GLASNIK MATEMATIČKI
Vol. 25(2) (1991)

A NIL-EXTENSION OF A REGULAR SEMIGROUP

S. Bogdanović and M. Ćirić, Beograd

Abstract. In this paper some characterizations of nil-extensions of regular semigroups are given, Further, we characterize retractive nil-extensions of some special types of regular semigroups.

1. Introduction and preliminaries

The nil-extensions of a completely simple semigroup are described in [7] and [9]. A nil-extension of a completely regular semigroup is described in [3]. Here, a characterization of a nil-extension of a regular semigroup is given. We characterize nil-extensions of a semilattice of left and right groups and of a semilattice of groups. For the last, we show that these are retractive. Moreover, the characterizations of nil-extensions of a band and retractive nil-extensions of completely simple semigroups and of left groups are given.

By $\mathbf{Z}^+$ we denote the set o all positive integers. A semigroup S is *π-regular* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^nSa^n$. A semigroup S is *completely π-regular* if for every $a \in S$ there exist $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^nxa^n$ and $a^nx = xa^n$. A semigroup S is called a *semigroup of Galbiati-Veronesi*, or simply *GV-semigroup*, if S is π-regular and every regular element of S is completely regular [10]. A semigroup S is a *π-group* if S is a nil-extension of a group. A semigroup S with zero 0 is a *nil-semiroup* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. By *nil-extension* we mean an ideal extension by a nil-semigroup. An ideal extension S of a semigroup T is a *retract extension* (or *retractive extension*) if there exists a homomorphism $\bar{\varphi}$ of S onto T such that $\varphi(t) = t$ for all $t \in T$. Such a homomorphism we call a *retraction*. A semigroup S is *weakly commutative* if for every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in bSa$ [14]. A semigroup S is an *LR-semigroup* if fo every $a,b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in Sa \cup bS$ [6]. By Reg(S) (Gr(S),

Mathematics subject classification (1980): 13B02
Key words and phrases: Nil-extension, regular semigroup, GV–semigroup, completely π-regular semigroup, LR–semigroup.

Intra(S), E(S)) we denote the set of regular (completely regular, intra-regular, idempotent) elements of a semigroup S. By G_e we denoe the maximal subgroup of a semigroup S having e as its identity. In any seiroup S define a relation $\mathcal{K}$ on S by

$$a\mathcal{K}b \Leftrightarrow (\exists m,n\in Z^+)\ a^m = b^n.$$

Then $\mathcal{K}$ is an equivalence relation. The $\mathcal{K}$-class containing an element a is denote by K_a. If S is a periodic semigroup, then

$$S = \bigcup_{e\in E(S)} K_e.$$

For undefined notions and notations we refer to [1] and [13].

The following results will be used in the next investigations.

LEMMA. *Let x be an element of a semigroup S such that x^n lies in a subgroup G_e of S, $e\in E(S)$, for some $n\in Z^+$. Then*

(1) $ex = xe\in G_e$;

(2) $x^m\in G_e$ *for every* $m\geq n$.

(See Theorem I 4.3. [1]). The last Lemma we will call *Munn's lemma* (W. D. Munn [12]). □

2. A nil-extension of a regular semigroup

The following theorem characterizes a nil-extension of a regular semigroup.

THEOREM 2.1. *A semigroup S is a nil-extension of a regular semigroup if and only if for every $x,a,y\in S$ there exists $n\in Z^+$ such that*

$$xa^ny \in xa^nySxa^ny. \tag{2.1}$$

Proof. Let S be a nil-extension of a regular semigroup K and let $x,a,y\in S$. Then there exists $n\in Z^+$ such that $a^n\in K$ and, since K is an ideal of S, then $xa^ny\in K$. Thus

$$xa^ny\in xa^nyKxa^ny\subseteq xa^nySxa^ny,$$

so (2.1) holds.

Conversely, let (2.1) holds. Let $a\in S$, then there exists $n\in Z^+$ such that $a^{n+2}\in a^{n+2}Sa^{n+2}$, so $n^{n+2}\in$ Reg(S) and thus Reg$(S)\neq\emptyset$. Let $a\in$ Reg(S) and let $z\in S$. Then

$$\begin{aligned} az &= axaz && \text{, for some } x\in S\\ &= a(xa)^n z && \text{, for every } n\in Z^+\\ &\in a(xa)^n zSa(xa)^n z, && \text{for some } n\in Z^+ \text{ (by (2.1))}\\ &= azSaz. \end{aligned}$$

Therefore, $az \in \mathrm{Reg}(S)$, so $\mathrm{Reg}(S)$ is a right ideal of S. In a similar way we show that $\mathrm{Reg}(S)$ is a left ideal of S. Thus $\mathrm{Reg}(S)$ is an ideal of S, so S is a nil-extension of a regular semigroup. □

In [6] the authors considered semilattice of left and right groups and semilattice of nil-extensions of left and right groups. Here, we describe a nil-extension of a semilattice of left and right groups.

THEOREM 2.2. *A semigroup S is a nil-extension of a semilattice of left and right groups if and only if for every $a,b \in S$ and $x,y \in S^1$ there exists $n \in \mathbb{Z}^+$ such that*

$$x(ab)^n y \in x(ab)^n ySx(ba)^n y \cup x(ba)^n ySx(ab)^n y. \tag{2.2}$$

Proof. Let S be a nil-extension of K and let K be a semilattice of left and right groups. Then, it is clear that S is π-regular and that $\mathrm{Reg}(S) = \mathrm{Gr}(S) = K$, so S is a GV-semigroup. By Therems 4.1. and 3.1 [6] we obtain that S is a semilattice Y of semigroups S_α, $\alpha \in Y$, where S_α is a nil-extension of K_α and K_α is a left or a right group.

Let $a \in S_\alpha$, $b \in S_\beta$, $\alpha, \beta \in Y$. Then there exists $n \in \mathbb{Z}^+$ such that $(ab)^n$, $(ba)^n \in K_{\alpha\beta}$. Let $x, y \in S^1$. Assume that $x, y \in S$. Then $x \in S_\gamma$, $y \in S_\delta$ for some $\gamma, \delta \in Y$. Since K is an ideal of S, then $x(ab)^n y, x(ba)^n y \in K \cap S_{\alpha\beta\gamma\delta} = K_{\alpha\beta\gamma\delta}$. Then by Lemma 1. [8] we have that

$$x(ab)^n y \in x(ab)^n y K_{\alpha\beta\gamma\delta}\, x(ba)^n y \subseteq x(ab)^n ySx(ba)^n y,$$

if $K_{\alpha\beta\gamma\delta}$ is a left group, and

$$x(ab)^n y \in x(ba)^n y K_{\alpha\beta\gamma\delta}\, x(ab)^n y \subseteq x(ba)^n ySx(ab)^n y,$$

if $K_{\alpha\beta\gamma\delta}$ is a right group. The similar proof we have if $x=1$ or $y=1$. Thus, (2.2) holds.

Conversely, let (2.2) holds. Let $x, a, y \in S$, then there exists $n \in \mathbb{Z}^+$ such that

$$xa^{2n}y \in xa^{2n}ySxa^{2n}y,$$

whence, by Theorem 2.1., we have that S is a nil-extension of a regular semigroup K. Let $a, b \in K$. Then by (2.2) it follows that there exists $n \in \mathbb{Z}^+$ such that

$$(ab)^n \in (ab)^n S(ba)^n \bigcup (ba)^n S(ab)^n.$$

Moreover, $(ab)^n \in K$ and since K is a regular semigroup, then $(ab)^n = (ab)^n x(ab)^n$ for some $x \in K$. Sine K is an ideal of S then we have that

$$\begin{aligned}(ab)^n &\in (ab)^n S(ab)^n S(ba)^n \bigcup (ba)^n S(ab)^n S(ab)^n \\ &\subseteq (ab)^n K(ba)^n \bigcup (ba)^n K(ab)^n.\end{aligned}$$

Let $a\in S_\alpha$, $b\in S_\beta$. Then there exists $n\in Z^+$ such that $(ab)^n$, $(ba)^n\in K$. Let $x,y\in S^1$. Assume that $x,y\in S$, i.e. $x\in S_\gamma$, $y\in S_\delta$, $\gamma,\delta\in Y$. Since K is an ideal of S, then $x(ab)^n y, x(ba)^n y\in K\bigcap S_{\alpha\beta\gamma\delta}=G_{\alpha\beta\gamma\delta}$, where $G_{\alpha\beta\gamma\delta}$ is a group and it is an ideal of $S_{\alpha\beta\gamma\delta}$ ~~is a group,~~ then by Lemma VI 2.1.1. [1] we have that

$$x(ab)^n y\in x(ba)^n y(x(ba)^n y)^{-1} S_{\alpha\beta\gamma\delta}(x(ba)^n y)^{-1} x(ba)^n y$$

$$\subseteq x(ba)^n ySx(ba)^n y.$$

The similar proof we have if $x=1$ or $y=1$. Thus, (*iii*) holds.

(*iii*)$\Rightarrow$(*i*). Let (*iii*) holds. Let $x,a,y\in S$. Then there exists $n\in Z^+$ such that $xa^{2n}y\in xa^{2n}ySxa^{2n}y$, so by Theorem 2.1. S is a nil-extension of a regular semigroup K. By (*iii*) it follows that K is weakly commutative, so by Theorem 3.2. [4] (or Theorem 2.2. [10]) we have that K is a semilattice Y of π-groups K_α, $\alpha\in Y$. Let K_α be a nil-extension of a group G_α, $\alpha\in Y$. Let $a\in K_\alpha$. Since K is regular, then $a=axa$ for some $x\in K$. It is clear that $ax, xa\in K_\alpha\bigcap E(S)=\{e_\alpha\}$, where e_α is the identity element of the group G_α. Thus $a=ae_\alpha\in K_\alpha G_\alpha\subseteq G_\alpha$, so $K_\alpha=G_\alpha$. Therefore K is a semilattice of groups, so S is a nil-extension of a semilattice of groups. □

COROLLARY 2.2. *A semigroup S is a nil-extension of a group if and only if*

$$(2.3)\qquad (\forall a,b\in S)(\exists n\in Z^+)\ a^n\in b^nSb^n.$$

Proof. Let S be a nil-extension of a group G. Let $a,b\in S$. Then there exists $n\in Z^+$ such that $a^n, b^n\in G$, so

$$a^n=b^n(b^n)^{-1}a^n(b^n)^{-1}b^n\in b^nSb^n.$$

Conversely, let (2.3) holds. Then S is π-regular. Let $e,f\in E(S)$. Then by (2.3) it follows that

$$e=fxf \text{ and } f=eye \text{ for some } x,y\in S,$$

whence

$$e=fxf=fxff=ef=eeye=eye=f.$$

Thus, S contains exactly one idempotent, so S is a nil-extension of a group. □

COROLLARY 2.3. *A semigroup S is a nil-extension of a periodic group if and only if*

$$(\forall a,b\in S)(\exists n\in Z^+)\ a^n=b^n. \qquad \square$$

Let Y be a semilattice, for every $\alpha\in Y$ let S_α be a semigroup and assume that the semigroups S_α are pairwise disjoint. For every pair $\alpha,\beta\in Y$

such that $\alpha \geqslant \beta$, let $\psi_{\alpha,\beta}: S_\alpha \to S_\beta$ be a homomorphism such that $\psi_{\alpha,\alpha}$ is the identity mapping and

$$\psi_{\alpha,\beta}\,\psi_{\beta,\gamma} = \psi_{\alpha,\gamma} \text{ if } \alpha > \beta > \gamma.$$

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ with the multiplication

$$a * b = (a\,\psi_{\alpha,\alpha\beta})\,(b\,\psi_{\beta,\alpha\beta}) \quad (a \in S_\alpha,\ b \in S_\beta).$$

Then S is a semilattice Y of semigroups S. The semigroup S which can be constructed in this way we call the *strong semilattice of semiroups* S_α and we denote it by

$$S = [Y; S_\alpha; \psi_{\alpha,\beta}].$$

A semigroup S is called an *inflation* of a semigroup T if T is a subsemigroup of S, $S^2 \subseteq T$ and there exists a retraction of S onto T.

THEOREM 2.4. *S is an inflation of a semilattice of groups if and only if S is a strong semilattice of inflations of groups.*

Proof. Let S be an inflation of a semilattice of groups. Then by Corollary 3. [2] (see Proposition [16]) it follows that S^2 is a semilattice of groups. Let $e \in E(S)$ and $x \in S$. Then $ex, xe \in S^2$ and by Theorem IV 2.1. [11] we have that

$$ex = e(ex) = (ex)e = e(xe) = (xe)e = xe,$$

so the idempotents from S are central. By Theorem 7.2. [10] it follows that S is a semilattice of π-groups S_α, $\alpha \in Y$. It is clear that S_α is an inflation of a group G_α, $\alpha \in Y$.

For $\alpha \geqslant \beta$ define a function $\psi_{\alpha,\beta}: S_\alpha \to S_\beta$ by:

$$x\,\psi_{\alpha,\beta} = \begin{cases} x, & \alpha = \beta, \\ e_\beta x, & \alpha \neq \beta, \end{cases}$$

where e_β is the identity element of a group G_β. Let $x, y \in S_\alpha$ and let $\alpha \neq \beta$. Then

$$(x\,\psi_{\alpha,\beta})\,(y\,\psi_{\alpha,\beta}) = e_\beta x e_\beta y = e_\beta e_\beta xy = e_\beta xy = (xy)\,\psi_{\alpha,\beta}.$$

Hence, $\psi_{\alpha,\beta}$ is a homomorphism. Let $\alpha > \beta > \gamma$ and $x \in S_\alpha$. Then

$$(x\,\psi_{\alpha,\beta})\,\psi_{\beta,\gamma} = (e_\beta x)\,\psi_{\beta,\gamma} = e_\gamma(e_\beta x) = (e_\gamma e_\beta)\,x = $$
$$= e_\gamma x = x\,\psi_{\alpha,\gamma}$$

so for $\alpha > \beta > \gamma$ it is $\psi_{\alpha,\beta}\,\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$.

Let $x \in S_\alpha$, $y \in S_\beta$. Then $xy \in S_\alpha S_\beta \subseteq G_{\alpha\beta}$, so $xy = e_{\alpha\beta}xy = e_{\alpha\beta}xe_{\alpha\beta}y = (x\,\psi_{\alpha,\alpha\beta})\,(y\,\psi_{\beta,\alpha\beta})$. Hence, S is a strong semilattice Y of inflations of groups S_α, $\alpha \in Y$.

Conversely, let $S=[Y; S_\alpha, \psi_{\alpha,\alpha}]$ be a strong semilattice of inflations of groups. Let S_α be an inflation of a group G_α, $\alpha\in Y$. Let $x\in S_\alpha$, $y\in S_\beta$. Then

$$xy=(x\psi_{\alpha,\alpha\beta})(y\psi_{\alpha,\alpha\beta})\in S_{\alpha\beta}S_{\alpha\beta}\subseteq G_{\alpha\beta}.$$

Hence $S^2=G=\bigcup_{\alpha\in Y} G_\alpha$ is a (strong) semilattice of groups. By Corollary 3. [2] (Proposition [16]) it follows that S is an inflation of a semilattice of groups. □

LEMMA 2.1. *Let S be a strong semilattice of π-groups. Then S is a (retractive) nil-extension of a semilattice of groups.*

Proof. Let $S=[Y; S_\alpha, \mathscr{C}_{\alpha,\beta}]$ be a strong semilattice of π-groups S_α. Let G_α be a group-ideal of S_α, $\alpha\in Y$. Let $a\in G_\alpha$ and $b\in S_\beta$. Then

$$ab=(a\mathscr{C}_{\alpha,\alpha\beta})(b\mathscr{C}_{\beta,\alpha\beta})\in G_{\alpha\beta}$$

since $a\mathscr{C}_{\alpha,\alpha\beta}\in G_{\alpha\beta}$. In a similar way we prove that $ba\in G_{\alpha\beta}$. Hence, Reg(S) is an ideal of S, and, clearly, Reg(S) is a semilattice of groups. Thus, S is a nil-extension of a semilattice of groups. □

REMARK. If S is a strong semilattice of completely archimedean semigroups, then S is a nil-extension of a strong semilattice of completely simple semigroups.

3. Retractive nil-extension of a completely simple semigroup

LEMMA 3.1. *Let S be a nil-extension of a union of groups T. Then every retraction $\mathscr{C}: S\to T$ has the following representation*

$$\mathscr{C}(a)=ea, \quad a\in S,$$

where $a^m\in G_e$ for some $m\in \mathbb{Z}^+$.

Proof. Let $\mathscr{C}: S\to T$ be a retraction and let $a\in S$. Then there exists $m\in \mathbb{Z}^+$ and $e\in E(S)$ such that $a^m\in G_e$, and by Theorem I 4.2. [1] we have that $G_e\cap G_f=\emptyset$ for $e,f\in E(S)$, $e\neq f$. Let $\mathscr{C}(a)\in G_f$ for some $f\in E(S)$. Then

$$a^m=\mathscr{C}(a^m)=(\mathscr{C}(a))^m\in G_f,$$

so $f=e$, i.e. $\mathscr{C}(a)\in G_e$. Now we obtain

$$\mathscr{C}(a)=e\mathscr{C}(a)=\mathscr{C}(e)\mathscr{C}(a)=\mathscr{C}(ea)=ea. \qquad \square$$

THEOREM 3.1. *S is a retractive nil-extension of a completely simple semigroup if and only if S is a completely π-regular semigroup and for every $a,b,c\in S$ there exists $n\in \mathbb{Z}^+$ such that*

(3.1) $$a^nc\in ea^nbSa^ncf \quad \text{and} \quad ca^n\in fca^nbSa^ne,$$

where $a^k\in G_e$ and $c^r\in G_f$ for some $k,r\in \mathbb{Z}^+$.

Proof. Let S be a retractive nil-extension of a completely simple semigroup K with the retraction $\mathscr{C}$. Then by Theorem 1. [7] we have that for every $a, b \in S$ there exists $n \in Z^+$ and $x \in S$ such that $a^n = a^n bxa^n$. Now, for every $c \in S$ we have that

$$a^n c = a^n bxa^n c \quad \text{and} \quad ca^n = ca^n bxa^n,$$

so

$$(a^n c) = \mathscr{C}(a^n bxa^n c) = \mathscr{C}(a^n)\mathscr{C}(bxa^n)\mathscr{C}(c)$$

and

$$(ca^n) = \mathscr{C}(ca^n bxa^n) = \mathscr{C}(c)\mathscr{C}(a^n bx)\mathscr{C}(a^n)$$

Since $a^k \in G_e$, $c^r \in G_f$ for some $k, r \in Z^+$, then by Lemma 3.1. we have that

$$\mathscr{C}(a^n) = ea^n \quad \text{and} \quad \mathscr{C}(c) = fc = cf$$

(Theorem I 4.3. [1]) and since $\mathrm{Reg}(S) = K$ in an ideal of S, then $a^n, a^n c$, ca^n, bxa^n, $a^n bx \in K$, so

$$a^n c = \mathscr{C}(a^n c) = \mathscr{C}(a^n)\mathscr{C}(bxa^n)\mathscr{C}(c) = ea^n bxa^n cf$$

and

$$ca^n = \mathscr{C}(ca^n) = \mathscr{C}(c)\mathscr{C}(a^n bx)\mathscr{C}(a^n) = fca^n bxa^n e.$$

Hence, (3.1) holds.

Conversely, by (3.1) we obtain

$$a^n c = ea^n c = a^n cf \quad \text{and} \quad ca^n = fca^n = ca^n e. \tag{3.2}$$

Let $a, b \in S$. Then $a^{n+1} \in SbS$ for some $n \in Z^+$, so S is an archimedean semigroup. Since S is a completely π-regular, then by Theorem VI 2.2.1. [1] S is a nil-extension of a completely simple semigroup K. Define a function $\mathscr{C}: S \to K$ by

$$\mathscr{C}(a) = ea \quad \text{if} \quad a^k \in G_e \text{ for some } k \in Z^+.$$

Let $a, b \in S$ and $a^k \in G_e$ for some $e, f \in E(S)$ and $k, r \in Z^+$. Let $(ab)^t \in G_g$ for some $g \in E(S)$ and $t \in Z^+$. Then

$$\begin{aligned}
\mathscr{C}(a)\mathscr{C}(b) &= eafb \\
&= e^n abf \qquad \text{(Munn's lemma)} \\
&= e^n abgf \qquad \text{(by (3.2))} \\
&= eabgf \\
&= ea(bg)^{p+1}((bg)^{-1})^p f, \quad \text{for all } p \in Z^+, \\
&= a(bg)^{p+1}((bg)^{-1})^p f, \quad \text{for some } p \in Z^+, \text{ ((3.2))},
\end{aligned}$$

$$
\begin{aligned}
&= abgf \\
&= gabf \qquad \text{(Munn's lemma)} \\
&= ((ga)^{-1})^q (ga)^{q+1} bf, \quad \text{for all } q \in \mathbf{Z}^+, \\
&= ((ga)^{-1})^q (ga)^{q+1} bf, \quad \text{for some } q \in \mathbf{Z}^+, \ ((3.2)), \\
&= gab \\
&= \mathscr{C}(ab),
\end{aligned}
$$

so $\mathscr{C}$ is a homomorphism. It is clear that $\mathscr{C}(\mathscr{C}(x)) = \mathscr{C}(x)$ for all $x \in S$. Hence, S is a retractive nil-extension of a completely simple semigroup. □

THEOREM 3.2. *A semigroup S is a retractive nil-extension of a left group if and only if S is completely π-regular semigroup and for every $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$ca^n \in gca^n S a^n bf \tag{3.3}$$

where $b^r \in G_f$ and $c^t \in G_g$ for some $r, t \in \mathbf{Z}^+$.

Proof. Let S be a retractive nil-extension of a left group K with the retraction $\mathscr{C}$. Let $a, b \in S$. Then $a^s \in K$ for some $s \in \mathbf{Z}^+$ and there exists $p \in \mathbf{Z}^+$ such that

$$a^{sp} = a^{sp} u a^{sp} b \quad \text{for some } u \in S,$$

(Theorem 2. [7]). Now we have that $ca^{sp} = ca^{sp} u a^{sp} b$, so

$$
\begin{aligned}
ca^{sp} = \mathscr{C}(ca^{sp}) &= \mathscr{C}(c)\,\mathscr{C}(a^{sp} u a^{sp})\,\mathscr{C}(b) \\
&= gca^{sp} u a^{sp} fb \quad \text{(by Lemma 3.1. and since } K \text{ is an ideal} \\
&= gca^{sp} u a^{sp} bf \qquad \text{(Munn's lemma)}
\end{aligned}
$$

where $c^t \in G_g$ and $b^r \in G_f$ for some $t, r \in \mathbf{Z}^+$. Hence, (3.3) holds.

Conversely, by (3.3) it follows that $ef = e$ for every $e, f \in \mathrm{E}(S)$, so by Theorem 2. [7] we have that S is a nil-extension of a left group K.

Define a function $\mathscr{C}: S \to K$ by:

$$\mathscr{C}(x) = ex \quad \text{if} \quad x^k \in G_e \text{ for some } k \in \mathbf{Z}^+,\ e \in \mathrm{E}(S).$$

Let $a, b \in S$ and $a^k \in G_e$, $b^r \in G_f$, $(ab)^t \in G_g$ for some $k, r, t \in \mathbf{Z}^+$ and some $e, f, g \in \mathrm{E}(S)$. Then

$$
\begin{aligned}
\mathscr{C}(a)\,\mathscr{C}(b) &= eafb \\
&= aefb \qquad \text{(Munn's lemma)} \\
&= aeb \qquad \text{(since } \mathrm{E}(S) \text{ is a left zero band)}
\end{aligned}
$$

$= eab$ (Munn's lemma)

$= egab$ (since $E(S)$ is a left zero band)

$= eabg$ (Munn's lemma)

$= ea(bg)^{p+1}((bg)^{-1})^p$, for all $p \in Z^+$,

$= a(bg)^{p+1}((bg)^{-1})^p$ (by (3.3) we have that $bca^m = ca^m$ for some $m \in Z^+$)

$= abg$

$= gab$ (Munn's lemma)

$= \mathscr{C}(ab)$.

Hence, $\mathscr{C}$ is a homomorphism and it is clear that $\mathscr{C}(\mathscr{C}(x)) = \mathscr{C}(x)$ for all $x \in S$, so S is a retractive nil-extension of a left group. □

4. A nil-extension of a band

THEOREM 4.1. *A semigroup S is a nil-extension of a band if and only if for every $x, a, y \in S$ there exists $n \in Z^+$ such that*

(4.1) $$xa^n y = (xa^n y)^2.$$

Proof. Let S be a nil-extension of a band E. Let $x, a, y \in S$. Then there exists $n \in Z^+$ such that $a^n \in E$. Since E is an ideal of S, then $xa^n y \in E$, so (4.1) holds.

Conversely, let (4.1) holds. Then for every $a \in S$ there exists $n \in Z^+$ such that $a^{n+2} = a^{2(n+2)}$, so S is periodic. Let $x \in S$ and $e \in E(S)$. Then

$$\begin{aligned} xe &= xe^n e &&\text{, for every } n \in Z^+,\\ &= (xe^n e)^2 &&\text{, for some } n \in Z^+,\\ &= (xe)^2. \end{aligned}$$

In a similar way we prove that $(ex)^2 = ex$. Thus, $E(S)$ is an ideal of S, so S is a nil-extension of a band $E(S)$. □

S is a *singular band* if it is a left zero band or a right zero band. A band S is an *LR-band* if $xy = xyx$ or $xy = yxy$ for all $x, y \in S$ [6].

LEMMA 4.1. [6] *S is a rectangular LR-band if and only if S is a singular band.* □

LEMMA 4.2. [6] *S is an LR-band if and only if S is a semilattice of singular bands.* □

THEOREM 4.2. *A semigroup S is a nil-extension of an LR-band (semilattice of singular bands) if and only if for every $a, b \in S$ and $x, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that*

$$x(ab)^n y = x(ab)^n yxbay \quad \text{or} \quad xy(ab)^n y = xbayx(ab)^n y. \tag{4.2}$$

Proof. Let S be a nil-extension of an LR-band E. Then S is π-regular and $\mathrm{Reg}(S) = \mathrm{E}(E)$, so by Theorem 6. [5] we have that S is a semilattice Y of nil-extensions S_α of rectangular bands E_α, $\alpha \in Y$. Since E is an LR-band, then by Lemma 4.1. it follows that E_α is a singular band for every $\alpha \in Y$. Let $a \in S_\alpha$, $b \in S_\beta$, $x \in S_\gamma$, $y \in S_\delta$, for $\alpha, \beta, \gamma, \delta \in Y$. Then there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in E$ and since E is an ideal of S then

$$x(ab)^n y, x(ab)^n yxbay, xbayx(ab)^n y \in E \cap S_{\alpha\beta\gamma\delta} = E_{\alpha\beta\gamma\delta}.$$

By Lemma VI 2.2.1. [1] we have that

$$x(ab)^n y = x(ab)^n y(xbay)^2 x(ab)^n y,$$

and since $E_{\alpha\beta\gamma\delta}$ is a singular band, then we obtain (4.2). The similar proof we have if $x=1$ or $y=1$.

Conversely, let (4.2) holds. Then for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^{2n} = a^{2n+2}$, so S is periodic. Let $x \in S$ and $e \in \mathrm{E}(S)$. Then

$$\begin{aligned} xe &= (ee)^n && \text{, for every } n \in \mathbf{Z}^+, \\ &= x(ee)^n xee && \text{, for some } n \in \mathbf{Z}^+ \text{ (by (4.2))}, \\ &= (xe)^2 \end{aligned}$$

or

$$\begin{aligned} xe &= x(ee)^n && \text{, for every } n \in \mathbf{Z}^+, \\ &= xeex(ee)^n && \text{, for some } n \in \mathbf{Z}^+ \text{ (by (4.2))}, \\ &= (xe)^2. \end{aligned}$$

Hence, $xe = (xe)^2$, and, in a similar way, we obtain that $(ex)^2 = ex$. Thus $\mathrm{E}(S)$ is an ideal of S.

Let $e, f \in \mathrm{E}(S)$. Then

$$\begin{aligned} ef &= (ef)^n && \text{, for every } n \in \mathbf{Z}^+, \\ &= (ef)^n fe && \text{, for some } n \in \mathbf{Z}^+ \text{ (by (4.2))}, \\ &= efe \end{aligned}$$

or

$$\begin{aligned} ef &= (ef)^n && \text{, for every } n \in \mathbf{Z}^+, \\ &= fe(ef)^n && \text{, for some } n \in \mathbf{Z}^+ \text{ (by (4.2))}, \\ &= fef. \end{aligned}$$

Hence, $E(S)$ is an LR-band, so S is a nil-extension of an LR-band. □

COROLLARY 4.1. *A semigroup S is a nil-extension of a semilattice of left zero bands if and only if for every $a, b \in S$ and $x, y \in S^1$ there exists $n \in Z^+$ such that*

$$x(ab)^n y = x(ab)^n yxbay. \quad \square$$

A semigroup S is an *ordinal summ Y of semigroups S_α, $\alpha \in y$* if Y is a chain and for any $a \in S_\alpha$, $b \in S_\beta$, $\alpha > \beta$ implies that $ab = ba = b$.

A semigroup S is an *R-band* (*Rédei's band*) if $xy \in \{x, y\}$ for all $x, y \in S$. A semigroup S is a *WR-band* (*weakened Rédei's band*) if $xy \in \{x, y\}$ or $yx \in \{x, y\}$ for all $x, y \in S$ [6]. A semigroup S is a *LWR-band* (*left weakened Rédei's band*) if $xy = x$ or $yx = y$ for all $x, y \in S$. Analogously we define an *RWR-band* (*right weakened Rédei's band*) [6].

The following Lemmas characterize WR-, R- and LWR-bands.

LEMMA 4.3. [6] *S is a rectangular WR-band if and only is S is a singular band.* □

LEMMA 4.4. [6] *S is a WR-band if and only is S is a chain of singular bands.* □

LEMMA 4.5. [6] *S is a R-band if and only is S is a ordinal sum of singular bands.* □

LEMMA 4.6. [6] *S is a LWR-band if and only is S is a chain of left zero bands.* □

LEMMA 4.7. [6] *A semigroup S is a chain of rectangular bands if and only if*

$$x = xyx \quad or \quad y = yxy$$

for all $x, y \in S$. □

THEOREM 4.3. *A semigroup S is a nil-extension of a chain of rectangular bands if and only if for every $a, b \in S$ and $x, y \in S^1$ there exists $n \in Z^+$ such that*

$$xa^n y = (xa^n ba^n y)^2 \quad or \quad xb^n y = (xb^n ab^n y)^2. \tag{4.3}$$

Proof. Let S be a nil-extension of a chain of rectangular bands E. Then S is π-regular and $\mathrm{Reg}(S) = E(S)$, so by Theorem 6. [5] it follows that S is a semilattice Y of nil-extensions S_α of rectangular bands E_α, $\alpha \in Y$. By Lemma 4.7. we have that Y is a chain. Let $a \in S_\alpha$, $b \in S_\beta$, and let $x, y \in S$. Then there exists $n \in Z^+$ such that $a^n, b^n \in E$. Assume that $\alpha \leqslant \beta$. Then $a^n b \in S$, so by Lemma VI 2.2.1. [1] we have that $a^n ba^n = a^n ba^n a^n = a^n$, whence

$$xa^n y = (xa^n y)^2 = (xa^n ba^n y)^2.$$

The similar proof we have if $\beta \leqslant \alpha$. Thus, (4.3) holds.

Conversely, let (4.3) holds. Then for every $a\in S$ there exists $n\in \mathbf{Z}^+$ such that $a^n = a^{2(2n+1)}$, so S is periodic. Let $x\in S$ and $e\in \mathrm{E}(S)$. Then

$$\begin{aligned} xe &= xe^n && \text{, for every } n\in \mathbf{Z}^+,\\ &= (xe^n ee^n)^2 && \text{, for some } n\in \mathbf{Z}^+ \text{ (by (4.3))},\\ &= (xe)^2 && , \end{aligned}$$

and similarly $ex = (ex)^2$. Thus, $\mathrm{E}(S)$ is an ideal of S. Let $e,f\in \mathrm{E}(S)$. Then there exists $n\in \mathbf{Z}^+$ such that

$$e = e^n e^n f e^n = efe \quad \text{or} \quad f = f^n = f^n e f^n = fef,$$

so by Lemma 4.7. $\mathrm{E}(S)$ is a chain of rectangular bands. Therefore, S is a il-extension of a chain of rectangular bands. □

THEOREM 4.4. *A semigroup S is a nil-extension of a WR-band if and only if for every $a,b\in S$ and $x,y\in S^1$ there exists $n\in \mathbf{Z}^+$ such that*

(4.4) $$xa^n b^n y = (xa^n y)^2 \quad or \quad xa^n b^n y = (xb^n y)^2 \quad or$$
$$xb^n a^n y = (xa^n y)^2 \quad or \quad xb^n a^n y = (xb^n y)^2.$$

Proof. Let S be a nil-extension of a WR-band E. Let $a,b\in S$ and $x,y\in S^1$. Then there exists $n\in \mathbf{Z}^+$ such that $a^n, b^n\in E$, so

$$a^n b^n = a^n \quad or \quad a^n b^n = b^n \quad or \quad b^n a^n = a^n \quad or \quad b^n a^n = b^n.$$

Since E is an ideal of S, then it follows (4.4).

Conversely, let (4.4) holds. Then for every $a\in S$ there exists $n\in \mathbf{Z}^+$ such that $a^{2n+1} = a^{2n+2}$, so S is periodic. Let $x\in S$ and $e\in \mathrm{E}(S)$. Then

$$\begin{aligned} xe &= xe^n e^n && \text{, for every } n\in \mathbf{Z}^+,\\ &= (xe^n)^2 && \text{, for some } n\in \mathbf{Z}^+ \text{ (by (4.4))},\\ &= (xe)^2 && , \end{aligned}$$

and similarly $ex = (ex)^2$. Thus, $\mathrm{E}(S)$ is an ideal of S. By (4.4) it follows that $\mathrm{E}(S)$ is a WR-band, so S is a nil-extension of a WR-band. □

COROLLARY 4.2. *A semigroup S is a nil-extension of an LWR-band (chain of left zero bands) if and only if for every $a,b\in S$ and $x,y\in S^1$ there exists $n\in \mathbf{Z}^+$ such that*

$$xa^n b^n y = (xa^n y)^2 \quad or \quad xb^n a^n y = (xb^n y)^2. \qquad \square$$

COROLLARY 4.3. *A semigroup S is a nil-extension of an R-band if and only if for every $a,b\in S$ and $x,y\in S^1$ there exists $n\in \mathbf{Z}^+$ such that*

$$xa^n b^n y = (xa^n y)^2 \quad or \quad xa^n b^n y = (xb^n y)^2. \qquad \square$$

5. A retractive nil-extension of a band

THEOREM 5.1. *A semigroup* S *is a retractive nil-extension of a band if and only if for every* $a, b\in S$ *and* $x, y\in S^1$ *there exists* $n\in Z^+$ *such that*

$$x(ab)^{n+k}y=(xa^{n+k}b^{n+k}y)^2 \tag{5.1}$$

for every $k\in Z^+$.

Proof. Let S be a retractive nil-extension of a band E with the retraction $\mathscr{C}$. Then $\mathrm{Reg}(S)=\mathrm{E}(S)$, so S is a union of nil-semigroups.

Let $a, b\in S$ and $x, y\in S^1$. Then there exists $n\in Z^+$ such that $(ab)^n, a^n, b^n\in E$. By Lemma 3.1. we have that $\mathscr{C}$ has the representation

$$\mathscr{C}(z)=e \quad \text{if} \quad z\in K_e$$

Let $a^n=e$, $b^n=f$, $e,f\in E$. Then $\mathscr{C}(a)=e$ and $\mathscr{C}(b)=f$, whence

$$(ab)^n=\mathscr{C}((ab)^n)=(\mathscr{C}(ab))^n=\mathscr{C}(ab)=\mathscr{C}(a)\mathscr{C}(b)=ef=a^nb^n.$$

Since $(ab)^{n+k}=(ab)^n$, $a^{n+k}=a^n$ and $b^{n+k}=b^n$ for every $k\in Z^+$, then

$$(ab)^{n+k}=a^{n+k}b^{n+k},$$

and since E is an ideal of S, then it follows (5.1).

Conversely, let (5.1) holds. Then for every $x, a, y\in S$ there exists $n\in Z^+$ such that

$$xa^{2n}y=(xa^{2n}y)^2,$$

so by Theorem 4.1. S is a nil-extension of a band $E=\mathrm{E}(S)$. Define a function $\mathscr{C}: S\to E$ by

$$\mathscr{C}(x)=e \quad \text{if} \quad x\in K_e,\ e\in E.$$

Let $a, b\in S$. Then there exists $n\in Z^+$ such that

$$(ab)^{n+k}=(a^{n+k}b^{n+k})^2$$

for every $k\in Z^+$, i.e.

$$(ab)^p=(a^pb^p)^2$$

for every $p\in Z^+$, $p\geqslant n+1$. Moreover, there exists $m\in Z^+$, $m\geqslant 2$ such that $(ab)^m$, $a^m, b^m\in E$, whence

$$\mathscr{C}(ab)=(ab)^m=(ab)^{nm}=(a^{nm}b^{nm})^2=(a^mm^m)^2=a^mb^m=\mathscr{C}(a)\mathscr{C}(b),$$

since $nm\geqslant n+1$. Thus $\mathscr{C}$ is a retraction, so S is a retractive nil-extension of a band. □

THEOREM 5.2. *A semigroup S is a retractive nil-extension of a rectangular band if and only if for every $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(ac)^{n+k} = a^{n+k} b c^{n+k} \tag{5.2}$$

for every $k \in \mathbf{Z}^+$.

Proof. Let S be a retractive nil-extension of a rectangular band E with the retraction $\mathscr{C}$. Then S is periodic and $\mathrm{Reg}(S) = E = \mathrm{E}(S)$, so S is a union of nilsemigroups and by Lemma 3.1. we have that $\mathscr{C}$ has the representation

$$\mathscr{C}(x) = e \quad \text{if} \quad x \in K_e,\ e \in E.$$

Let $a, b, c \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $(ac)^n, a^n, c^n \in E$ and $(ac)^{n+k} = (ac)^n$, $a^{n+k} = a^n$ and $c^{n+k} = c^n$ for every $k \in \mathbf{Z}^+$. Now we have that

$$\begin{aligned} a^{n+k} b c^{n+k} &= \mathscr{C}(a^{n+k} b c^{n+k}) && \text{, since } E \text{ is an ideal,} \\ &= (\mathscr{C}(a))^{n+k} \mathscr{C}(b) (\mathscr{C}(c))^{n+k} \\ &= \mathscr{C}(a) \mathscr{C}(b) \mathscr{C}(c) \\ &= \mathscr{C}(a) \mathscr{C}(c) && \text{, by Proposition 3.1. [11],} \\ &= \mathscr{C}(ac) = (ac)^n = (ac)^{n+k}. \end{aligned}$$

Thus (5.2) holds.

Conversely, let (5.2) holds. Then for every $a \in S$ there exists $m \in \mathbf{Z}^+$ such that $a^{2m} = a^{2m+1}$, so S is periodic. Let $x \in S$ and $e \in \mathrm{E}(S)$. Then

$$\begin{aligned} exe &= e^m x e^m && \text{, for every } m \in \mathbf{Z}^+, \\ &= (ee)^m && \text{, for some } m \in \mathbf{Z}^+ \text{ (by (5.2)),} \\ &= e. \end{aligned}$$

Thus, $exe = e$ for every $x \in S$ and $e \in \mathrm{E}(S)$, whence $\mathrm{E}(S)$ is a rectangular band and $\mathrm{E}(S)$ is an ideal of S.

Define a function $\mathscr{C} : S \to \mathrm{E}(S)$ by

$$\mathscr{C}(x) = e \quad \text{if} \quad x \in K_e,\ e \in \mathrm{E}(S).$$

Let $a, b \in S$. Then there exists $m \in \mathbf{Z}^+$, $m \geqslant 2$, such that $(ab)^m, a^m, b^m \in \mathrm{E}(S)$ and there exists $n \in \mathbf{Z}^+$ such that

$$(ab)^{n+k} = a^{n+k} a b^{n+k}$$

for every $k \in \mathbf{Z}^+$, i.e.

$$(ab)^p = a^{p+1} b^p$$

for every $p \in \mathbf{Z}^+$, $p \geq n+1$. Now we have that

$$\mathscr{C}(ab) = (ab)^m = (ab)^{nm} = a^{nm+1}b^{nm} = a^m b^m = \mathscr{C}(a)\mathscr{C}(b),$$

since $nm \geq n+1$. Thus $\mathscr{C}$ is a retraction, so S is a retractive nil-extension of a rectangular band. □

THEOREM 5.3. *A semigroup S is a retractive nil-extension of a left zero band if and only if for every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(ab)^n = a^{2n+1}. \tag{5.3}$$

Proof. Let S be a retractive nil-extension of a left zero band E with the retraction $\mathscr{C}$. Then S is periodic, $\mathrm{Reg}(S) = E = \mathrm{E}(S)$, so S is a union of nil-semigroups and by Lemma 3.1. it follows that $\mathscr{C}$ has the representation

$$\mathscr{C}(x) = e \quad \text{if} \quad x \in K_e,\ e \in E.$$

Let $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $(ab)^n n\ a^b$, $b^b \in E$, so

$$(ab)^n = \mathscr{C}((ab)^n) = (\mathscr{C}(ab))^n = \mathscr{C}(ab) = \mathscr{C}(a)\mathscr{C}(b) =$$
$$= \mathscr{C}(a) = a^n = a^{2n+1},$$

since E is a left zero band. Thus (5.3) holds.

Conversely, let (5.3) holds. Then for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^{2n} = a^{2n+1}$, so S is periodic. Let $x \in S$ and $e \in \mathrm{E}(S)$. Then there exists $n \in \mathbf{Z}^+$ such that

$$(ex)^n = e^{2n+1} = e.$$

Moreover, there exists $m \in \mathbf{Z}^+$ such that

$$(ex)^{2m} = (ex)^{2m+1},$$

whence $(ex)^{2m} \in \mathrm{E}(S)$ and $(ex)^p = (ex)^{2m}$ for every $p \in \mathbf{Z}^+$, $p \geq 2m$. Now we obtain that

$$(ex)^{2mn} = e^{2n} = e$$

and

$$(ex)^{2mn+1} = (ex)^{2mn}ex = eex = ex.$$

So $ex = (ex)^{2m} \in \mathrm{E}(S)$, and by this it follows that $ex = e$. Now $xe = xee = x(ex)e = (xe)^2$. Thus, $\mathrm{E}(S)$ is an ideal of S and $\mathrm{E}(S)$ is a left zero band.

Define a function $\mathscr{C}: S \to \mathrm{E}(S)$ by

$$\mathscr{C}(x) = e \quad \text{if} \quad x \in K_e,\ e \in \mathrm{E}(S).$$

Let $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $(ab)^n$, $a^n \in \mathrm{E}(S)$ and there exists $m \in \mathbf{Z}^+$ such that

$$(ab)^m = a^{2m+1}.$$

Then

$$\mathscr{C}(ab) = (ab)^n = (ab)^{mn} = a^{n(2m+1)} = a^n = \mathscr{C}(a) = \mathscr{C}(a)\,\mathscr{C}(b).$$

Therefore, S is a retractive nil-extension of a left zero band. □

THEOREM 5.4. *A semigroup S is a retractive nil-extension of an LR-band (semilattice of singular bands) if and only if for every $a, b \in S$ and $x, y \in S^1$ there exists $n \in \mathbf{Z}^+$ such that*

$$\text{(5.4)}\qquad \begin{aligned} x(ab)^{n+k}y &= xa^{n+k}b^{n+k}yxb^{n+k}a^{n+k}y \\ x(ab)^{n+k}y &= xb^{n+k}a^{n+k}yxa^{n+k}b^{n+k}y \end{aligned} \qquad \text{or}$$

for every $k \in \mathbf{Z}^+$.

Proof. Let S be a retractive nil-extension of an LR-band E with the retraction $\mathscr{C}$. Then S is periodic and $\mathrm{Reg}(S) = E = \mathrm{E}(S)$, so S is a union of nil-semigroups and by Lemma 3.1. we have that $\mathscr{C}$ has the representation [illegible] that

$$x(ab)^m y = a(ab)^m yxbay \qquad \text{or} \qquad x(ab)^m y = xbayx(ab)^m y.$$

Moreover, there exists $n \in \mathbf{Z}^+$ such that $(ab)^n, a^n, b^n \in E$, and then $\mathscr{C}(a) = a^n$, $\mathscr{C}(b) = b^n$ and $\mathscr{C}(ab) = (ab)^n = \mathscr{C}(a)\,\mathscr{C}(b) = a^n b^n$. Let $x(ab)^m y = x(ab)^m yxbay$. Then

$$x(ab)^m y = x(ab)^m y(xbay)^s$$

for every $s \in \mathbf{Z}^+$. Since there exists $p \in \mathbf{Z}^+$ such that $(xbay)^p \in E$, then $x(ab)^m y = x(ab)^m y(xbay)^p \in E$, so

$$\begin{aligned} x(ab)^m y &= \mathscr{C}(x(ab)^m y) = \mathscr{C}(x(ab)^m y(xbay)^p) = \\ &= \mathscr{C}(x)(\mathscr{C}(ab))^m \mathscr{C}(y)(\mathscr{C}(x)\mathscr{C}(b)\mathscr{C}(a)\mathscr{C}(y))^p = \\ &= \mathscr{C}(x)\mathscr{C}(ab)\mathscr{C}(y)\mathscr{C}(x)\mathscr{C}(b)\mathscr{C}(a)\mathscr{C}(y) = \\ &= \mathscr{C}(x)a^n b^n \mathscr{C}(y)\mathscr{C}(x)b^n a^n \mathscr{C}(y) = \\ &= \mathscr{C}(xa^n b^n yxb^n a^n y) = xa^n b^n yxb^n a^n y \end{aligned}$$

and

$$\begin{aligned} x(ab)^m y &= \mathscr{C}(x(ab)^m y) = \mathscr{C}(x)(\mathscr{C}(ab))^m \mathscr{C}(y) = \\ &= \mathscr{C}(x)\mathscr{C}(ab)\mathscr{C}(y) = \mathscr{C}(x)(ab)^n \mathscr{C}(y) = \\ &= \mathscr{C}((ab)^n y) = x(ab)^n y. \end{aligned}$$

Hence $x(ab)^n y = xa^n b^n yxb^n a^n y$, and since $(ab)^{n+k} = (ab)^n$, $a^{n+k} = a^n$, $b^{n+k} = b^n$ for every $k \in \mathbf{Z}^+$, then the fist identity from (5.4) holds. In a similar way we prove that, if $x(ab)^m y = xbayx(ab)^m y$ the second identity from (5.4) holds. Thus, in any cases (5.4) holds.

Conversely, let (5.4) holds. Then for every $x, a, y \in S$ there exists $m \in \mathbf{Z}^+$ such that $xa^{2m}y = (xa^{2m}y)^2$, so by Theorem 4.1. we have that S is a nil-extension of a band E. Let $e, f \in E$. Then by (5.4) it follows that $ef = efe$ or $ef = fef$, so E is an LR-band.

Define a function $\mathscr{C}: S \to E$ by

$$\mathscr{C}(x) = e \quad \text{if} \quad x \in K_e,\ e \in E.$$

Let $a, b \in S$. Then there exists $m \in \mathbf{Z}^+$, $m \geqslant 2$ such that $(ab)^m, a^m, b^m \in E$, and $\mathscr{C}(ab) = (ab)^m$, $\mathscr{C}(a) = a^m$, $\mathscr{C}(b) = b^m$. Moreover, by (5.4) it follows that there exists $n \in \mathbf{Z}^+$ such that

$$(ab)^{n+k} = a^{n+k} b^{n+k} a^{n+k} \quad \text{or} \quad (ab)^{n+k} = b^{n+k} a^{n+k} b^{n+k}$$

for every $k \in \mathbf{Z}^+$, i.e.

$$(ab)^p = a^p b^p a^p \quad \text{or} \quad (ab)^p = b^p a^p b^p$$

for every $p \in \mathbf{Z}^+$, $p \geqslant n+1$. Now we have that

$$\mathscr{C}(ab) = (ab)^m = (ab)^{mn} = a^{mn} b^{mn} a^{mn} = a^m b^m a^m$$

or

$$\mathscr{C}(ab) = (ab)^m = (ab)^{mn} = b^{mn} a^{mn} b^{mn} = b^m a^m b^m.$$

Let $a^m = e$, $b^m = f$, $e, f \in E(S)$. Assume that $ef = efe$. Let $(ab)^m = \mathscr{C}(ab) = b^m a^m b^m = fef$. By $ef = efe$ it follows that $fef = (fe)^2 = fe$. Hence, $(ab)^m = fe$. In a similar way we prove that $af = ef$ or $af = fe$ (since $af \in E$, then $(af)^k = af$ for every $k \in \mathbf{Z}^+$). Assume that $af = fe$. Then

$$ef = a^m f = a^{m-1} af = a^{m-1} fe = a^{m-2} afe = a^{m-2} fee = a^{m-2} fe =$$
$$= \ldots = afe = fee = fe = af.$$

Hence, $af = ef$. Similarly we show that $be = fe$. Now we have that $abe = afe = efe = ef$ and

$$fe = fee = (ab)^m e = (ab)^{m-1} abe = (ab)^{m-1} ef = (ab)^{m-2} abef =$$
$$= (ab)^{m-2} eff = (ab)^{m-2} ef = \ldots = abef = ef,$$

whence $(ab)^m = ef$, so

$$\mathscr{C}(ab) = (ab)^m = ef = \mathscr{C}(a)\mathscr{C}(b).$$

The similar proof we have in the other cases. Therefore $\mathscr{C}$ is a retraction, so S is a retractive nil-extension of an LR-band. □

COROLLARY 5.1. *A semigroup S is a retractive nil-extension of a semilattice of left zero bands if and only if for every $a,b\in S$ and $x,y\in S^1$ there exists $n\in \mathbf{Z}^+$ such that*

$$x(ab)^{n+k}y=xa^{n+k}b^{n+k}yxb^{n+k}a^{n+k}y$$

for every $k\in \mathbf{Z}^+$. □

THEOREM 5.5. *A semigroup S is a retractive nil-extension of an R-band if and only if for every $a,b\in S$ and $x,y\in S^1$ there exists $n\in \mathbf{Z}^+$ such that*

$$(5.5)\qquad x(ab)^n y=(xa^n y)^2 \quad or \quad x(ab)^n y=(xb^n y)^2.$$

Proof. Let S be a retractive nil-extension of an R-band E with the retraction $\mathscr{C}$. Then S is periodic and $\mathrm{Reg}(S)=E=\mathrm{E}(S)$, so S is a union of nil-semigroups and by Lemma 3.1. $\mathscr{C}$ has the representation

$$\mathscr{C}(x)=e \quad \text{if} \quad x\in K_e,\ e\in E.$$

Let $a,b\in S$ and $x,y\in S^1$. Then there exists $n\in \mathbf{Z}^+$ such that $a^n, b^n, (ab)^n\in E$, so

$$(ab)^n=\mathscr{C}((ab)^n)=(\mathscr{C}(ab))^n=\mathscr{C}(ab)=\mathscr{C}(a)\mathscr{C}(b)=a^nb^n,$$

and since E is an R-band and an ideal of S, then it follows (5.5).

Conversely, let (5.5) holds. Then for every $a\in S$ there exists $n\in \mathbf{Z}^+$ such that $a^{2n+1}=a^{2n+2}$, so S is periodic. Let $x\in S$ and $e\in \mathrm{E}(S)$. Then

$$\begin{aligned} xe &= x(ee)^n && \text{, for every } n\in \mathbf{Z}^+,\\ &=(xe^n)^2 && \text{, for some } n\in \mathbf{Z}^+ \text{ (by (5.5))},\\ &=(xe)^2, \end{aligned}$$

and in a similar way we obtain that $(ex)^2=ex$. Thus $\mathrm{E}(S)$ is an ideal of S. Let $e,f\in \mathrm{E}(S)$. Then $(ef)^n=ef$ for every $n\in \mathbf{Z}^+$, so by (5.5) it follows that $\mathrm{E}(S)$ is an R-band.

Define a function $\mathscr{C}: S\to \mathrm{E}(S)$ by

$$\mathscr{C}(x)=e \qquad \text{if} \qquad x\in K_e,\ e\in \mathrm{E}(S).$$

Let $a,b\in S$. Then there exists $m\in \mathbf{Z}^+$ such that $a^m, b^m, (ab)^m\in \mathrm{E}(S)$. Let $a^m=e$, $b^m=f$, $e,f\in \mathrm{E}(S)$. By (5.5) it follows that there exists $n\in \mathbf{Z}^+$ such that $(ab)^n=a^{2n}$ or $(ab)^n=b^{2n}$.

Assume that $ef=e$. Let $(ab)^n=b^{2n}$. Then $(ab)^{mn}=b^{2mn}=f$. Moreover

$$\begin{aligned} af &=(af)^k && \text{, for every } k\in \mathbf{Z}^+,\\ &=a^{2k} && \text{, for some } k\in \mathbf{Z}^+ \text{ (by (5.5))},\\ &=e && \text{(since } af\in \mathrm{E}(S)) \end{aligned}$$

or

$$\begin{aligned} af &= (af)^k &&, \text{ for every } k\in Z^+, \\ &= f^{2k} &&, \text{ for some } k\in Z^+ \text{ (by (5.5))}, \\ &= f &&. \end{aligned}$$

Assume that $af=f$. Then $a^t f=f$ for every $t\in Z^+$, so $e=ef=f$. Hence $af=e$. In a similar way we obtain that $be\in\{e,f\}$. Let $be=e$. Then $abe=ae=e$ so $(ab)^p e=e$ for every $p\in Z^+$. Now we have that

$$\begin{aligned} (ab)^{mn} &= f = ff = (ab)^{mn} f = (ab)^{mn-1} abf = (ab)^{mn-1} af = \\ &= (ab)^{mn-1} e = e. \end{aligned}$$

Let $be=f$. Then $abe=af=e$ so $(ab)^p e$ for every $p\in Z^+$, and similarly we obtain that $(ab)^{mn}=e$. Thus,

$$(ab)^{mn}=e=ef.$$

The similar proof we have if $ef=f$. Therefore

$$\mathscr{C}(ab)=(ab)^m=(ab)^{mn}=ef\mathscr{C}(a)\mathscr{C}(b).$$

Thus, $\mathscr{C}$ is a retraction, so S is a retractive nil-extension of an R-band $E(S)$. □

REFERENCES

[1] *S. Bogdanović*, Semigroups with a system of subsemigroups, Inst. of Math. Novi Sad 1985.

[2] *S. Bogdanović*, Inflation of a union of groups, Mat. vesnik 37 (1985), 351–355.

[3] *S. Bogdanović*, Nil-extensions of a completely regular semigroup, Proc. of the conference "Algebra and Logic", Sarajevo 1987, Univ. of Novi Sad 1988.

[4] *S. Bogdanović*, Semigeroups of Galbiati-Veronesi, Proc. of the conference "Algebra and Logic", Zagreb 1984, Univ. of Novi Sad 1985, 3–20.

[5] *S. Bogdanović*, Semigeroups of Galbiati-Veronesi II, Facta Universitatis (Niš), Ser. Math. Inform. 2 (1987), 61–66.

[6] *S. Bogdanović*and *M. Ćirić*, , Semigeroups of Galbiati-Veronesi III, Facta Universitatis (Niš) ~~(to appear)~~ 4 (1989), 1–14.

[7] *S. Bogdanović* and *S. Milić*, A nil-extension of a completely simple semigroup, Publ. Inst. Math. 36 (50), 1984, 45–50.

[8] *S. Bogdanović* and *B. Stamenković*, Semigroups in which S^{n+1} is a semilattice of right groups, Note di Matematica ~~(to appear)~~ 8 (1988), 155–172.

[9] *J. L Galbiati* e *M. L Veronesi*, Sui semigruppi che sono un band di t-semigruppi, Instituto Lombardo (Rend. Sc.) A114 (1980), 217–234.

[10] *J. L Galbiati* e *M. L Veronesi*, Sui semigruppi quasi regolari, Instituto Lombardo (Rend. Sc.) 116, (1982).

[11] *J. M. Howie*, An introduction to Semigroup Theory, Academic Press, New York, 1976.

[12] *W. D. Munn*, Pseudoinverses in semigroups, Proc. Camb. Phil. Soc., 57 (1961), 247–250.
[13] *M. Petrich*, Introduction to semigroups, Merill Publ. Comp. Ohio 1973.
[14] *M. Petrich*, The maximal semilattice decomposition of a semigroup, Math. Zeitschr. 85 (1964), 68–82.
[15] *L Rédei*, Algebra I, Pergomon Press, Oxford, 1967, pp. 81.
[16] *T. Tamura*, Semigroups satisfying identity $xy=f(x,y)$, Pacific J. of Math. 31 (1969), 513–521.

(Received May 22, 1989)

MATEMATIČKI INSTITUT
11000 *Beograd*
Knez Mihailova 35
Yugoslavia

NIL-PROŠIRENJE REGULARNE POLUGRUPE

S. Bogdanović i *M. Ćirić*, Beograd

Sadržaj

U članku su dane neke karakterizacije nil-proširenja regularnih polugrupa. Nadalje, karakteriziraju se retraktivna nil-proširenja nekih specijalnih tipova regularnih polugrupa.