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SEMILATTICES OF WEAKLY LEFT ARCHIMEDEAN SEMIGROUPS

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Abstract. By the well-known result of A. H. Clifford, any band of left Archimedean semigroups is a semilattice of matrices (rectangular bands) of left Archimedean semigroups. The converse of this assertion don't hold, i.e. the class of semilattices of matrices of left Archimedean semigroups is larger than the class of bands of left Archimedean semigroups. In this paper we characterize semilattices of matrices of left Archimedean semigroups, and especially matrices of left Archimedean semigroups. The obtained results generalize the some results of M. S. Putcha and L. N. Shevrin.

Bands of left (also right and two-sided) Archimedean semigroups form important classes of semigroups studied by a number of authors. General characterizations of these semigroups have been given by M. S. Putcha [16], and in the completely π-regular case by L. N. Shevrin [17]. Some new characterizations of bands of left Archimedean semigroups and of bands of nil-extensions of left simple semigroups have been given recently by the authors [6]. By the well-known result of A. H. Clifford, any band of left Archimedean semigroups is a semilattice of matrices (rectangular bands) of left Archimedean semigroups. The converse of this assertion don't hold, i.e. the class of semilattices of matrices of left Archimedean semigroups is larger than the class of bands of left Archimedean semigroups. In this paper we give a complete characterization of semigroups having a semilattice decomposition whose components are matrices of left Archimedean semigroups. Moreover, we describe such components in the general and some special cases. For the related results see [7], [12] and [13]. For more informations about semilattice-matrix decompositions of semigroups the reader is reffered

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to [10] and [11]. The obtained results generalize the above quoted results of M. S. Putcha and L. N. Shevrin.

Throughout this paper $\mathbf{Z}^+$ will denote the set of positive integers. The division relations $|$ and $\underset{l}{|}$ on a semigroup S are defined by

$$a \mid b \;\Leftrightarrow\; (\exists x, y \in S^1)\; b = xay, \qquad a \underset{l}{|} b \;\Leftrightarrow\; (\exists x \in S^1)\; b = xa,$$

and that the relations $\longrightarrow$ and $\xrightarrow{l}$ on S are defined by

$$a \longrightarrow b \;\Leftrightarrow\; (\exists n \in \mathbf{Z}^+)\; a \mid b^n, \qquad a \xrightarrow{l} b \;\Leftrightarrow\; (\exists n \in \mathbf{Z}^+)\; a \underset{l}{|} b^n.$$

The relation $\xrightarrow{r}$ on S is defined dually. For $n \in \mathbf{Z}^+$, $\xrightarrow{l}{}^{n}$ will denote the n-th power of $\xrightarrow{l}$, and $\xrightarrow{l}{}^{\infty}$ will denote the transitive closure of $\xrightarrow{l}$. For an element a of a semigroup S we define sets $\Lambda_n(a)$, $n \in \mathbf{Z}^+$, and $\Lambda(a)$ by

$$\Lambda_n(a) = \{x \in S \mid a \xrightarrow{l}{}^{n} x\}, \qquad \Lambda(a) = \{x \in S \mid a \xrightarrow{l}{}^{\infty} x\},$$

and the equivalence relations Λ_n, $n \in \mathbf{Z}^+$, and λ on S by

$$a \,\lambda_n\, b \;\Leftrightarrow\; \Lambda_n(a) = \Lambda_n(b), \qquad a \,\lambda\, b \;\Leftrightarrow\; \Lambda(a) = \Lambda(b),$$

[3]. For undefined notions and notations we refer to [1], [2] and [14].

First we prove the following theorem:

Theorem 1. *The following conditions on a semigroup S are equivalent:*

(i) *λ is a matrix congruence on S;*
(ii) *λ is a right zero band congruence on S;*
(iii) *$(\forall a, b, c \in S)\; abc \xrightarrow{l}{}^{\infty} ac$;*
(iv) *$(\forall a, b \in S)\; aba \xrightarrow{l}{}^{\infty} a$;*
(v) *$(\forall a, b \in S)\; ab \xrightarrow{l}{}^{\infty} b$;*
(vi) *S is a disjoint union of all its principal left radicals;*
(vii) *$\xrightarrow{l}{}^{\infty}$ is a symmetric relation on S.*

Proof. (i) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (i). This follows immediately.

(iv) $\Rightarrow$ (v). For all $a, b \in S$, $ab \xrightarrow{l} bab$, so by (iv), $ab \xrightarrow{l}{}^{\infty} b$.

(v) $\Rightarrow$ (ii). Let $a, b \in S$ such that $a \,\lambda\, b$, and $x \in S$. By (v), $\Lambda(ax) = \Lambda(x) = \Lambda(bx)$ and $\Lambda(xa) = \Lambda(a) = \Lambda(b) = \Lambda(xb)$. Therefore, λ is a congruence. Clearly, it is a right zero band congruence.

(ii) $\Rightarrow$ (vi). Let S be a right zero band B of semigroups S_i, $i \in B$, which are λ-classes of S. Assume $a \in S$. Then $a \in S_i$, for some $i \in B$, and since S_i is a completely semiprime left ideal of S (Lemma 4 [3]), then $\Lambda(a) \subseteq S_i$. On the other hand, if $b \in S_i$, then $b \,\lambda\, a$, so $b \in \Lambda(b) = \Lambda(a)$, whence $S_i \subseteq \Lambda(a)$. Therefore, $\Lambda(a) = S_i$, so (vi) holds.

(vi) $\Rightarrow$ (vii). Let $a, b \in S$ such that $a \xrightarrow{l}{}^{\infty} b$. Then $b \in \Lambda(a)$, whence $\Lambda(a) \cap \Lambda(b) \neq \varnothing$, so by (vi), $\Lambda(a) = \Lambda(b)$. Therefore, $b \xrightarrow{l}{}^{\infty} a$.

(vii) $\Rightarrow$ (v). For all $a, b \in S$, $b \xrightarrow{l} ab$, so by (vii), $ab \xrightarrow{l}{}^{\infty} b$. $\square$

Corollary 1. *The following conditions on a semigroup S are equivalent:*

(i) λ_n *is a matrix congruence on S;*
(ii) λ_n *is a right zero band congruence on S;*
(iii) $(\forall a, b \in S)\ \Lambda_n(a) \subseteq \Lambda_n(aba)$;
(iv) $(\forall a, b \in S)\ \Lambda_n(b) \subseteq \Lambda_n(ab)$;
(v) $\xrightarrow{l}{}^{n}$ *is a symmetric relation on S.*

Lemma 1. *Let ξ be a band congruence on a semigroup S contained in $\stackrel{l}{\text{—}}$, where $\stackrel{l}{\text{—}} = \xrightarrow{l} \cap \xrightarrow{l}{}^{-1}$. Then any ξ-class of S is a left Archimedean semigroup.*

Recall that a semigroup S is called *left Archimedean* if $a \xrightarrow{l} b$, for all $a, b \in S$. Here we introduce a more general notion: a semigroup S will be called *weakly left Archimedean* if $ab \xrightarrow{l} b$, for all $a, b \in S$. *Weakly right Archimedean semigroups* are defined dually. A semigroup S is *weakly t-Archimedean* (or *weakly two-sided Archimedean*) if it is both weakly left and weakly right Archimedean, i.e. if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in abSba$.

We give the following characterization of semilattices of weakly left Archimedean semigroups:

Theorem 2. *A semigroup S is a semilattice of weakly left Archimedean semigroups if and only if*

$$a \longrightarrow b \quad \Rightarrow \quad ab \xrightarrow{l} b,$$

for all $a, b \in S$.

Proof. Let S be a semillatice Y of weakly left Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. If $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, then $\beta \leq \alpha$, whence $b, ba \in S_\beta$. Now, $b^n \in S_\beta bab \subseteq Sab$, for some $n \in \mathbf{Z}^+$, since S_β is weakly left Archimedean. Therefore, $ab \xrightarrow{l} b$.

Conversely, let for all $a, b \in S$, $a \longrightarrow b$ implies $ab \xrightarrow{l} b$. Assume $a, b \in S$. Since $a \longrightarrow ab$, then by the hypothesis, $a^2b \xrightarrow{l} ab$, i.e. $(ab)^n \in Sa^2b \subseteq Sa^2S$, for some $n \in \mathbf{Z}^+$. Now, by Theorem 1 [9], S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. Further, assume $\alpha \in Y$, $a, b \in S_\alpha$. Then $a \longrightarrow b$, so by the hypothesis, $ab \xrightarrow{l} b$ in S, and by Lemma 11 (c) [3], $ab \xrightarrow{l} b$ in S_α. Therefore, S_α is weakly left Archimedean. $\square$

Corollary 2. *A semigroup S is a semilattice of weakly t-Archimedean semigroups if and only if*

$$a \longrightarrow b \quad \Rightarrow \quad ab \xrightarrow{l} b \;\;\&\;\; ba \xrightarrow{r} b,$$

for all $a, b \in S$.

The components of the semilattice decomposition treated in Theorem 2 will be characterized by the next theorem. Namely, we will give a description of weakly left Archimedean semigroups.

Theorem 3. *The following conditions on a semigroup S are equivalent:*

(i) *S is weakly left Archimedean;*
(ii) *S is a matrix of left Archimedean semigroups;*
(iii) *S is a right zero band of left Archimedean semigroups;*
(iv) *$\xrightarrow{l}$ is a symmetric relation on S.*

Proof. (i) $\Rightarrow$ (iv). Let $a, b \in S$ such that $a \xrightarrow{l} b$, i.e. $b^n = xa$, for some $n \in \mathbf{Z}^+$, $x \in S$. By (i), $a^m = yxa = yb^n$, for some $m \in \mathbf{Z}^+$, $y \in S$, whence $b \xrightarrow{l} a$.

(iv) $\Rightarrow$ (i). This follows by the proof for (vii) $\Rightarrow$ (v) of Theorem 1.

(iv) $\Rightarrow$ (iii). Let $a, b, c \in S$ such that $a \xrightarrow{l} b$ and $b \xrightarrow{l} c$. By (iv), $c \xrightarrow{l} b$, so $b^n = xa = yc$, for some $n \in \mathbf{Z}^+$, $x, y \in S$. Since (iv) $\Leftrightarrow$ (i), then there exists $m \in \mathbf{Z}^+$, $z \in S$ such that $c^m = z(yc) = zb^n = zxa \in Sa$. Therefore, $a \xrightarrow{l} c$, so $\xrightarrow{l}$ is transitive, i.e. $\xrightarrow{l} = \xrightarrow{l}{}^\infty$. Now, by Theorem 1, $\lambda_1 = \lambda$ is a right zero band congruence. By Lemma 1, λ_1-classes are left Archimedean semigroups.

(iii) $\Rightarrow$ (ii). This follows immediately.

(ii) $\Rightarrow$ (i). Let S be a matrix B of left Archimedean semigroups S_i, $i \in B$. Then for $a, b \in S$, $a, aba \in S_i$, for some $i \in B$, whence $a^n \in S_i aba \subseteq Sba$, for some $n \in \mathbf{Z}^+$. $\square$

Recall that the relation $\xrightarrow{t}$ on a semigroup S is defined by $\xrightarrow{t} = \xrightarrow{l} \cap \xrightarrow{r}$. Now by Theorem 3 and its dual we obtain the following corollary:

Corollary 3. *The following conditions on a semigroup S are equivalent:*

(i) *S is weakly t-Archimedean;*
(ii) *S is a matrix of t-Archimedean semigroups;*
(iii) $\xrightarrow{t}$ *is a symmetric relation on S;*
(iv) $\xrightarrow{l}$ *and* $\xrightarrow{r}$ *are symmetric relations on S.*

By the following theorem we characterize matrices of nil-extensions of left simple semigroups.

Theorem 4. *The following conditions on a semigroup S are equivalent:*

(i) *S is weakly left Archimedean and left π-regular;*
(ii) *S is weakly left Archimedean and intra-π-regular;*
(iii) *S is a matrix of nil-extensions of left simple semigroups;*
(iv) *S is a right zero band of nil-extensions of left simple semigroups;*
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in S(ba)^n$;
(vi) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sb^n a$.

Proof. (i) $\Rightarrow$ (iv). This follows by Theorem 3 and Theorem 4.1 [15], since the components of any band decomposition of a left π-regular semigroup are also left π-regular.

(iv) $\Rightarrow$ (iii). This follows immediately.

(iii) $\Rightarrow$ (ii). This follows by Theorem 3, since a nil-extension of a left simple semigroup is intra-π-regular.

(ii) $\Rightarrow$ (i). By Theorem 3, S is a right zero band B of left Archimedean semigroups S_i, $i \in B$. Let $a \in Intra(S)$, i.e. $a = xa^2y$, for some $x, y \in S$. Then $a = (xa)^k ay^k$, for each $k \in \mathbf{Z}^+$. Further, $a \in S_i$, for some $i \in B$, and clearly, $y \in S_i$, so $y^k = za^2$, for some $k \in \mathbf{Z}^+$, $z \in S$, since S_i is left Archimedean. Therefore, $a = (xa)^k ay^k = (xa)^k aza^2$, whence $a \in LReg(S)$, so by Theorem 1 [5], S is left π-regular.

(iv) $\Rightarrow$ (vi). Let S be a right zero band B of semigroups S_i, $i \in B$, and for each $i \in B$, let S_i be a nil-extension of a left simple semigroup K_i. Since (v) $\Leftrightarrow$ (i), then S is a nil-extension of a left completely simple semigroup K. Clearly, $K = LReg(S) = \bigcup_{i\in B} K_i$. Now, for $a, b \in S$, $a \in S_i$, $b \in S_j$, for some $i, j \in B$, and $a^n \in K_i$, $b^n \in K_j$, for some $n \in Z$, whence $b^n a \in S_i \cap K = K_i$, so $a^n \in K_i b^n a \subseteq Sb^n a$.

(vi) $\Rightarrow$ (v). Assume $a, b \in S$. By (*vii*), there exists $n \in \mathbf{Z}^+$ such that $a^n \in S(ab)^n a \subseteq S(ba)^n$.

(v) $\Rightarrow$ (i). This follows immediately. $\square$

Let T be a semigroup of a semigroup S. A mapping φ of S onto T is a *right retraction* of S onto T if $a\varphi = a$, for each $a \in T$, and $(ab)\varphi = a(b\varphi)$, for all $a, b \in S$. *Left retractions* are defined dually. A mapping φ of S onto

T is a *retraction* of S onto T if it is a homomorphism and $a\varphi = a$, for each $a \in T$. If T is an ideal of S, then φ is a retraction of S onto T if and only if it is both left and right retraction of S onto T. An ideal extension S of a semigroup T is a (*left, right*) *retractive* extension of T if there exists a (*left, right*) *retraction* of S onto T.

Some characterizations of matrices of nil-extensions of left groups have been given by L. N. Shevrin in [17]. By the next theorem we prove that such semigroups are exactly right retractive nil-extensions of completely simple semigroups. In this way we generalize some results of S. Bogdanović and S. Milić [7], J. L. Galbiati and M. L. Veronesi [12] and A. Mărkuş [13].

Theorem 5. *The following conditions on a semigroup S are equivalent:*

(i) *S is a right retractive nil-extension of a completely simple semigroup;*
(ii) *S is weakly left Archimedean and has an idempotent;*
(iii) *S is a matrix of nil-extensions of left groups;*
(iv) *S is a right zero band of nil-extensions of left groups;*
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^n S(ba)^n$*;*
(vi) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^n S b^n a$*.*

Proof. (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii). This follows immediately.

(ii) $\Rightarrow$ (i). By Theorem 4.1 [15]. S is a nil-extension of a simple semigroup K, so it is intra-π-regular and by Theorem 1 [5], S is left π-regular, it is a right zero band B of semigroups S_i, $i \in B$, and for each $i \in B$, S_i is a nil-extension of a left simple semigroup K_i. Further, $K = Intra(S) = LReg(S) = \bigcup_{i\in B} K_i$, by Theorem 1 [5], since the components of any band decomposition of a left π-regular semigroup are also left π-regular. Thus, K is left completely simple, so it is completely simple, since it has an idempotent. Thus, for each $i \in B$, K_i is a left group, so by Theorem VI 3.1 [1] (or Theorem 3.7 [2]), it has a right identity e_i. Define a mapping φ of S onto K by:

$$a\varphi = ae_i \qquad\qquad \text{if } a \in S_i,\ i \in B.$$

Clearly, $a\varphi = a$, for each $a \in K$. Further, for $a, b \in S$, $a \in S_i$, $b \in S_j$, for some $i, j \in B$, and $ab \in S_j$, whence $(ab)\varphi = (ab)e_j = a(be_j) = a(b\varphi)$. Therefore, φ is a right retraction of S onto K.

(i) $\Rightarrow$ (vi). Let S be a right retractive nil-extension of a completely simple semigroup K, and let K be a right zero band B of left groups K_i, $i \in B$. Let $a, b \in S$. Then $a^n, b^n \in K$, for some $n \in \mathbf{Z}^+$, and $a^n \in K_i$, $b^n \in K_j$, for some $i, j \in B$. If $a\varphi \in K_l$, for some $l \in B$, since $a^{n+1} \in K_i$, then $a^{n+1} = a^{n+1}\varphi = a^n(a\varphi) \in K_i K_l \subseteq K_l$, whence $l = i$. Thus, $a\varphi \in K_i$, so $b^n a = (b^n a)\varphi = b^n(a\varphi) \in K_j K_i \subseteq K_i$. Therefore, $a^n, b^n a \in K_i$, so by the dual of Lemma 1.1 [8], $a^n \in a^n K_i b^n a \subseteq a^n S b^n a$.

(vi) $\Rightarrow$ (v). For $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^n S(ab)^n a = a^n Sa(ba)^n \subseteq a^n S(ba)^n$.

(v) $\Rightarrow$ (iv). This follows by Theorem 4. $\square$

Corollary 4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a completely simple semigroup;*
(ii) *S is weakly t-Archimedean and intra-π-regular;*
(iii) *S is weakly t-Archimedean and has an idempotent;*
(iv) *S is a matrix of π-groups;*
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in (ab)^n S(ba)^n$.

Let us introduce the following notations for some classes of semigroups:

Notation	Class of semigroups	Notation	Class of semigroups
$\mathcal{LA}$	*left Archimedean*	$\mathcal{M}$	*rectangular bands*
$\mathcal{B}$	*bands*	$\mathcal{S}$	*semilattices*

and by $\mathcal{X}_1 \circ \mathcal{X}_2$ we denote the Mal'cev product of classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of semigroups. Let

$$\mathcal{LA} \circ \mathcal{M}^{k+1} = \left(\mathcal{LA} \circ \mathcal{M}^k\right) \circ \mathcal{M}, \quad k \in \mathbf{Z}^+.$$

Now we can state the following:

Problem. *Describe the structure of semigroups from the following classes*

$$\mathcal{LA} \circ \mathcal{M}^{k+1}, \quad \left(\mathcal{LA} \circ \mathcal{M}^{k+1}\right) \circ \mathcal{B}, \quad \left(\mathcal{LA} \circ \mathcal{M}^{k+1}\right) \circ \mathcal{S}.$$

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