. (M. Petrich [69]) *A relation θ on a semigroup S is a semilattice congruence if and only if $\theta = \Theta(\mathcal{A})$, for some nonempty family $\mathcal{A}$ of filters of S.*

Theorem 2.7. (M. Petrich [69]) *The smallest semilattice congruence on a semigroup S equals the congruence $\Theta(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all filters of S.*

Another proofs of the previous two theorems have been given by the authors in [21].

The role of completely prime ideals and filters in semilattice decompositions of semigroups can be explained by Theorem 1.7. Namely, the two-element chain Y_2 is, up to an isomorphism, the unique subdirectly irreducible semilattice, and any homomorphism of a semigroup S onto Y_2 determines a partition of S whose one component is a completely prime ideal and other is a filter of S. This approach has been used by M. Petrich in [72], 1973.

M. Petrich [69], 1964, also gave a method to construct the principal filters of a semigroup:

Theorem 2.8. (M. Petrich [69]) *The principal filter of a semigroup S generated by an element $a \in S$ can be computed using the following formulas:*

$$N_1(a) = \langle a \rangle\,, \quad N_{n+1}(a) = \langle \{x \in S \mid N_n(a) \cap J(x) \neq \varnothing\} \rangle\,,\ n \in \mathbf{Z}^+$$
$$N(a) = \bigcup_{n \in \mathbf{Z}^+} N_n(a).$$

The third approach to the greatest decompositions of semigroups is the one of T. Tamura from [117], 1972. Using the *division relation* $\mid$ on a semigroup S defined by:

$$a \mid b \;\;\Leftrightarrow\;\; b \in S^1 a S^1,$$

T. Tamura defined the relation $\longrightarrow$ on S by:

$$a \longrightarrow b \;\;\Leftrightarrow\;\; (\exists n \in \mathbf{Z}^+)\; a \mid b^n,$$

and he gave an efficient characterization of the smallest semilattice congruence on a semigroup:

Theorem 2.9. (T. Tamura [117]) *The smallest semilattice congruence on a semigroup S equals the natural equivalence of the relation $\longrightarrow^\infty$.*

Another proof of this theorem has been given by T. Tamura [118], 1973.

Three different characterizations of the smallest semilattice congruence on a semigroup have been also obtained by M. S. Putcha in [79], 1973, and [80], 1975.

Theorem 2.10. (M. S. Putcha [80]) *The smallest semilattice congruence on a semigroup S equals the equivalence on S generated by the relation $xy \equiv xyx \equiv yx$, for all $x, y \in S^1$.*

Another proof of this theorem has been given by T. Tamura [119], 1973.

Theorem 2.11. (M. S. Putcha [80]) *The smallest semilattice congruence on a semigroup S equals the relation $-\!\!-^\infty$, where $-\!\!- = \longrightarrow \cap \longrightarrow^{-1}$.*

Theorem 2.12. (M. S. Putcha [79]) *The smallest semilattice congruence on a semigroup S equals the relation θ on S defined by: $a\,\theta\, b$ if and only if for all $x, y \in S^1$ there exists a semilattice indecomposable subsemigroup T of S such that $xay, xby \in T$.*

An approach to semilattice decompositions of semigroups, different to the one of M. Petrich and T. Tamura, has been developed by M. Ćirić and S. Bogdanović in [21]. As we will see later, the results obtained there explained the connections between the above presented results of M. Petrich and T. Tamura. M. Ćirić and S. Bogdanović [21] started from the completely semiprime ideals and they first gave the following representations of the principal radicals of a semigroup:

Theorem 2.13. (M. Ćirić and S. Bogdanović [21]) *The principal radical of a semigroup S generated by an element $a \in S$ has the following representation:*

$$\Sigma(a) = \{x \in S \mid a \longrightarrow^{\infty} x\}.$$

Theorem 2.14. (M. Ćirić and S. Bogdanović [21]) *The principal radical of a semigroup S generated by an element $a \in S$ can be computed using the following formulas:*

$$\Sigma_1(a) = \sqrt{SaS},\ \Sigma_n(a) = \sqrt{S\Sigma_n(a)S},\ n \in \mathbf{Z}^+,\ \Sigma(a) = \bigcup_{n\in\mathbf{Z}^+} \Sigma_n(a).$$

Recall that $\mathcal{I}d^{\mathbf{cs}}(S)$ denotes the lattice of all completely semiprime ideals of a semigroup S. By means of Theorems 2.13 and 2.9, the authors in [21] obtained the following characterization of the smallest semilattice congruence on a semigroup:

Theorem 2.15. (M. Ćirić and S. Bogdanović [21]) *The smallest semilattice congruence on a semigroup S equals the equivalence $\Theta(\mathcal{I}d^{\mathbf{cs}}(S))$.*

A characterization of the greatest semilattice homomorphic image of a semigroup has been given by M. Ćirić and S. Bogdanović [21], through principal radicals of a semigroup:

Theorem 2.16. (M. Ćirić and S. Bogdanović [21]) *If a, b is any pair of elements of a semigroup S, then*

$$\Sigma(a) \cap \Sigma(b) = \Sigma(ab),$$

i.e. the set Σ_S of all principal radicals of S, partially ordered by inclusion, is a semilattice and it is the greatest semilattice homomorphic image of S.

As a consequence of the previous theorem, the authors in [21] proved the next theorem without use of the Zorn's lemma arguments:

The next theorem, which gives a connection between Theorems 2.15 and 2.5, has been proved by M. Petrich [72], 1973. Another proof of this theorem, without use of the Zorn's lemma arguments, has been given by the authors in [21], as a consequence of Theorem 2.16.

Theorem 2.17. (M. Petrich [72]) *Any completely semiprime ideal of a semigroup S is the intersection of some family of completely prime ideals of S.*

In other words, Theorem 2.17 says that the set of completely prime ideals of a semigroup S is meet-dense in $\mathcal{I}d^{\mathbf{cs}}(S)$.

Another consequence of Theorem 2.16 is the next theorem which gives a representation of the principal filters better than the one from Theorem 2.8.

Theorem 2.18. (M. Ćirić and S. Bogdanović [21]) *The principal filter of a semigroup S generated by an element a has the following representation:*

$$N(a) = \{x \in S \mid x \longrightarrow^{\infty} a\}.$$

The components of the greatest semilattice decomposition of a semigroup are characterized by the next theorem, which is clearly a consequence of Theorems 2.13, 2.18 and 2.9.

Theorem 2.19. (M. Petrich [72]) *The component of the greatest semilattice decomposition of a semigroup S containing an element a of S is precisely the subsemigroup $\Sigma(a) \cap N(a)$.*

The most significant theorem of the theory of semilattice decompositions of semigroup is probably the theorem of T. Tamura [110], 1956, on atomicity of semilattice decompositions of semigroups, given here as Theorem 2.20. Note that another proofs of this theorem have been given by T. Tamura in [112], 1964, by means of the concept of "contents", in [117], 1972, using the relation $\longrightarrow^{\infty}$, in [118], 1973, and [120], 1975, by M. Petrich [69], 1964, by means of completely prime ideals, and by M. S. Putcha [79], 1973, using the relation defined in Theorem 2.12 and the subsemigroups of the form $C(a_1, \dots, a_n)$.

Theorem 2.20. (T. Tamura [110]) *Any component of the greatest semilattice decomposition of a semigroup is a semilattice indecomposable semigroup.*

Semilattice indecomposable semigroups have been described by T. Tamura [117] and M. Petrich [69], [72], by the following

Theorem 2.21. *The following conditions on a semigroup S are equivalent:*

(i) *S is semilattice indecomposable;*
(ii) $(\forall a, b \in S)\ a \longrightarrow^{\infty} b$*;*
(iii) *S has no proper completely semiprime ideals;*
(iv) *S has no proper completely prime ideals.*

The equivalence of conditions (i) and (ii) has been established by T. Tamura [117], 1972, (i) $\Leftrightarrow$ (iii) has been proved by M. Petrich [69], 1964, and (i) $\Leftrightarrow$ (iv) by M. Petrich [72], 1973.

Note that in the class of semilattice indecomposable semigroup the mostly investigated were Archimedean semigroups, defined by: $a \longrightarrow b$, for all elements a and b. Semilattices of such semigroups have been studied by many authors. The most important results from this area have been obtained by M. S. Putcha [79], 1973, T. Tamura [116], 1972, M. Ćirić and S. Bogdanović

[19], 1993, and [21], S. Bogdanović and M. Ćirić [6], 1992, and [14], and L. N. Shevrin [99] and [100], 1994. For more informations about semilattices of Archimedean semigroups the reader is also referred to the survey paper of S. Bogdanović and M. Ćirić [8], 1993, or their book [7], 1993.

2.2. The lattice of semilattice decompositions

T. Tamura [120] got an idea of studying semilattice decompositions of a semigroup through quasi-orders on this semigroup having some suitable properties. We say that a quasi-order ξ on a semigroup S is *positive* if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$. These quasi-orders have been introduced by B. M. Schein [88], 1965, and they were since studied from different points of view by T. Tamura, M. S. Putcha, S. Bogdanović, M. Ćirić and other. By a *half-congruence* T. Tamura in [120], 1975, called a compatible quasi-order on a semigroup, and by a *lower-potent* quasi-order he called a quasi-order ξ on a semigroup satisfying the condition: $a^2\,\xi\,a$, for all elements a. Using these notions, T. Tamura proved the following theorem:

Theorem 2.22. (T. Tamura [120]) *The lattice of semilattice congruences on a semigroup S is isomorphic to the lattice of positive lower-potent half-congruences on S.*

As the authors noted in [23], the notion "lower-potent half-congruence" in Theorem 2.22 can be replaced by "quasi-order satisfying the *cm*-property". Recall from Section 1.1 that a relation ξ on a semigroup S satisfies the *common multiple property*, briefly the *cm-property*, if for all $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$. Using this notion, introduced by T. Tamura in [116], 1972, Theorem 2.22 can be written as follows:

Theorem 2.23. *The lattice of semilattice congruences on a semigroup S is isomorphic to the lattice of positive quasi-orders on S satisfying the cm-property.*

Using the Tamura's approach, the authors in [23] connected semilattice decompositions of a semigroup with some sublattices of the lattice $\mathcal{I}d^{\mathbf{cs}}(S)$ of completely simple ideals of a semigroup. Recall from Section 1.1 that a subset K of a lattice L is *meet-dense* in L if any element of L can be written as the meet of some family of elements of K. We will say that a sublattice L of $\mathcal{I}d^{\mathbf{cs}}(S)$ satisfies the *completely prime ideal property*, shortly the *cpi-property*, if the set of completely prime ideals from L is meet-dense in L, i.e. if any element of L can be written as the intersection of some family of completely prime ideals from L. As we seen before, this property was proved for $\mathcal{I}d^{\mathbf{cs}}(S)$ by Theorem 2.17. M. Ćirić and S. Bogdanović [23] showed that the *cpi*-property plays a crucial role in semilattice decompositions of semigroups:

Theorem 2.24. (M. Ćirić and S. Bogdanović [23]) *The lattice of semilattice decompositions of a semigroup S is isomorphic to the lattice of complete 1-sublattices of $\mathcal{I}d^{\mathbf{cs}}(S)$ satisfying the cpi-property.*

Another connection of semilattice decompositions of a semigroup, with some sublattices of the lattice of subsets of a semigroup, has been established by S. Bogdanović and M. Ćirić in [12]. There they proved the following theorem:

Theorem 2.25. (S. Bogdanović and M. Ćirić [12]) *The lattice of semilattice decompositions of a semigroup S is isomorphic to the lattice of complete 1-sublattices of $\mathcal{P}(S)$ whose principal elements are filters of S.*

For more informations about the role of quasi-orders in semilattice decompositions of semigroups we refer to another survey paper of S. Bogdanović and M. Ćirić [16].

3. Band decompositions

Although the existence of the greatest band decomposition has been established by T. Tamura and N. Kimura in [123], 1955, by the theorem which is given here as Theorem 1.3, there are no sufficiently efficient characterizations of the greatest band decomposition of a semigroup in the general case. But, there are very nice descriptions of greatest decompositions for some special types of band decompositions, as semilattice decompositions, treated in the previous chapter, matrix decompositions, where left zero band and right zero band decompositions are included, and normal band decompositions, where left normal band and right normal band decompositions are included. This chapter is devoted to the results concerning the greatest matrix decomposition of a semigroup, which will be presented in Section 3.1, and to the results concerning the greatest normal band decomposition of a semigroup, which will be presented in Section 3.2.

Matrix decompositions, as well as left zero band and right zero band decompositions, have appeared first in studying of completely simple semigroups. Namely, by the famous Rees-Sushkevich theorem on matrix representations of completely simple semigroups, any completely simple semigroup can be decomposed into a matrix of groups, and also into a left zero band of right groups and into a right zero band of left groups. First general results concerning these decompositions have been obtained by P. Dubreil

[41], 1951, who constructed the smallest left zero band congruence on a semigroup, and by G. Thierrin [128], 1956, who characterized the components of the greatest left zero band decomposition of a semigroup. The general theory of matrix decompositions of semigroups has been founded by M. Petrich in [70], 1996. These results will be a topic of Section 3.1.

Normal bands have been introduced by M. Yamada and N. Kimura [133], 1958, whereas left normal bands have been first defined and studied by V. V. Vagner [129], 1962, and B. M. Schein [86], 1963, and [87], 1965. The general results concerning left normal band, right normal band and normal band decompositions of a semigroup, presented in Section 3.2, have been obtained by M. Petrich in [71], 1966.

For additional informations about matrix and normal band decompositions the reader is referred to the book of M. Petrich [73], 1977.

3.1. Matrix decompositions

As we noted before, the first general result concerning left zero band decompositions of a semigroup is the one of P. Dubreil [41], 1951. Define the relations $\overset{l}{\approx}$ and $\overset{r}{\approx}$ on a semigroup S by:

$$a \overset{l}{\approx} b \;\Leftrightarrow\; L(a)\cap L(b)\neq\varnothing, \qquad a \overset{r}{\approx} b \;\Leftrightarrow\; R(a)\cap R(b)\neq\varnothing, \;\; (a,b\in S).$$

The relation $\overset{r}{\approx}$ has been introduced in above mentioned paper of P. Dubreil, where he proved the following theorem:

Theorem 3.1. (P. Dubreil [41]) *The smallest left zero band congruence on a semigroup S equals the relation $\overset{r}{\approx}{}^{\infty}$.*

The components of the greatest left zero band decomposition of a semigroup have been first described by G. Thierrin [128], 1956, by the following theorem:

Theorem 3.2. (G. Thierrin [128]) *The components of the greatest left zero band decomposition of a semigroup S are the minimal left consistent right ideals.*

Other characterizations of the greatest left zero band decomposition of a semigroup have been obtained by M. Petrich in [70], 1966. In this paper he proved the following two theorems:

Theorem 3.3. (M. Petrich [70]) *A relation θ on a semigroup S is a left zero band congruence on S if and only if $\theta=\Theta(\mathcal{A})$, for some nonempty family $\mathcal{A}$ of left consistent right ideals of S.*

Theorem 3.4. *The smallest left zero band congruence on a semigroup S equals the relation $\Theta(\mathcal{RI}d^{\mathbf{lc}}(S))$.*

The key theorem in theory of matrix decompositions of semigroups is the next theorem, proved by M. Petrich in [70], 1966, which gives a connection between left zero band, right zero band and matrix congruences on a semigroup:

Theorem 3.5. (M. Petrich [70]) *The intersection of a left zero band congruence and a right zero band congruence on a semigroup S is a matrix congruence on S.*

Conversely, any matrix congruence on S can be written uniquely as the intersection of a left zero band congruence and a right zero band congruence on S.

Combining Theorems 3.1 and 3.5, the following characterization of the smallest matrix congruence on a semigroup follows:

Theorem 3.6. (M. Petrich [70]) *The smallest matrix congruence on a semigroup S equals the relation $\overset{l}{\approx}{}^{\infty} \cap \overset{r}{\approx}{}^{\infty}$.*

Combining Theorem 3.3 and its dual, M. Petrich [70] obtained the following two theorems:

Theorem 3.7. (M. Petrich [70]) *A relation θ on a semigroup S is a matrix congruence on S if and only if $\theta = \Theta(\mathcal{A})$, for some nonempty subset $\mathcal{A}$ of $\mathcal{X}$, where $\mathcal{X} = \mathcal{LI}d^{\mathbf{rc}}(S) \cup \mathcal{RI}d^{\mathbf{lc}}(S)$.*

Theorem 3.8. (M. Petrich [70]) *The smallest matrix congruence on a semigroup S equals the relation $\Theta(\mathcal{X})$, where $\mathcal{X} = \mathcal{LI}d^{\mathbf{rc}}(S) \cup \mathcal{RI}d^{\mathbf{lc}}(S)$.*

M. Petrich in [70] also gave an alternative approach to matrix decompositions of semigroups, through so-called quasi-consistent subsemigroups. Namely, by a *quasi-consistent* subset of a semigroup S he defined a completely semiprime subset A of S satisfying the condition: for all $x, y, z \in S$, $xyz \in A$ if and only if $xy \in A$. Quasi-consistent subsemigroups of a semigroup M. Petrich connected with left consistent right ideals and right consistent left ideals by the following theorem:

Theorem 3.9. (M. Petrich [70]) *The intersection of a left consistent right ideal and a right consistent left ideal of a semigroup S is a quasi-consistent subsemigroup of S.*

Conversely, any quasi-consistent subsemigroup of S can be written uniquely as the intersection of a left consistent right ideal and a right consistent left ideal.

Using the previous theorem, matrix congruences on a semigroup can be characterized through quasi-consistent subsemigroups of a semigroup as follows:

Theorem 3.10. (M. Petrich [70]) *A relation θ on a semigroup S is a matrix congruence on S if and only if $\theta = \Theta(\mathcal{A})$, for some nonempty family $\mathcal{A}$ of the set of quasi-consistent subsemigroups of S.*

Theorem 3.11. (M. Petrich [70]) *The smallest matrix congruence on a semigroup S equals the relation $\Theta(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all quasi-consistent subsemigroups of S.*

Using Theorem 3.5 and the fact that the join of any left zero band congruence and any right zero band congruence on a semigroup equals the universal congruence on this semigroup, the lattice of matrix congruences on a semigroup can be characterized in the following way:

Theorem 3.12. *The lattice of matrix congruences on a semigroup S is isomorphic to the direct product of the lattice of left zero band congruences and the lattice of right zero band congruences on S.*

A characterization of the lattice of right zero band decompositions of a semigroup can be obtained through left consistent right ideals of a semigroup, modifying the results of S. Bogdanović and M. Ćirić [13] to semigroups without zero. For related results concerning semigroups with zero we refer to Section 4.2.

Until the end of this section we will consider only semigroups without zero, because the definition of the lattice $\mathcal{RI}d(S)$ is different for semigroups with and without zero, and the set of right consistent left ideals of a semigroup with zero is one-element.

Theorem 3.13. *The poset $\mathcal{RI}d^{\mathbf{lc}}(S)$ of left consistent right ideals of a semigroup $S \neq S^0$ without zero is a complete atomic Boolean algebra and $\mathcal{RI}d^{\mathbf{lc}}(S) = \mathfrak{B}(\mathcal{RI}d(S))$.*

Theorem 3.14. *The lattice of left zero band decompositions of a semigroup $S \neq S^0$ is isomorphic to the lattice of complete Boolean subalgebras of $\mathcal{RI}d^{\mathbf{lc}}(S)$.*

The role of left zero band decompositions of a semigroup in direct decompositions of the lattice of right ideals of this semigroup is demonstrated by the following two theorems:

Theorem 3.15. *The lattice $\mathcal{RI}d(S)$ of right ideals of a semigroup $S \neq S^0$ is a direct product of lattices L_α, $\alpha \in Y$, if and only if S is a left zero band of semigroups S_α, $\alpha \in Y$, and $L_\alpha \cong \mathcal{RI}d(S_\alpha)$, for any $\alpha \in Y$.*

Theorem 3.16. *If S_α, $\alpha \in Y$, are components of the greatest left zero band decomposition of a semigroup $S \neq S^0$, then the lattice $\mathcal{RI}d(S)$ can be decomposed into a direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$, which are directly indecomposable.*

3.2. Normal band decompositions

In the introduction of Chapter 3 we said that the general theory of normal band decompositions of semigroups, including here left normal band and right normal band decompositions, has been founded by M. Petrich in [71], 1966. The methods used in this paper has been obtained by combination of the methods which M. Petrich used in [69], in studying of semilattice decompositions, and the ones used in [70], in studying of matrix decompositions.

M. Petrich in [71] defined a *left* (*right*) *normal complex* of a semigroup S as a nonempty subset A of S which is a left (right) consistent right (left) ideal of the smallest filter $N(A)$ of S containing A, and he defined a *normal complex* of S as a subset A of S which is a quasi-consistent subsemigroup of $N(A)$. He also introduced the following relations on a semigroup S: for a nonempty subset A of S, Φ_A is the equivalence relation on S whose classes are nonempty sets among the sets A, $N(A) - A$ and $S - N(A)$, and for a nonempty family $\mathcal{A}$ of subsets of S, $\Phi(\mathcal{A})$ is the equivalence relation on S defined by:

$$\Phi(\mathcal{A}) = \bigcap_{A \in \mathcal{A}} \Phi_A.$$

Theorem 3.17. (M. Petrich [71]) *A relation θ on a semigroup S is a left normal band congruence on S if and only if $\theta = \Phi(\mathcal{A})$, for some nonempty family $\mathcal{A}$ of left normal complexes of S.*

Theorem 3.18. (M. Petrich [71]) *The smallest left normal band congruence on a semigroup S equals the relation $\theta = \Phi(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all left normal complexes of S.*

In order to study normal band congruences on a semigroup through left normal band congruences and right normal band congruences, M. Petrich proved the following theorem, similar to Theorem 3.5 concerning matrix congruences:

Theorem 3.19. (M. Petrich [71]) *The intersection of a left normal band congruence and a right normal band congruence on a semigroup S is a normal band congruence on S.*

Conversely, any normal band congruence on S can be written as the intersection of the smallest left normal band congruence and the smallest right normal band congruence on S containing it.

Using this theorem, M. Petrich [71] characterized normal band congruences and the smallest normal band congruence on a semigroup by the following two theorems:

Theorem 3.20. (M. Petrich [71]) *A relation θ on a semigroup S is a normal band congruence on S if and only if $\theta = \Phi(\mathcal{A})$, for some nonempty subset $\mathcal{A}$ of $\mathcal{X}$, where $\mathcal{X}$ denotes the set of all left normal complexes and all right normal complexes of S.*

Theorem 3.21. (M. Petrich [71]) *The smallest normal band congruence on a semigroup S equals the relation $\theta = \Phi(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all left normal complexes and all right normal complexes of S.*

In Theorem 3.20, $\mathcal{X}$ cannot be replaced by the set of all normal complexes, but this can be done in Theorem 3.21:

Theorem 3.22. (M. Petrich [71]) *The smallest normal band congruence on a semigroup S equals the relation $\theta = \Phi(\mathcal{X})$, where $\mathcal{X}$ denotes the set of all normal complexes of S.*

4. Decompositions of semigroups with zero

The first known type of decompositions of semigroups with zero have been orthogonal decompositions, called also 0-direct unions, which have been first defined and studied by E. S. Lyapin in [60] and [61], 1950, and Š. Schwarz [90], 1951. After that, they have been studied by many authors, mainly as orthogonal sums of completely 0-simple semigroups. General study of orthogonal decompositions of semigroups with zero has done by S. Bogdanović and M. Ćirić in [10], 1995, and [13]. The results obtained there will be a topic of Section 4.1. Among these results we emphasize Theorem 4.8 on atomicity of orthogonal decompositions.

Decompositions of a semigroup with zero into a left, right and matrix sum of semigroups have been first defined and studied by S. Bogdanović and M. Ćirić in [13]. The results concerning these decompositions obtained in this paper will be presented in Section 4.2. We also give Theorem 4.21 which establish connections between the decompositions into a left, right and matrix sum, and orthogonal decompositions inside the lattice of 0-decompositions of a semigroup with zero. Note also that some decompositions of semigroups with zero, similar to decomposition into a matrix sum, have been considered by O. Steinfeld in [105].

Quasi-semilattice decompositions of a semigroup with zero, which are carried by partially ordered sets, appeared recently in the paper of M. Ćirić and S. Bogdanović [26]. These decompositions will be considered in Section 4.3.

Note finally that decompositions into a left, right and matrix sum of semigroups, and quasi-semilattice decompositions of semigroups with zero are generalizations (or analogues) of left zero band, right zero band, matrix and semilattice decompositions, respectively, as showed by Theorems 4.22 and 4.28. Orthogonal sums have no such analogue.

4.1. Orthogonal decompositions

In studying of orthogonal decompositions of semigroups with zero, S. Bogdanović and M. Ćirić [10], 1995, has started from the notion of 0-consistent ideal. They defined a *0-consistent* ideal of a semigroup $S = S^0$ as an ideal A having the property that $A^\bullet$ is a consistent subset of S. They denoted by $\mathcal{I}d^{\mathbf{0c}}(S)$ the set of all 0-consistent ideals of a semigroup $S = S^0$ and they proved the following theorem:

Theorem 4.1. (S. Bogdanović and M. Ćirić [10]) *The poset $\mathcal{I}d^{\mathbf{0c}}(S)$ of all 0-consistent ideals of a semigroup $S = S^0$ is a complete atomic Boolean algebra and $\mathcal{I}d^{\mathbf{0c}}(S) = \mathfrak{B}(\mathcal{I}d(S))$.*

Furthermore, any complete atomic Boolean algebra is isomorphic to the Boolean algebra of 0-consistent ideals of some semigroup with zero.

Using this theorem, S. Bogdanović and M. Ćirić [10] obtained the following theorem concerning orthogonal decompositions:

Theorem 4.2. (S. Bogdanović and M. Ćirić [10]) *Any semigroup $S = S^0$ has a greatest orthogonal decomposition and its summands are all the atoms in $\mathcal{I}d^{\mathbf{0c}}(S)$.*

Another approach to orthogonal decompositions, through certain equivalence relations, has done by S. Bogdanović and M. Ćirić in [13]. A 0-restricted and 0-consistent equivalence relation on a semigroup $S = S^0$ will be called an *orthogonal equivalence*. This name will be justified by the role of these equivalences in orthogonal decompositions, which will be demonstrated in Theorem 4.4. Namely, the authors proved in [13] the following two theorems:

Theorem 4.3. (S. Bogdanović and M. Ćirić [13]) *The poset of orthogonal equivalences on a semigroup $S = S^0$ is a complete sublattice of the lattice $\mathcal{E}(S)$.*

Theorem 4.4. (S. Bogdanović and M. Ćirić [13]) *The poset of orthogonal decompositions of a semigroup $S = S^0$ is a complete lattice and it is dually isomorphic to the lattice of orthogonal equivalences on S.*

Note that the sumands in an orthogonal decomposition of a semigroup $S = S^0$ are precisely the nonzero classes of the related orthogonal equivalence, with the zero adjoined, and vice versa.

By Theorems 4.3 and 4.4 we deduce the following:

Theorem 4.5. *The lattice of orthogonal decompositions of a semigroup* $S = S^0$ *is a complete sublattice of the partition lattice of* S.

The lattice of orthogonal decompositions has been also characterized by some Boolean subalgebras of $\mathcal{I}d^{\mathbf{0c}}(S)$ as follows:

Theorem 4.6. (S. Bogdanović and M. Ćirić [13]) *The lattice of orthogonal decompositions of a semigroup* $S = S^0$ *is isomorphic to the lattice of complete Boolean subalgebras of* $\mathcal{I}d^{\mathbf{0c}}(S)$.

Note that any complete Boolean subalgebra of $\mathcal{I}d^{\mathbf{0c}}(S)$ is atomic and its atoms are precisely the summands in the related orthogonal decomposition of S, and vice versa.

To describe the smallest orthogonal equivalence on a semigroup with zero, S. Bogdanović and M. Ćirić in [10] defined the relation $\sim$ on a semigroup $S = S^0$ by

$$a \sim b \quad \Leftrightarrow \quad J(a) \cap J(b) \neq 0, \quad \text{for } a, b \in S^\bullet, \qquad\qquad 0 \sim 0,$$

and they proved the following:

Theorem 4.7. (S. Bogdanović and M. Ćirić [10]) *The smallest orthogonal equivalence on a semigroup* $S = S^0$ *equals the relation* $\sim^\infty$.

Note also that the lattice of orthogonal equivalences on S is the principal dual ideal of the lattice $\mathcal{E}^\bullet(S)$ of 0-restricted equivalence relations on S, generated by $\sim^\infty$. Since $\mathcal{E}^\bullet(S)$ is the principal ideal of $\mathcal{E}(S)$, generated by the equivalence relation χ on S determined by the partition $\{S^\bullet, 0\}$, then the lattice of orthogonal equivalences on S is precisely the (closed) interval $[\sim^\infty, \chi]$ of $\mathcal{E}(S)$.

The main theorem of the theory of orthogonal decompositions of semigroups with zero is the theorem on atomicity of orthogonal decompositions, proved by S. Bogdanović and M. Ćirić in [10], 1995. This is the following theorem:

Theorem 4.8. (S. Bogdanović and M. Ćirić [10]) *The summands of the greatest orthogonal decomposition of a semigroup* $S = S^0$ *are orthogonally indecomposable semigroups.*

S. Bogdanović and M. Ćirić in [13] also observed that orthogonal decompositions of a semigroup $S = S^0$ are closely connected with direct decompositions of the lattice of ideals of S. This connection is demonstrated by the following three theorems:

Theorem 4.9. *The lattice $\mathcal{I}d(S)$ of ideals of a semigroup $S = S^0$ is a direct product of lattices L_α, $\alpha \in Y$, if and only if S is an orthogonal sum of semigroups S_α, $\alpha \in Y$, and $L_\alpha \cong \mathcal{I}d(S_\alpha)$, for any $\alpha \in Y$.*

Theorem 4.10. (S. Bogdanović and M. Ćirić [13]) *The lattice $\mathcal{I}d(S)$ of ideals of a semigroup $S = S^0$ is directly indecomposable if and only if S is orthogonally indecomposable.*

Theorem 4.11. (S. Bogdanović and M. Ćirić [13]) *If S_α, $\alpha \in Y$, are summands of the greatest orthogonal decomposition of a semigroup $S = S^0$, then the lattice $\mathcal{I}d(S)$ can be decomposed into a direct product of lattices $\mathcal{I}d(S_\alpha)$, $\alpha \in Y$, which are directly indecomposable.*

4.2. Decompositions into a left, right and matrix sum

In studying of decompositions of semigroups with zero into a left sum of semigroups, S. Bogdanović and M. Ćirić in [13] used the methods similar to the ones used in studying of orthogonal decompositions. At first, they considered equivalence relations on a semigroup with zero which we call here left sum equivalences. Namely, a 0-restricted, left 0-consistent equivalence relation on a semigroup $S = S^0$ will be called an *left sum equivalence*. *Right sum equivalences* on S are defined dually. These names will be explained by the role of these equivalences in decompositions of S into a left sum and a right sum of semigroups, respectively, as demonstrated in Theorem 4.13. But, first we give the following theorem:

Theorem 4.12. (S. Bogdanović and M. Ćirić [13]) *The poset of left sum equivalences on a semigroup $S = S^0$ is a complete sublattice of the lattice $\mathcal{E}(S)$.*

Theorem 4.13. (S. Bogdanović and M. Ćirić [13]) *The poset of decompositions of a semigroup $S = S^0$ into a left sum of semigroups is a complete lattice and it is dually isomorphic to the lattice of left sum equivalences on S.*

As in orthogonal decompositions, the sumands in a decomposition of a semigroup $S = S^0$ into a left sum are the nonzero classes of the related left sum equivalence, with the zero adjoined, and vice versa.

By Theorems 4.12 and 4.13 we obtain the following:

Theorem 4.14. *The lattice of decompositions of a semigroup $S = S^0$ into a left sum of semigroups is a complete sublattice of the partition lattice of S.*

To characterize the smallest left sum equivalence on a semigroup, the authors used the relation $\overset{r}{\sim}$ defined by G. Lallement and M. Petrich [59], 1966, on a semigroup $S = S^0$ by:

$$a \overset{r}{\sim} b \quad \Leftrightarrow \quad R(a) \cap R(b) \neq 0, \quad \text{for } a, b \in S^{\bullet}, \qquad\qquad 0 \overset{r}{\sim} 0.$$

The relation $\overset{\ell}{\sim}$ on S is defined dually. Using the above relation, S. Bogdanović and M. Ćirić [13] characterized the smallest left sum equivalence as follows:

Theorem 4.15. (S. Bogdanović and M. Ćirić [13]) *The smallest left sum equivalence on a semigroup $S = S^0$ equals the relation $\overset{r}{\sim}{}^{\infty}$.*

As in orthogonal equivalences, the set of left sum equivalences on a semigroup $S = S^0$ equals the interval $[\overset{r}{\sim}{}^{\infty}, \chi]$ of the lattice $\mathcal{E}(S)$.

Instead of 0-consistent ideals, used in studying of orthogonal decompositions, in studying of decompositions of a semigroup with zero into a left sum of semigroups, S. Bogdanović and M. Ćirić used in [13] the notion of the left 0-consistent right ideal. Namely, they defined a right ideal A of a semigroup $S = S^0$ to be *left 0-consistent* if $A^{\bullet}$ is a left consistent subset of S. The set of all left 0-consistent ideals of a semigroup they denoted by $\mathcal{RI}d^{\mathbf{l0c}}(S)$ and they proved the following two theorems:

Theorem 4.16. (S. Bogdanović and M. Ćirić [13]) *The poset $\mathcal{RI}d^{\mathbf{l0c}}(S)$ of all left 0-consistent right ideals of a semigroup $S = S^0$ is a complete atomic Boolean algebra and $\mathcal{RI}d^{\mathbf{l0c}}(S) = \mathfrak{B}(\mathcal{RI}d(S))$.*

Theorem 4.17. (S. Bogdanović and M. Ćirić [13]) *The lattice of decompositions of a semigroup $S = S^0$ into a left sum of semigroup is isomorphic to the lattice of complete Boolean subalgebras of $\mathcal{RI}d^{\mathbf{l0c}}(S)$.*

As in orthogonal decompositions, the summands in a decomposition of a semigroup $S = S^0$ into a left sum of semigroups are precisely the atoms in the related complete Boollean subalgebra of $\mathcal{RI}d^{\mathbf{l0c}}(S)$, which is atomic, and vice versa.

As S. Bogdanović and M. Ćirić in [13] observed, the previous two theorems can be applied to direct decompositions of the lattice of right ideals of a semigroup with zero:

Theorem 4.18. *The lattice $\mathcal{RI}d(S)$ of right ideals of a semigroup $S = S^0$ is a direct product of lattices L_α, $\alpha \in Y$, if and only if S is a left sum of semigroups S_α, $\alpha \in Y$, and $L_\alpha \cong \mathcal{RI}d(S_\alpha)$, for any $\alpha \in Y$.*

Theorem 4.19. (S. Bogdanović and M. Ćirić [13]) *If S_α, $\alpha \in Y$, are the summands of the greatest decomposition of a semigroup $S = S^0$ into a left sum of semigroups, then the lattice $\mathcal{LI}d(S)$ can be decomposed into a direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$, which are directly indecomposable.*

Note that the interval $[0, S_\alpha]$ in Theorem 4.19 cannot be replaced by the lattice $\mathcal{RI}d(S_\alpha)$, in contrast to Theorem 4.11.

In order to characterize decompositions of a semigroup with zero into a matrix sum of semigroups, S. Bogdanović and M. Ćirić consider in [13] equivalence relations that are the intersection of a left sum equivalence and a right sum equivalence, which will be called here *matrix sum equivalences*, and they proved the following theorems:

Theorem 4.20. (S. Bogdanović and M. Ćirić [13]) *The poset of matrix sum equivalences on a semigroup $S = S^0$ is a complete lattice.*

Theorem 4.21. (S. Bogdanović and M. Ćirić [13]) *The poset of decompositions of a semigroup $S = S^0$ into a matrix sum of semigroups is a complete lattice and it is dually isomorphic to the lattice of matrix sum equivalences on S.*

Note that the sumands in a decomposition of a semigroup $S = S^0$ into a matrix sum are exactly the nonzero classes of the related matrix sum equivalence, with the zero adjoined, and vice versa.

Note also that the previous two theorems give a connection between the decompositions into a left sum, decompositions into a right sum and decompositions into a matrix sum. The authors in [13] established a similar connection between the decompositions into a left sum, decompositions into a right sum and orthogonal decompositions. This connection is given by the following theorem:

Theorem 4.22. (S. Bogdanović and M. Ćirić [13]) *The join in $\mathcal{E}(S)$ of any left sum equivalence and any right sum equivalence on a semigroup $S = S^0$ is an orthogonal equivalence on S.*

Especially, the join of $\overset{r}{\sim}{}^{\infty}$ and $\overset{\ell}{\sim}{}^{\infty}$ equals $\sim^{\infty}$.

The above quoted results can be summarized by the following theorem:

Theorem 4.23. *In the partition lattice of a semigroup $S = S^0$, the meet of any decomposition of S into a left sum and any decomposition of S into a right sum of semigroups is an orthogonal decomposition, and its join is a decomposition into a matrix sum of semigroups.*

Especially, the meet of the greatest decomposition of S into a left sum and the greatest decomposition of S into a right sum of semigroups is the greatest *orthogonal decomposition of S, and its join is the greatest decomposition of S into a matrix sum of semigroups.*

Note finally that decompositions of a semigroup with zero into a left sum, right sum and matrix sum of semigroups can be treated as generalizations of left zero band, right zero band and matrix decompositions, respectively. This follows by the following theorem:

Theorem 4.24. *The lattice of left zero band* (*right zero band, matrix*) *decompositions of a semigroup S is isomorphic to the lattice of decompositions into a left* (*right, matrix*) *sum of semigroups of a semigroup T arising from S by adjoining the zero.*

4.3. Quasi-semilattice decompositions

Studying of quasi-semilattice decompositions of semigroups with zero began in the paper of M. Ćirić and S. Bogdanović [20], 1994. In this paper, some notions which appears in studying of semilattice decompositions of semigroups the authors modified for semigroups with zero. Namely, the authors defined a *0-positive* quasi-order on a semigroup $S = S^0$ as a quasi-order ξ having the property that for $a, b \in S$, $ab \neq 0$ implies $a\,\xi\, ab$ and $b\,\xi\, ab$, they defined a quasi-order ξ on S to satisfy the *0-cm-property* if for all $a, b, c \in S$, $ab \neq 0$, $a\,\xi\, c$ and $b\,\xi\, c$ implies $ab\,\xi\, c$, and they proved the following theorem:

Theorem 4.25. (M. Ćirić and S. Bogdanović [20]) *The poset of left 0-restricted positive quasi-orders on a semigroup $S = S^0$ satisfying the 0-cm-property and the poset of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property are complete lattices and they are isomorphic.*

Further, M. Ćirić and S. Bogdanović defined in [20] a *completely 0-semiprime* ideal of a semigroup $S = S^0$ as an ideal A of S having the property that $A^\bullet$ is a completely semiprime subset of S. Similarly, A is said to be *completely 0-prime* if $A^\bullet$ is a completely prime subset. The set of all completely 0-semiprime ideals of S, denoted by $\mathcal{I}d^{\mathbf{c0s}}(S)$ is clearly a complete lattice. A sublattice L of $\mathcal{I}d^{\mathbf{c0s}}(S)$ is defined to satisfy the *c-0-pi-property* if the set of completely 0-prime ideals from L is meet-dense in L, i.e. if any element of L can be written as the intersection of some family of completely 0-prime ideals from L. Using these notions, M. Ćirić and S. Bogdanović [20] proved the following theorem:

Theorem 4.26. (M. Ćirić and S. Bogdanović [20]) *For a semigroup $S = S^0$, the poset of complete 0,1-sublattices of the lattice $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi-property is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property.*

The investigation of quasi-semilattice decompositions of semigroups with zero M. Ćirić and Bogdanović continued in [26], where they proved the following three theorems that characterize the lattice of quasi-semilattice decompositions of a semigroup with zero:

Theorem 4.27. (M. Ćirić and S. Bogdanović [20]) *The poset of quasi-semilattice decompositions of a semigroup $S = S^0$ is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property.*

Theorem 4.28. (M. Ćirić and S. Bogdanović [26]) *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ it is dually isomorphic to the lattice of left 0-restricted positive quasi-orders on S satisfying the 0-cm-property.*

Theorem 4.29. (M. Ćirić and S. Bogdanović [26]) *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ is isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi-property.*

We finish this chapter by the theorem which give a connection between quasi-semilattice decompositions of semigroups with zero and semilattice decompositions. Note that this connection is incorporated in the name of quasi-semilattice decompositions.

Theorem 4.30. (M. Ćirić and S. Bogdanović [26]) *The lattice of semilattice decompositions of a semigroup S is isomorphic to the lattice of quasi-semilattice decompositions of the semigroup T arising from S by adjoining the zero.*

5. Yet other decompositions

In this paper we talk about yet other types of decompositions having the greatest one.

The topic of Section 5.1 will be $\cup$-decompositions, introduced and first studied by L. N. Shevrin [93], 1961, as a powerful tool in studying of lattices of subsemigroups of a semigroup. We quote the theorem considering the properties of the poset of $\cup$-decompositions, the theorem on atomicity of these decompositions, as Theorem 5.3, and also three theorems on application of $\cup$-decompositions to direct decompositions of the lattice of subsemigroups of a semigroup. For informations on other applications of $\cup$-decompositions, and related $\cup$-band decompositions, in studying of the lattice of subsemigroups of a semigroup the reader is referred to the books of L. N. Shevrin and A. Ya. Ovsyanikov [102], 1990, and [103], 1991, their survey article [101], 1983, and the book of M. Petrich [73], 1977. Note that L. N. Shevrin used the names "strong decomposition" and "strong band decomposition" for these decompositions. But, because the notion "strong band of semigroups" has been also used for other concepts of the semigroup theory, here we use the names used also in the book of M. Petrich [73], 1977.

Ordinal decompositions, treated in Section 5.2, came out from studying of linearly ordered groups in the papers of F. Klein-Barmen [55] and [56], 1942, and [57], 1948, and A. M. Kaufman [51] and [52], 1949. They have been introduced by A. M. Kaufman [51], 1949, where they have been called successively-annihilating sums (bands) of semigroups. General study of these decompositions has done by E. S. Lyapin in his book [62], 1960, where he showed that the poset of ordinal decompositions of any semigroup is a complete sublattice of the partition lattice of this semigroup, and proved the theorem on atomicity of ordinal decompositions, given here as Theorem 5.8. Here we also present the results of M. Ćirić and S. Bogdanović that characterize lattices of ordinal decompositions of semigroups. For more applications on applications of ordinal decompositions see the books: E. S. Lyapin [62], 1960, M. Petrich [73], 1977, and L. N. Shevrin and A. Ya. Ovsyanikov [102], 1990, and [103], 1991.

I-matrix decompositions have arisen in the paper of G. Lallement and M. Petrich [59], 1966, as a generalization of matrix decompositions. The very nice results obtained in this paper will be presented in Section 5.3. For some applications of such decompositions see the papers of J. Fountain and M. Petrich [43], 1986, and [44], 1989.

The last section of this chapter is devoted to semilattice-matrix decompositions of semigroups. These decompositions have been first studied by A. H. Clifford [29], 1941, who proved that unions of groups (completely regular semigroups) are semilattices of completely simple semigroups, which are in fact semilattices of matrices of groups. After that, semilattice-matrix decompositions have been studied by many authors, for example by P. Chu, Y, Guo and X. Ren [28], 1989, L. N. Shevrin [100], 1994, S. Bogdanović and M. Ćirić [11], 1995, and [15], and other. By the well-known theorem of D. McLean [64], 1954, and A. H. Clifford [30], 1954, on the decomposition of a band into a semilattice of rectangular bands, semilattice-matrix decompositions can be treated as generalizations of band decompositions, and in many papers these decompositions have been used to make preparations for band decompositions. Here we present some general properties of these decompositions discovered by M. Ćirić and S. Bogdanović in [27].

5.1. $\bigcup$-decompositions

We said in the introduction of this paper that general study of ∪-decompositions has been done by L. N. Shevrin in [96], 1965. There he has obtained the result that can be formulated in the following way:

Theorem 5.1. (L. N. Shevrin [96]) *The poset of ∪-decompositions of a semigroup S is a principal ideal of the partition lattice of S.*

In the same paper L. N. Shevrin considered also ∪-band decompositions and some their special types. Namely, for any subvariety $\mathcal{V}$ of the variety of bands a ∪-$\mathcal{V}$*-band decomposition* of a semigroup S is defined as a decomposition which is both ∪-decomposition and $\mathcal{V}$-band decomposition. By Theorems 5.1 and 1.6 the following theorem follows:

Theorem 5.2. *For any subvariety $\mathcal{V}$ of the variety of bands, the poset of ∪-$\mathcal{V}$-band decompositions of a semigroup S is a principal ideal of the partition lattice of S.*

L. N. Shevrin [96] also proved the theorem on atomicity of ∪-decompositions, which is given below.

Theorem 5.3. (L. N. Shevrin [96]) *The components of the greatest ∪-decomposition of a semigroup S are ∪-indecomposable.*

Among the numerous applications of ∪-decompositions in studying of lattices of subsemigroups of a semigroup we emphasize the application to decompositions of these lattices into a direct product, which is demonstrated by the following three theorems:

Theorem 5.4. (L. N. Shevrin [94]) *The lattice* $\mathrm{Sub}\,(S)$ *of subsemigroups of a semigroup S is a direct product of lattices L_α, $\alpha \in Y$, if and only if S has a ∪-decomposition into subsemigroups S_α, $\alpha \in Y$, and* $\mathrm{Sub}\,(S_\alpha) \cong L_\alpha$, *for any $\alpha \in Y$.*

Theorem 5.5. (L. N. Shevrin [96]) *The lattice* $\mathrm{Sub}\,(S)$ *of subsemigroups of a semigroup S is directly indecomposable if and only if S is ∪-indecomposable.*

Theorem 5.6. (L. N. Shevrin [96]) *If S_α, $\alpha \in Y$, are the components of the greatest ∪-decomposition of a semigroup S, then the lattice* $\mathrm{Sub}\,(S)$ *of subsemigroups of S can be decomposed into a direct product of lattices* $\mathrm{Sub}\,(S_\alpha)$, *$\alpha \in Y$, which are directly indecomposable.*

We advise the reader to compare the previous three theorems with Theorems 4.9–4.11, concerning direct decompositions of the lattice of ideals of a semigroup with zero, Theorems 4.18 and 4.19, concerning direct decompositions of the lattice of right ideals of a semigroup with zero, and Theorems 3.15 and 3.16, concerning direct decompositions of the lattice of right ideals of a semigroup without zero.

5.2. Ordinal decompositions

General study of ordinal decompositions has been made by E. S. Lyapin in his book [62] from 1960. There he showed the following property of the poset of ordinal decompositions:

Theorem 5.7. (E. S. Lyapin [62]) *The poset of ordinal decompositions of a semigroup S is a complete sublattice of the partition lattice of S.*

E. S. Lyapin [62] also proved the very important theorem on atomicity of ordinal decompositions, whose another proof has been given by M. Ćirić and S. Bogdanović in [25].

Theorem 5.8. (E. S. Lyapin [62]) *The components of the greatest ordinal decomposition of a semigroup S are ordinally indecomposable.*

To characterize the lattice of ordinal decompositions, M. Ćirić and S. Bogdanović [25] have used the next theorem, obtained in their earlier paper [23], which gives a characterization of the poset of chain decompositions of a semigroup through completely prime ideals.

Theorem 5.9. (M. Ćirić and S. Bogdanović [23]) *The poset of chain decompositions of a semigroup S is isomorphic to the poset of complete 1-sublattices of $\mathcal{I}d^{\mathbf{cs}}(S)$ consisting of completely prime ideals of S.*

Note that another characterization of the poset of chain decompositions can be given by filters as follows:

Theorem 5.10. (S. Bogdanović and M. Ćirić [12]) *The poset of chain decompositions of a semigroup S is isomorphic to the poset of complete 0,1-sublattices of $\mathcal{P}(S)$ consisting of filters of S.*

M. Ćirić and S. Bogdanović [25] defined a *strongly prime* ideal of a semigroup S as an ideal P of S having the property that for all $x, y \in S$, $xy = p \in P$ implies that either $x = p$ or $y = p$ or $x, y \in P$, and they proved that the set of all strongly prime ideals of a semigroup S, denoted by $\mathcal{I}d^{\mathbf{sp}}(S)$, is a complete 1-sublattice of the lattice $\mathcal{I}d(S)$ of ideals of S. Moreover, they gave the following characterization of the lattice of ordinal decompositions of a semigroup:

Theorem 5.11. (M. Ćirić and S. Bogdanović [25]) *The lattice of ordinal decompositions of a semigroup S is isomorphic to the lattice of complete 1-sublattices of $\mathcal{I}d^{\mathbf{sp}}(S)$.*

5.3. *I*-matrix decompositions

If θ is a congruence on a semigroup S and S/θ is a rectangular 0-band, then θ is said to be an *I-matrix congruence*, where I is an ideal of S which is the θ-class that is the zero of S/θ. The corresponding decomposition is an *I-matrix decomposition* of S, and I is called a *matrix ideal* of S. G. Lallement and M. Petrich [59] defined a *quasi-completely prime* ideal of a semigroup S as an ideal I satisfying the condition that for all $a, b, c \in S$, $abc \in I$ implies that either $ab \in I$ or $bc \in I$, and they proved the following theorem:

Theorem 5.12. (G. Lallement and M. Petrich [59]) *An ideal I of a semigroup S is a matrix ideal if and only if it is prime and quasi-completely prime.*

To characterize I-matrix congruences, G. Lallement and M. Petrich [59] introduced the following notions: if A is a nonempty subset of a semigroup S, then an equivalence relation θ on S is called a *left A-equivalence* if the following conditions hold:

(1) A is a θ-class of S;
(2) θ is a left congruence;
(3) for all $x, y \in S$, $xy \notin A$ implies $xy\,\theta\,x$.

A *right A-equivalence* is defined dually. Necessary and sufficient conditions for existence of a left A-equivalence and a right A-equivalence on a semigroup have been determined by the following theorem:

Theorem 5.13. (G. Lallement and M. Petrich [59]) *Let A be a subset of a semigroup S. Then there exists a left A-equivalence and a right A-equivalence if and only if A is a quasi-completely prime ideal of S.*

The following theorem has been also proved in [59]:

Theorem 5.14. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the poset of left I-equivalences on S is a complete sublattice of the lattice of left congruences on S.*

G. Lallement and M. Petrich [59] characterized the smallest I-equivalence in three ways. At first, they defined a *left I-complex* of a semigroup S as a nonempty subset A of S having the following properties:

(1) $A \cap I = \varnothing$;

(2) A is a left consistent subset of S;
(3) $A \cup I$ is a right ideal of S.

A *right I-complex* has been defined dually. For an element $a \in S - I$, let $C(a)$ they denoted the smallest left I-complex of S containing a, i.e. the intersection of all left I-complexes of S containing a, called the *principal left I-complex* of S generated by a, and they proved the following

Theorem 5.15. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the relation θ on S defined by:*

$$a\,\theta\,b \quad\Leftrightarrow\quad a, b \in I \;\text{ or }\; C(a) = C(b) \qquad (a, b \in S),$$

equals the smallest left I-equivalence on S.

The second and third characterization of the smallest left I-equivalence on a semigroup have been given by the following two theorems:

Theorem 5.16. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the relation θ on S defined by*

$$a\,\theta\,b \quad\Leftrightarrow\quad (\forall x \in S)\;(axa \in I \quad\Leftrightarrow\quad bxb \in I) \qquad (a, b \in S),$$

equals the smallest I-matrix congruence on S.

Theorem 5.17. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the smallest left I-equivalence equals the principal left congruence L_I.*

Following the ideas used by M. Petrich in studying of matrix decompositions, G. Lallement and M. Petrich proved in [59] the next theorem, similar to Theorem 3.5.

Theorem 5.18. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the intersection of a left I-equivalence and a right I-equivalence is a I-matrix congruence on S.*

Conversely, any I-matrix congruence on S can be written uniquely as the intersection of a left I-equivalence and a right I-equivalence on S.

Using the previous theorem and Theorem 5.14, G. Lallement and M. Petrich proved also in [59] the following two theorems:

Theorem 5.19. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S. Then the poset of I-matrix congruences on S is a complete sublattice of* $\mathrm{Con}\,(S)$.

Theorem 5.20. (G. Lallement and M. Petrich [59]) *Let I be a matrix ideal of a semigroup S and let θ denote the smallest I-matrix congruence on S. Then $\theta = R_I \cap L_I = R_K = L_K$, where $K = \{x \in S \mid x^2 \notin I\}$.*

5.4. Semilattice-matrix decompositions

Let a semigroup S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for any $\alpha \in Y$, let S_α be a left zero band (right zero band, matrix) of semigroups S_i, $i \in I_\alpha$. M. Ćirić and S. Bogdanović [27] called the partition $\{S_i \mid i \in I\}$, where $I = \bigcup_{\alpha \in Y} I_\alpha$, a *semilattice-left* (*semilattice-right, semilattice-matrix*) *decomposition* of S, or briefly *s-l-* (*s-r-, s-m-*)*decomposition*. If θ denotes the equivalence relation determined by this partition and if ϱ denotes the semilattice congruence determined by the partition $\{S_\alpha \mid \alpha \in Y\}$, then θ is called a *semilattice-left* (*semilattice-right, semilattice-matrix*) *equivalence* on S *carried by* ϱ, or briefly *s-l-* (*s-r-, s-m-*)*equivalence*, and ϱ is called a *carrier* of θ. Clearly, an equivalence relation θ on a semigroup S contained in a semilattice-congruence ϱ on S is a s-l-(s-r-, s-m-) equivalence carried by ϱ if and only if for all $a, b \in S$, $a\,\varrho\,b$ implies $ab\,\theta\,a$ ($a\,\varrho\,b$ implies $ab\,\theta\,b$, $a\,\varrho\,b$ implies $aba\,\theta\,a$).

M. Ćirić and S. Bogdanović studied in [27] some general properties of s-l-, s-r- and s-m-equivalences and their carriers, and they proved the next four theorems. Note that Theorem 5.24 is similar to Theorems 3.5, 3.19 and 5.18.

Theorem 5.21. (M. Ćirić and S. Bogdanović [27]) *The set of s-l-* (*s-r-, s-m-*) *equivalences on a semigroup S carried by a semilattice congruence ϱ on S is a closed interval of $\mathcal{E}(S)$.*

Theorem 5.22. (M. Ćirić and S. Bogdanović [27]) *The set of carriers of a s-l-*(*s-r-, s-m-*)*equivalence θ on a semigroup S is a convex subset, with the smallest element, of the lattice of semilattice congruences on S.*

Theorem 5.23. (M. Ćirić and S. Bogdanović [27]) *The poset of all a s-l-*(*s-r-, s-m-*)*equivalences on a semigroup S is a complete lattice.*

Theorem 5.24. (M. Ćirić and S. Bogdanović [27]) *The intersection of a s-l-equivalence and a s-r-equivalence on a semigroup S is a s-m-equivalence.*

Conversely, any s-m-equivalence can be written, uniquely up to a carrier, as the intersection of a s-l-equivalence and a s-r-equivalence.

References

[1] O. Anderson, *Ein Bericht uber Structur abstracter Halbgruppen*, Thesis, Hamburg, 1952.

[2] G. Birkhoff, *Lattice theory*, Amer. Math. Soc, Coll. Publ. Vol. 25, (3rd. edition, 3rd. printing), Providence, 1979.

[3] G. Birkhoff, *Subdirect unions in universal algebra*, Bull. Amer. Math. Soc. **50** (1944), 764–768.

[4] S. Bogdanović, *Semigroups with a system of subsemigroups*, Inst. of Math., Novi Sad, 1985.

[5] S. Bogdanović and M. Ćirić, $\mathcal{U}_{n+1}$*-semigroups*, Contributions MANU **11** (1990), no. 1–2, 9–23.

[6] S. Bogdanović and M. Ćirić, *Semigroups in which the radical of every ideal is a subsemigroup*, Zb. rad. Fil. fak. Niš, Ser. Mat. **6** (1992), 129–135.

[7] S. Bogdanović and M. Ćirić, *Semigroups*, Prosveta, Niš, 1993. (in Serbian)

[8] S. Bogdanović and M. Ćirić, *Semilattices of Archimedean semigroups and (completely) π-regular semigroups* I *(A survey)*, Filomat (Niš) **7** (1993), 1–40.

[9] S. Bogdanović and M. Ćirić, *A new approach to some greatest decompositions of semigroups (A survey)*, Southeast Asian Bull. Math. **18** (1994), no. 3, 27–42.

[10] S. Bogdanović and M. Ćirić, *Orthogonal sums of semigroups*, Israel J. Math. **90** (1995), 423–428.

[11] S. Bogdanović and M. Ćirić, *Semilattices of weakly left Archimedean semigroups*, ibid, this volume.

[12] S. Bogdanović and M. Ćirić, *Positive quasi-orders with the common multiple property on a semigroup*, in: Proc. Math. Conf. in Priština 1994, Lj. D. Kočinac ed, Priština, 1995, pp. 1–6.

[13] S. Bogdanović and M. Ćirić, *Decompositions of semigroups with zero*, Publ Inst. Math. (Beograd) (to appear).

[14] S. Bogdanović and M. Ćirić, *Semilattices of left completely Archimedean semigroups* (to appear).

[15] S. Bogdanović and M. Ćirić, *A note on left regular semigroups*, Publ. Math. Debrecen (to appear).

[16] S. Bogdanović and M. Ćirić, *Quasi-orders and semilattice decompositions of semigroups (A survey)* (to appear).

[17] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, 1983.

[18] M. Ćirić and S. Bogdanović, *Rédei's bands of periodic π-groups*, Zbornik rad. Fil. fak. Niš, Ser. Mat. **3** (1989), 31–42.

[19] M. Ćirić and S. Bogdanović, *Decompositions of semigroups induced by identities*, Semigroup Forum **46** (1993), 329–346.

[20] M. Ćirić and S. Bogdanović, *The lattice of positive quasi-orders on a semigroup* II, Facta Univ. (Niš), Ser. Math. Inform. **9** (1994), 7–17.

[21] M. Ćirić and S. Bogdanović, *Semilattice decompositions of semigroups*, Semigroup Forum (to appear).

[22] M. Ćirić and S. Bogdanović, *Orthogonal sums of 0-σ-simple semigroups*, Acta. Math. Hung (to appear).

[23] M. Ćirić and S. Bogdanović, *The lattice of positive quasi-orders on a semigroup*, Israel J. Math. (to appear).

[24] M. Ćirić and S. Bogdanović, *Posets of $\mathfrak{C}$-congruences*, Algebra Universalis (to appear).

[25] M. Ćirić and S. Bogdanović, *Ordinal sums of semigroups*, Univ. Beograd Publ. Elektr. Fak, Ser. Mat. (to appear).

[26] M. Ćirić and S. Bogdanović, *Quasi-semilattice decompositions of semigroups with zero* (to appear).

[27] M. Ćirić and S. Bogdanović, *Semilattice-matrix decompositions of semigroups* (to appear).

[28] P. Chu, Y. Guo and X. Ren, *The semilattice (matrix)-matrix(semilattice) decomposition of the quasi-completely orthodox semigroups*, Chinese J. of Contemporary Math. **10** (1989), no. 4, 425–438.

[29] A. H. Clifford, *Semigroups admitting relative inverses*, Annals of Math. **42** (1941), no. 2, 1037–1049.

[30] A. H. Clifford, *Bands of semigroups*, Proc. Amer. Math. Soc. **5** (1954), no. 2, 499–504.

[31] A. H. Clifford, *Naturally totally ordered commutative semigroups*, Amer. J. Math. **76** (1954), no. 3, 631–646.

[32] A. H. Clifford, *Review of M. Yamada* [132], Math. Reviews **17** (1956), 584.

[33] A. H. Clifford, *Totally ordered commutative semigroups*, Bull. Amer. Math. Soc. **64** (1958), no. 6, 305–316.

[34] A. H. Clifford, *Completion of semi-continuous ordered commutative semigroups*, Duke Math. J. **26** (1959), no. 1, 41–60.

[35] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol 1, Amer. Math. Soc., 1961.

[36] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol 2, Amer. Math. Soc., 1967.

[37] R. Croisot, *Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples*, Ann. Sci. Ecole Norm. Sup. **70** (1953), no. 3, 361–379.

[38] R. Croisot, *Équivalences principales bilatères définies dans un demi-groupe*, J. Math. Pures Appl. **36** (1957), no. 9, 373–417.

[39] J. Dieudonné, *Sur le socle d'un anneaux et les anneaux simple infinis*, Bull. Soc. Math. France **70** (1942), 46–75.

[40] P. Dubreil, *Contribution a la theorie des demi-groupes*, Mem. Acad. Sci. Inst. France (2) **63** (1941), no. 3, 1–52.

[41] P. Dubreil, *Contribution a la theorie des demi-groupes* II, Rend. Mat. e appl. **10** (1951), 183–200.

[42] M. Ego, *Structure des demi-groupes dont le treillis des sous-demi-groupes est modulaire ou semi-modulaire*, C. R. Acad. Sci. Paris **254** (1962), 1723–1725.

[43] J. Fountain and M. Petrich, *Completely 0-simple semigroups of quotients*, J. Algebra **101** (1986), 365–402.

[44] J. Fountain and M. Petrich, *Completely 0-simple semigroups of quotients* III, Math. Proc. Camb. Phil. Soc. **105** (1989), 263–275.

[45] G. Grätzer, *Universal Algebra*, Van Nostrand, Princeton, 1968.

[46] J. A. Green, *On the structure of semigroups*, Ann. of Math. **54** (1951), 163–172.

[47] T. Hall, *On the natural order of $\mathcal{T}$-class and of idempotents in a regular semigroup, Glasgow Math. J.* **11** (1970), 167–168.

[48] J. M. Howie, *An introduction to semigroup theory*, Acad. Press, New York, 1976.

[49] K. Iséki, *Contribution to the theory of semigroups* IV, Proc. Japan Acad. **32** (1956), 430–435.

[50] P. R. Jones, *Semilattice decompositions and the congruence lattice of semigroups*, preprint (1981).

[51] A. M. Kaufman, *Successively-annihilating sums of associative systems*, Uch. Zap. Leningrad. Gos. Ped. Inst. **86** (1949), 149–165. (in Russian)

[52] A. M. Kaufman, *Structure of some holoids*, Uch. Zap. Leningrad. Gos. Ped. Inst. **86** (1949), 167–182. (in Russian)

[53] A. V. Kelarev, *Radicals and 0-bands of semigroups*, Stud. Sci. Math. Hung. **27** (1992), 125–132.

[54] N. Kimura, *On some existence theorems on multiplicative systems.* I. *Greatest quotients*, Proc. Japan Acad. **36** (1958), no. 6, 305–309.

[55] F. Klein-Barmen, *Über gewisse Halverbände und kommutative Semigruppen* I, Math. Z. **48** (1942), 275–288.

[56] F. Klein-Barmen, *Über gewisse Halverbände und kommutative Semigruppen* II, Math. Z. **48** (1942), 715–734.

[57] F. Klein-Barmen, *Ein Beitrag zur Theorie der linearen Holoide*, Math. Z. **51** (1948), 355–366.

[58] O. B. Kozhevnikov, *A generalization of the concept of complete regularity*, in: Associativnye dejstvija, Gos. Ped. Inst., Leningrad, 1983, pp. 50–56. (in Russian)

[59] G. Lallement and M. Petrich, *Décompositions I-matricielles d'un demi-groupe*, J. Math. Pures et Appl. **45** (1966), 67–117.

[60] E. S. Lyapin, *Normal complexes of associative systems*, Izv. Akad. Nauk SSSR **14** (1950), 179–192. (in Russian)

[61] E. S. Lyapin, *Semisimple commutative associative systems*, Izv. Akad. Nauk SSSR **14** (1950), 367–380. (in Russian)

[62] E. S. Lyapin, *Semigroups*, Fizmatgiz, Moscow, 1960, English transl. Amer. Math. Soc. 1968 (2nd edition). (in Russian)

[63] E. S. Lyapin, *Identities of successively-annihilating bands of semigroups*, Izv. Vysh. Uchebn. Zav. Mat. **1 (200)** (1979), 38–45. (in Russian)

[64] D. Mc Lean, *Idempotent semigroups*, Amer. Math. Monthly **6** (1954), 110–113.

[65] H. Mitsch, *Semigroups and their lattice of congruences*, Semigroup Forum **26** (1983), 1–64.

[66] W. D. Munn, *Decompositions of the lattice of congruences on a semigroup*, Proc. Edinburgh Math. Soc. **23** (1980), 193–198.

[67] Y. S. Park, J. P. Kim and M. G. Sohn, *Semiprime ideals in semigroups*, Math. Japonica **33** (1988), no. 2, 269–273.

[68] M. Petrich, *The maximal semilattice decomposition of a semigroup*, Bull. Amer. Math. Soc. **69** (1963), 342–344..

[69] M. Petrich, *The maximal semilattice decomposition of a semigroup*, Math. Zeitschr. **85** (1964), 68–82.

[70] M. Petrich, *The maximal matrix decomposition of a semigroup*, Portugaliae Math. **25** (1966), 15–33.

[71] M. Petrich, *Homomorphisms of a semigroup onto normal bands*, Acta. Sci. Math. Szeged **27** (1966), 185–196.

[72] M. Petrich, *Introduction to semigroups*, Merill, Ohio, 1973.

[73] M. Petrich, *Lectures in semigroups*, Akad. Verlag, Berlin, 1977.

[74] R. S. Pierce, *Homomorphisms of semigroups*, Ann. of Math. **59** (1954), no. 2, 287–291.

[75] R. Plemmons, *Semigroups with a maximal semigroup with zero homomorphic image*, Notices Amer. Math. Soc. **11** (1964), no. 7, 751.

[76] R. J. Plemmons and T. Tamura, *Semigroups with a maximal homomorphic image having zero*, Proc. Japan Acad. **41** (1965), 681–685.

[77] G. B. Preston, *Matrix representation of inverse semigroups*, J. Austral. Math. Soc. **9** (1969), 29–61.

[78] G. B. Preston, *Representations of inverse semigroups*, J. London Math. Soc. **29** (1954), 411–419.

[79] M. S. Putcha, *Semilattice decompositions of semigroups*, Semigroup Forum **6** (1973), 12–34.

[80] M. S. Putcha, *Minimal sequences in semigroups*, Trans. Amer. Math. Soc. **189** (1974), 93–106.

[81] M. S. Putcha, *Positive functions from* $\mathcal{S}$*-indecomposable semigroups into partially ordered sets*, Check. Math. J. **26 (101)** (1976), 161–170.

[82] M. S. Putcha, *On the maximal semilattice decomposition of the power semigroup of a semigroup*, Semigroup Forum **15** (1978), 263–267.

[83] M. S. Putcha and A. H. Schoenfeld, *Applications of algebraic and combinatoric techniques to a problem in geometry*, J. Pure Appl. Algebra **7** (1976), 235–237.

[84] D. Rees, *On semi-groups*, Proc. Cambridge Phil. Soc. **36** (1940), 387–400.

[85] B. M. Schein, *On subdirectly irreducible semigroups*, Doklady AN SSSR **144** (1962), 999–1002, Correction – **148** (1963), p. 996. (in Russian)

[86] B. M. Schein, *A contribution to the theory of restrictive semigroups*, Izv. vysh. Uchebn. Zaved, Mat. **2 (33)** (1963), 152–154. (in Russian)

[87] B. M. Schein, *Restrictive bisemigroups*, Izv. vysh. Uchebn. Zaved, Mat. **1 (44)** (1965), 168–179. (in Russian)

[88] B. M. Schein, *On certain classes of semigroups of binary relations*, Sibirsk. Mat. Zhurn. **6** (1965), 616–635. (in Russian)

[89] B. M. Schein, *Homomorphisms and subdirect decompositions of semigroups*, Pacific J. Math. **17** (1966), 529–547.

[90] Š. Schwarz, *Semigroups having a kernel*, Check. Math. J. **1 (76)** (1951), 259–301. (in Russian)

[91] Š. Schwarz, *On Gallois connections in the theory of characters of commutative semigroups*, Check. Math. J. **4** (1956), 296–313. (in Russian)

[92] Š. Schwarz, *Semigroups satisfying some weakened forms of the cancellation laws*, Mat.-fyz. čas. **6** (1956), 149–158. (in Slovakian)

[93] L. N. Shevrin, *Semigroups with some types of lattices of subsemigroups*, DAN SSSR **138** (1961), no. 4, 796–798. (in Russian)

[94] L. N. Shevrin, *Semigroups whose lattice of subsemigroups is decomposable into a direct product of lattices*, 5th vsesoyuzn koll. po obsh. algebre (Abstracts), Novosibirsk, 1963, p. 59. (in Russian)

[95] L. N. Shevrin, *Semigroups of finite width*, Theory of semigroups and its applications, V. V. Vagner (editor), Izd. Saratov. Univ., 1965, pp. 325–351. (in Russian)

[96] L. N. Shevrin, *Strong bands of semigroups*, Izv. Vysh. Uchebn. Zaved. Mat. **6 (49)** (1965), 156–165. (in Russian)

[97] L. N. Shevrin, *On the isomorphism of semigroups with relatively complemented subsemigroups*, Mat. Zap. Ural. Gos. Univ. **5** (1965), 92–100. (in Russian)

[98] L. N. Shevrin, *Semigroups*, in: general Algebra, L. A. Skornyakov (editor), Nauka, Moscow, 1991, pp. 11–191. (in Russian)

[99] L. N. Shevrin, *Theory of epigroups* I, Mat. Sbornik **185** (1994), no. 8, 129–160. (in Russian)

[100] L. N. Shevrin, *Theory of epigroups* II, Mat. Sbornik **185** (1994), no. 9, 153–176. (in Russian)

[101] L. N. Shevrin and A. Ya. Ovsyanikov, *Semigroups and their subsemigroup lattices*, Semigroup Forum **27** (1983), no. 1, 1–154.

[102] L. N. Shevrin and A. Ya. Ovsyanikov, *Semigroups and their subsemigroup lattices*, Vol. 1, Izd. Ural. Univ., Sverdlovsk, 1990. (in Russian)

[103] L. N. Shevrin and A. Ya. Ovsyanikov, *Semigroups and their subsemigroup lattices*, Vol. 2, Izd. Ural. Univ., Sverdlovsk, 1991. (in Russian)

[104] O. Steinfeld, *On semigroups which are unions of completely 0-simple semigroups*, Check. Math. J. **16** (1966), 63–69.

[105] O. Steinfeld, *Quasi-ideals in rings and semigroups*, Akad. Kiadó, Budapest, 1978.

[106] R. Šulka, *The maximal semilattice decomposition of a semigroup, radicals and nilpotency*, Mat. časopis **20** (1970), 172–180.

[107] A. K. Sushkevich, *Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit*, Math. Ann. **99** (1928), 30–50.

[108] A. K. Sushkevich, *Theory of generalized groups*, GNTI, Kharkov–Kiev, 1937. (in Russian)

[109] G. Szász, *Théorie des treillis*, Akad. Kiadó, Budapest, et Dunod, Paris, 1971.

[110] T. Tamura, *The theory of construction of finite semigroups* I, Osaka Math. J. **8** (1956), 243–261.

[111] T. Tamura, *Operations on binary relations and their applications*, Bull. Amer. Math. Soc. **70** (1964), 113–120.

[112] T. Tamura, *Another proof of a theorem concerning the greatest semilattice decomposition of a semigroup*, Proc. Japan Acad. **40** (1964), 777–780.

[113] T. Tamura, *The theory of operations on binary relations*, Trans. Amer. Math. Soc. **120** (1965), 343–358.

[114] T. Tamura, *Maximal or greatest homomorphic image of a given type*, Canad. J. Math. **20** (1968), 264–271.

[115] T. Tamura, *The study of closets and free contents related to semilattice decomposition of semigroups*, in: Semigroups, K. W. Folley (editor), Acad. Press, New York, 1969, pp. 221–259.

[116] T. Tamura, *On Putcha's theorem concerning semilattice of Archimedean semigroups*, Semigroup Forum **4** (1972), 83–86.

[117] T. Tamura, *Note on the greatest semilattice decomposition of semigroups*, Semigroup Forum **4** (1972), 255–261.

[118] T. Tamura, *Semilattice congruences viewed from quasi-orders*, Proc. Amer. Math. Soc. **41** (1973), 75–79.

[119] T. Tamura, *Remark on the smallest semilattice congruence*, Semigroup Forum **5** (1973), 277–282.

[120] T. Tamura, *Quasi-orders, generalized Archimedeaness, semilattice decompositions*, Math. Nachr. **68** (1975), 201–220.

[121] T. Tamura, *Semilattice indecomposable semigroups with a unique idempotent*, Semigroup Forum **24** (1982), 77–82.

[122] T. Tamura and N. Kimura, *On decompositions of a commutative semigroup*, Kodai Math. Sem. Rep. **4** (1954), 109–112.

[123] T. Tamura and N. Kimura, *Existence of greatest decomposition of a semigroup*, Kodai Math. Sem. Rep. **7** (1955), 83–84.

[124] T. Tamura and J. Shafer, *Another proof of two decomposition theorems of semigroups*, Proc. Japan Acad. **42** (1966), 685–687.

[125] G. Thierrin, *Quelques propriétiés des sous-groupoides consistants d'un demi-groupe abélien*, C. R. Acad. Sci. Paris **236** (1953), 1837–1839.

[126] G. Thierrin, *Sur quelques propriétiés de certainess classes de demi-groupes*, C. R. Acad. Sci. Paris **241** (1955), 1192–1194.

[127] G. Thierrin, *Sur quelques décompositions des groupoides*, C. R. Acad. Sci. Paris **242** (1956), 596–598.

[128] G. Thierrin, *Sur la théorie des demi-groupes*, Comment. Math. Helv. **30** (1956), 211–223..

[129] V. V. Vagner, *Restrictive semigroups*, Izv. vysh. Uchebn. Zaved, Mat. **6 (31)** (1962), 19–27. (in Russian)

[130] P. S. Venkatesan, *On a class of inverse semigroups*, Amer. J. Math. **84** (1962), 578–582.

[131] P. S. Venkatesan, *On decompositions of semigroups with zero*, Math. Zeitsch. **92** (1966), 164–174.

[132] M. Yamada, *On the greatest semilattice decomposition of a semigroup*, Kodai Math. Sem. Rep. **7** (1955), 59–62.

[133] M. Yamada and N. Kimura, *Note on idempotent semigroups* II, Proc. Japan Acad. **34** (1958), 110–112.

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