FILOMAT (Niš) 9:3 (1995), 611–617
Algebra, Logic & Discrete Mathematics
Niš, April 14–16, 1995.

SEMILATTICES OF HEREDITARY ARCHIMEDEAN SEMIGROUPS

Stojan Bogdanović, Miroslav Ćirić and Melanija Mitrović

ABSTRACT. In this paper we investigate semigroups whose any subsemigroup is Archimedean, called hereditary Archimedean, and semilattices of such semigroups. The obtained results generalize some results of J. L. Chrislock, M. Schutzenberger and M. V. Sapir and E. V. Suhanov.

T. E. Nordahl [14] studied commutative semigroups whose any proper ideal is a power-joined semigroup. C. S. H. Nagore [12] extended this result to quasi-commutative semigroups. Semigroups containing an idempotent and whose any subsemigroup is t-Archimedean have been studied by A. Cherubini and A. Varisco [9]. S. Bogdanović [2] studied weakly commutative semigroups whose any proper right ideal is power-joined. B. Pondeliček [15] described semigroups whose one-sided ideals are t-Archimedean semigroups. Semigroups whose any proper ideal is a power-joined semigroup have been studied by A. Nagy [13]. S. Bogdanović in [3] described semigroups whose proper (left) ideals are Archimedean (left Archimedean, t-Archimedean, power-joined) semigroups. S. Bogdanović and T. Malinović [8] studied semigroups whose any proper subsemigroup is right Archimedean (t-Archimedean). In this paper we study semigroups whose any subsemigroup is Archimedean, called hereditary Archimedean semigroups, and semilattices of such semigroups. We prove also a more general theorem concerning semigroups whose any subsemigroup is a semilattice of Archimedean semigroups. Note that semilattices of Archimedean semigroups have been studied by a number of authours. M. S. Putcha in [16] gave the first complete description of such semigroups. Another characterizations of semilattices of

Supported by Grant 0401B of RFNS through Math. Inst. SANU.

Archimedean semigroups have been given by T. Tamura [17], S. Bogdanović and M. Ćirić [4] and M. Ćirić and S. Bogdanović [10].

Throughout this paper, $Reg(S)$ ($E(S)$) will denote the set of regular (idempotent) elements of a semigroup S, and for $e \in E(S)$, G_e will denote the maximal subgroup of S with e as its identity. A semigroup S is said to be *π-regular* if for any $a \in S$, some power of a is regular.

For undefined notions and notations we refer to [1] and [5].

Recall that the *division relation* $\mid$ on a semigroup S is defined by

$$a \mid b \quad \Leftrightarrow \quad (\exists x, y \in S^1)\, b = xay,$$

and the relation $\longrightarrow$ is defined by

$$a \longrightarrow b \quad \Leftrightarrow \quad (\exists n \in \mathbf{Z}^+)\, a \mid b^n.$$

Also, on a semigroup S we define the relations $\uparrow$, $\underset{l}{\uparrow}$, $\underset{r}{\uparrow}$ and $\underset{t}{\uparrow}$ by

$$\begin{aligned}
a \uparrow b \quad &\Leftrightarrow \quad (\exists n \in \mathbf{Z}^+)\, b^n \in \langle a, b\rangle\, a\, \langle a, b\rangle\,, \\
a \underset{l}{\uparrow} b \quad &\Leftrightarrow \quad (\exists n \in \mathbf{Z}^+)\, b^n \in \langle a, b\rangle\, a, \\
a \underset{r}{\uparrow} b \quad &\Leftrightarrow \quad (\exists n \in \mathbf{Z}^+)\, b^n \in a\, \langle a, b\rangle\,, \\
a \underset{t}{\uparrow} b \quad &\Leftrightarrow \quad (\exists n \in \mathbf{Z}^+)\, a \underset{l}{\uparrow} b \;\&\; a \underset{r}{\uparrow} b
\end{aligned}$$

Clearly, $a \underset{t}{\uparrow} b$ if and only if $b^n \in a\, \langle a, b\rangle\, a$, for some $n \in \mathbf{Z}^+$.

A semigroup S is a *hereditary Archimedean* if $a \uparrow b$ for all $a, b \in S$. By a *hereditary left Archimedean semigroup* we mean a semigroup S satisfying the condition: $a \underset{l}{\uparrow} b$, for all $a, b \in S$. A *hereditary right Archimedean semigroup* is defined dually. A semigroup S is called *hereditary t-Archimedean* if it is both hereditary left Archimedean and hereditary right Archimedean. i.e. if $a \underset{t}{\uparrow} b$ for all $a, b \in S$.

The next lemma gives an explanation why we the notion "hereditary Archimedean" is used.

Lemma 1. [5] *A semigroup S is hereditary Archimedean if and only if any subsemigroup of S is Archimedean.*

Similar assertions hold for hereditary left, right or t-Archimedean semigroups.

T. Tamura in [23] proved that the class of all semigroups which are semilattices of Archimedean semigroups is not subsemigroup closed. By the following theorem we determine the greatest subsemigroup closed subclass of this class. In other words, we describe all semigroups having the property that any its subsemigroup is a semilattice of Archimedean semigroups.

Theorem 1. *Any subsemigroup of a semigroup S is a semilattice of Archimedean semigroups if and only if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that*
$$(ab)^n \in \langle a, b\rangle\, a^2\, \langle a, b\rangle\,.$$

Proof. If $a, b \in S$ and $T = \langle a, b\rangle$, then by Theorem 1 [11] it follows that
$$(ab)^m \in Ta^2T = \langle a, b\rangle\, a^2\, \langle a, b\rangle\,,$$
for some $m \in \mathbf{Z}^+$.

Conversely, if T is a subsemigroup of S and $a, b \in T$, then there exists $m \in \mathbf{Z}^+$ such that
$$(ab)^m \in \langle a, b\rangle\, a^2\, \langle a, b\rangle \subseteq Ta^2T,$$
so by Theorem 1 [11], T is a semilattice of Archimedean semigroups. $\square$

The main result of this paper is the following theorem which characterizes semilattices of hereditary Archimedean semigroups.

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of hereditary Archimedean semigroups;*
(ii) $(\forall a, b \in S)$ $a \longrightarrow b \;\Rightarrow\; a^2 \uparrow b$;
(iii) $(\forall a, b, c \in S)$ $a \longrightarrow c \;\&\; b \longrightarrow c \;\Rightarrow\; ab \uparrow c$;
(iv) $(\forall a, b, c \in S)$ $a \longrightarrow b \;\&\; b \longrightarrow c \;\Rightarrow\; a \uparrow c$.

Proof. (i) $\Rightarrow$ (ii). Let S be a semilattice Y of hereditary Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. Then $b, a^2b \in S_\alpha$, for some $\alpha \in Y$, so by the hypothesis we obtain that
$$b^n \in \left\langle b, a^2b\right\rangle a^2b \left\langle b, a^2b\right\rangle \subseteq \left\langle a^2, b\right\rangle a^2 \left\langle a^2, b\right\rangle.$$
Thus $a^2 \uparrow b$, so (ii) holds.

(ii) $\Rightarrow$ (iii). Assume $a, b, c \in S$ such that $a \longrightarrow c \;\&\; b \longrightarrow c$. Then by Theorem 5.5 [5], $ab \longrightarrow c$. Now by (ii) it follows $(ab)^2 \uparrow c$, whence $ab \uparrow c$.

(iii) $\Rightarrow$ (iv). By (iii) and Propositions 7 [22], $\longrightarrow$ is transitive. Assume $a, b, c \in S$ such that $a \longrightarrow b$ and $b \longrightarrow c$. Then $a \longrightarrow c$, so $a^2 \uparrow c$, by (iii), whence $a \uparrow c$.

(iv) $\Rightarrow$ (i). By (iv), $\longrightarrow$ is transitive, so by Proposition 7 [22], S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$.

Assume $\alpha \in Y$ and $a, b \in S_\alpha$. Then $a \longrightarrow b$ and $b \longrightarrow b$, whence $a \uparrow b$, by (iv). Therefore, S_α is hereditary Archimedean. Hence, (i) holds. $\square$

The next theorem gives a characterisation of semigroups which are chains of hereditary Archimedean semigroups.

Theorem 3. *A semigroup S is a chain of hereditary Archimedean semigroups if and only if*

$$ab \uparrow a \quad or \quad ab \uparrow b.$$

for all $a, b \in S$.

Proof. Let S be a chain Y of hereditary Archimedean semigroups S_α, $\alpha \in Y$. If $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, then $a, ab \in S_\alpha$ or $b, ab \in S_\beta$, whence

$$a^n \in \langle a, ab\rangle\, ab\, \langle a, ab\rangle \qquad \text{or} \qquad b^n \in \langle b, ab\rangle\, ab\, \langle b, ab\rangle$$

for some $n \in \mathbf{Z}^+$.

Conversely, by the hypothesis and Theorem 1 [7], S is a chain Y of Archimedean semigroups $S_\alpha, \alpha \in Y$. If $\alpha \in Y$ and $a, b \in S_\alpha$, then then there exists $n \in \mathbf{Z}^+$ such that $b^n \in S_\alpha a S_\alpha$, and by Theorem 2, $a^2 \uparrow b^n$, whence $a \uparrow b$. Thus, S_α is hereditary Archimedean. Hence, S is a chain of hereditary Archimedean semigroups. $\square$

Further we study semilattices of hereditary left Archimedean semigroups.

Theorem 4. *A semigroup S is a semilattice of hereditary left Archimedean semigroups if and only if for all $a, b \in S$,*

$$a \longrightarrow b \quad \Rightarrow \quad a \underset{l}{\uparrow} b.$$

Proof. Let S be a semilattice Y of hereditary left Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. Since $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, we then have that $\beta \leq \alpha$, so $b, ba \in S_\beta$. Now $ba \underset{l}{\uparrow} b$, whence $a \underset{l}{\uparrow} b$, which proves the direct part of the theorem.

Conversely, by the hypothesis and Theorem 3 (2) [18], S is a semilattice Y of left Archimedean semigroup S_α, $\alpha \in Y$. Assume $\alpha \in Y$ and $a, b \in S_\alpha$. Then $a \longrightarrow b$, whence $a \underset{l}{\uparrow} b$, by the hypothesis. Therefore, any S_α is hereditary left Archimedean, so S is a semilattice of hereditary left Archimedean semigroups. $\square$

Corollary 1. *A semigroup S is a semilattice of hereditary t-Archimedean semigroups if and only if for all $a, b \in S$,*

$$a \longrightarrow b \quad \Rightarrow \quad a \underset{t}{\uparrow} b.$$

Now we prove a theorem which generalizes a result of J. L. Chrislock [10].

Theorem 5. *The folowing conditions on a semigroup S are equivalent:*

(i) *S is hereditary Archimedean and π-regular;*
(ii) *S is hereditary Archimedean and has a primitive idempotent;*
(iii) *S is a nil-extension of a periodic completely simple semigroup;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n = (a^n b^n a^n)^n$.

Proof. (i) $\Rightarrow$ (ii). First we prove that

$$(1) \qquad (\forall a \in S)(\forall e \in E(S))(\exists n \in \mathbf{Z}^+)\ e = (eae)^n.$$

Indeed, for $a \in S$, $e \in E(S)$, $ea \uparrow e$, by (i), whence $e = (ea)^n$ or $e = (ea)^n e$, for some $n \in \mathbf{Z}^+$. However, in both of cases it follows that $e = (ea)^n e = (eae)^n$. Thus (1) holds.

Further, assume $a \in S$. Let $m \in \mathbf{Z}^+$ such that $a^m \in Reg(S)$ and let x be an inverse of a^n. Then $a^m x, xa^m \in E(S)$, so by (1) we obtain that

$$a^m x = (a^m x \cdot a \cdot a^m x)^n = (a^{m+1}x)^n,$$

for some $n \in \mathbf{Z}^+$, whence

$$\begin{aligned} a^m = a^m x a^m &= (a^{m+1}x)^n a^m = (a^{m+1}x)^{n-1} a^{m+1} x a^m = \\ &= (a^{m+1}x)^{n-1} a a^m x a^m = (a^{m+1}x)^{n-1} a^{m+1} = \\ &= (a^{m+1}x)^{n-2} a^{m+1} x a^{m+1} = (a^{m+1}x)^{n-2} a a^m x a^m a = \\ &= (a^{m+1}x)^{n-2} a a^m a = (a^{m+1}x)^{n-2} a^{m+2} = \cdots = \\ &= (a^{m+1}x)^{n-(n-1)} a^{m+(n-1)} = \\ &= a^{m+1} x a^{m+n-1} = a a^m x a^m a^{n-1} = \\ &= a a^m a^{n-1} = a^{m+n}. \end{aligned}$$

Thus, S is periodic, and by Theorem 3.14 [5], S has a primitive idempotent.

(ii) $\Rightarrow$ (iii). By Theorem 3.14 [5], S is a nil-extension of a completely simple semigroup K. But, K is hereditary Archimedean and regular, so it is periodic, by the proof of (i) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (iv). Assume $a, b \in S$. Then $a^k = e$ and $b^n = f$, for some $e, f \in E(S)$, $k \in \mathbf{Z}^+$. Further, $efe \in eSe = G_e$, by Lemma 3.13 [5], whence $(efe)^m = e$, for some $m \in \mathbf{Z}^+$. Now, for $n = km$ we obtain that $a^n = (a^n b^n a^n)^n$.

(iv) $\Rightarrow$ (i). This follows immediately. $\square$

Finaly we prove the following theorem which generalizes some results of M. V. Sapir and E. V. Suhanov [19] and M. Schutzenberger [20] (see L. N. Shevrin and E. V. Suhanov [21]).

Theorem 6. *The following conditions on a semigroup S are equivalent:*

(i) *S is π-regular and a semilattice of hereditary Archimedean semigroups;*

(ii) *S is a semilattice of nil-extensions of periodic completely simple semigroups;*

(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n = (ab)^n((ba)^n(ab)^n)^n$;

(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n = ((ab)^n(ba)^n(ab)^n)^n$.

Proof. (i) $\Rightarrow$ (ii). This follows immediately by Theorem 5.

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). This follows by Theorem 5.

(iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i). This follows immediately. $\square$

References

[1] S. Bogdanović, *Semigroups with system of subsemigroups*, Inst. of Math., Novi Sad, 1985.

[2] S. Bogdanović, *Q_r-semigroups*, Publ. Inst. Math. **29(43)** (1981), 15–21.

[3] S. Bogdanović, *Semigroups whose proper ideals are Archimedean semigroups*, Zb. rad. PMF Novi Sad **13** (1983), 289–296.

[4] S. Bogdanović and M. Ćirić, *Semigroups in which the radical of every ideal is a subsemigroup*, Zb. rad. Fil. fak. Niš, Ser. Mat. **6** (1992), 129–135.

[5] S. Bogdanović and M. Ćirić, *Polugrupe*, Prosveta, Niš, 1993.

[6] S. Bogdanović and M. Ćirić, *Semilattices of Archimedean semigroups and (completely) π-regular semigroups I* (*A survey*), FILOMAT (Niš) **7** (1993), 1–40.

[7] S. Bogdanović and M. Ćirić, *Chains of Archimedean semigroups* (*Semiprimary semigroups*), Indian J. Pure Appl. Math. **25** (1994), no. 3, 331–336.

[8] S. Bogdanović and T. Malinović, *Semigroups whose proper subsemigroups are* (*right*) *t-Archimedean*, Algebra and Logic, Cetinje, 1986, Inst. of Math., Novi Sad, 1988, pp. 1–14.

[9] A. Cherubini Spoletini e A. Varisco, *Sui semigruppi i cui soto semigruppi propri sono t-Archemedei*, Ist. Lombardo **112** (1978), 91–98.

[10] J. L. Chrislock, *A certain class of identities on semigroups*, Proc. Amer. Math. Soc. **21** (1969), 189–190.

[11] M. Ćirić and S. Bogdanović, *Decompositions of semigroups induced by identities*, Semigroup Forum **46** (1993), 329–346.

[12] M. Ćirić and S. Bogdanović, *Semilattice decompositions of semigroup*, Semigroup Forum (to appear).

[13] C. S. H. Nagore, *Quasi-commutative Q-semigroups*, Semigroup Forum **15** (1978), 189–193.

[14] A. Nagy, *Semigroups whose proper two-sided ideals are power-joined*, Semigroup Forum **25** (1982), 325–329.

[15] T. E. Nordahl, *Commutative semigroups whose proper subsemigroups are power-joined*, Semigroup Forum **6** (1973), 35–41.

[16] B. Pondeliček, *Semigroups whose proper one-sided ideals are t-Archimedean*, Mat. Vesnik **37 (3)** (1985), 315–321.

[17] M. S. Putcha, *Semilattice decompositions of semigroups*, Semigroup Forum **6** (1973), 12–34.

[18] M. S. Putcha, *Rings which are semilattices of Archimedean semigroups*, Semigroup Forum **23** (1981), 1–5.

[19] M. V. Sapir and E. V. Suhanov, *On varieties of periodic semigroups*, Izv. vysh. uchebn. zav, Mat. **4** (1981), 48–55. (in Russian)

[20] M. Schutzenberger, *Sur le produit de concatenation non ambigu*, Semigroup Forum **13** (1976), 47–75.

[21] L. N. Shevrin and E. V. Suhanov, *Structural aspects of the theory of varieties of semigroups*, Izv. vysh. uchebn. zav, Mat. **6** (1989), 3–39. (in Russian)

[22] T. Tamura, *On Putcha's theorem concerning semilattice of Archimedean semigroups*, Semigroup Forum **4** (1972), 255–261.

[23] T. Tamura, *Quasi-orders, generalized Archimedeaness, semilattice decompositions*, Math. Nachr. **68** (1975), 201–220.

Univeristy of Niš, Faculty of Economics, Trg JNA 11, 18000 Niš, Yugoslavia

E-mail address: root@eknux.eknfak.ni.ac.yu

Univeristy of Niš, Faculty of Philosophy, Ćirila i Metodija 2, 18000 Niš, Yugoslavia

E-mail address: mciric@archimed.filfak.ni.ac.yu

Univeristy of Niš, Faculty of Mechanical Engineering, Beogradska 14, 18000 Niš, Yugoslavia