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CHAINS OF ARCHIMEDEAN SEMIGROUPS (SEMIPRIMARY SEMIGROUPS)

STOJAN BOGDANOVIC AND MIROSLAV CIRIC

Institute of Mathematics, Knez Mihailova 35, *Beograd, Yogoslavia*

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We describe chains of Archimedean (right Archimedean, t-Archimedean) semigroups by radicals of ideals (right ideals, bi-ideals) and we prove that these semigroups are exactly semiprimary semigroups.

1. INTRODUCTION AND PRELIMINARIES

By $\mathbf{Z}^+$ we denote the set of all positive integers. If $a, b \in S$, then $a \mid b$ iff $b = xay$ for some $x, y \in S^{-1}$, $a \mid_r b$ iff $xa = b$ for some $x \in S^{-1}$, $a \mid_t b$ iff $xa = b$ for some $x \in S^1$, $a \mid_t b$ iff $a \mid_r b$ and $a \mid_l b$, $a \to b$ iff $a \mid b^i$ for some $i \in \mathbf{Z}^+$ and $a \xrightarrow{h} b$ iff $a \mid_h b^i$ for some $i \in \mathbf{Z}^+$, where h is r, l or t. A semigroup S is Archimedean (right Archimedean), t-Archimedean iff for all $a, b \in S$, $a \to b$ ($a \xrightarrow{r} b$, $a \xrightarrow{t} b$). By the radical of the subset A of a semigroup S we mean the set $\sqrt{A}$ defined by

$$\sqrt{A} = \{ x \in S \mid (\exists n \in \mathbf{Z}^+)\ x^n \in A \}.$$

A subset A of a semigroup S is completely prime iff

$$(\forall a, b \in S)\ ab \in A \Rightarrow a \in A \ \vee b \in A.$$

A subsemigroup B of a semigroup S is a bi-ideal of S if $BSB \subseteq B$.

For undefined notions and notations we refer to Gilmer and Mott[7] and Satyanarayana[12].

Semilattices of Archimedean semigroups well known as Putcha semigroups, were first completely described by Putcha[11] and afterwards by the authors[4, 6]. In this paper we consider chains of Archimedean semigroups and their relationship with semiprimary semigroups.

In our investigations the following result will be very useful

Theorem A[6] — The following conditions on a semigroup S are equivalent :

(i) S is a semilattice of Archimedian semigroups;

(ii) $(\forall a,\ b \in S)\ a^2 \twoheadrightarrow ab$;

(iii) $\sqrt{A}$ if an ideal, for every ideal A of S.

Satyanarayana[12, 13] studied primary semigroups and rings. For the related results see also Aubert[1], Gilmer and Mott[7], Lal[8] and Zariski and Samuel[14]. Bogdanović[2] introduced the following notion : A subset A of a semigroup S is semiprimary iff

$$(\forall x,\ y \in S)\ (\exists n \in \mathbb{Z}^+)\ xy \in A\ \Rightarrow\ x \in A \vee y^n \in A.$$

S is a *semiprimary semigroup* if all of its ideals are semiprimary. A simple characterization of semiprimary semigroups is given by the following

Lemma A[2] — A semigroup S is semiprimary if and only if

$$(\forall a,\ b \in S)\ ab \twoheadrightarrow a\ \vee\ ab \twoheadrightarrow b. \square$$

2. The Main Results

In this paper we prove that semiprimary semigroups are exactly chains of Archimedean semigroups.

Theorem 1 — The following conditions on a semigroup S are equivalent :

(i) S is a chain of Archimedean semigroups;

(ii) S is semiprimary;

(iii) $\sqrt{A}$ is a completely prime ideal, for every ideal A of S;

(iv) $\sqrt{A}$ is a completely prime subset of S, for every ideal A of S;

(v) S is a semilattice of Archimedean semigroups and completely prime ideals of S are totally ordered.

PROOF : (i) $\Rightarrow$ (ii). Let S be a chain Y of Archimedean semigroups S_α, $\alpha \in Y$. Let $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$. Then x, $xy \in S_\alpha$ or y, $xy \in S_\beta$, whence

$$x^n \in S_\alpha xy S_\alpha \subseteq SxyS \quad \text{or} \quad y^n \in S_\beta xy S_\beta \subseteq SxyS$$

for some $n \in \mathbb{Z}^+$. By Lemma A we have that the assertion holds.

(ii) $\Rightarrow$ (iii). Let S be a semiprimary semigroup. Let a, $b \in S$. Then by Lemma A there exists $n \in \mathbb{Z}^+$ such that

$$(ba)^n \in S(ba)(ab)S \quad \text{or} \quad (ab)^n \in S(ba)(ab)S$$

whence

$$(ab)^{n+1} \in Sa^2S \qquad \ldots (1)$$

so by Theorem A it follows that $\sqrt{A}$ is an ideal, for every ideal A of S. Let $ab \in \sqrt{A}$. Then from

$$a^n \in SabS \subseteq \sqrt{A} \quad \text{or} \quad b^n \in SabS \subseteq \sqrt{A}$$

for some $n \in \mathbf{Z}^+$, it follows that $a \in \sqrt{A}$ or $b \in \sqrt{A}$. Therefore, $\sqrt{A}$ is a completely prime ideal.

(iii) $\Rightarrow$ (iv). This implication follows immediately.

(iv) $\Rightarrow$ (ii). Let $\sqrt{A}$ be a completely prime subset of S, for every ideal A of S. Then $\sqrt{SabS}$ is a completely prime subset of S, for every $a, b \in S$. Since $a^2 b^2 \in SabS \in \sqrt{SabS}$, we then have that

$$a^2 \in \sqrt{SabS} \text{ or } b^2 \in \sqrt{SabS}$$

i.e.

$$a^{2k} \in SabS \text{ or } b^{2k} \in SabS$$

for some $k \in \mathbf{Z}^+$, whence by Lemma A it follows that S is a semiprimary semigroup.

(ii) $\Rightarrow$ (i). Let S be a semiprimary semigroup. Then we have that the condition (1) holds and by Theorem A, S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. Let $\alpha, \beta \in Y$. Assume that $a \in S_\alpha$, $b \in S_\beta$. Then by Lemma A there exists $n \in \mathbf{Z}^+$ such that

$$a^n \in SabS \text{ or } b^n \in SabS$$

whence $\alpha \leq \beta$ or $\beta \leq \alpha$. Thus Y is a chain.

(iii) $\Rightarrow$ (v). Let P_1 and P_2 be completely prime ideals of S. Suppose that $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. Then there exists $a \in P_1 - P_2$ and $b \in P_2 - P_1$, whence $ab \in P_1 \bigcap P_2 = \sqrt{P_1 \bigcap P_2}$, and by (iii), $a \in P_1 \bigcap P_2$ or $b \in P_1 \bigcap P_2$, which is not possible. Thereofore, completely prime ideals of S are totally ordered. That S is a semilattice of Archimdean semigroups follows by Theorem A.

(v) $\Rightarrow$ (iii). By Theorem A we have that $\sqrt{A}$ is an ideal, for every ideal A of S. By Theorem II 3.7 of Petrich[9] we have that $\sqrt{A} = \bigcap P_i$, $A \subseteq P_i$, where P_i are completely prime ideals. Assume that $a \notin \bigcap P_i$, $b \notin \bigcap P_i$. Then there exist P_j, P_k such that $a \notin P_j$ and $b \notin P_k$. By hypothesis, $P_j \subseteq P_k$ or $P_k \subseteq P_j$. Assume that $P_j \subseteq P_k$ (the case $P_k \subseteq P_j$. can be similarly treated). Then $a \notin P_j$ and $b \notin P_j$ and $ab \notin P_j$, since P_j is completely prime. Thus $ab \notin \bigcup P_i$ and by contradiction we have the assertion.□

Proposition 1 — Every right ideal of a semigroup S is semiprimary if and only if

$$(\forall a, b \in S)\ ab \xrightarrow{r} a \vee ab \xrightarrow{r} b. \qquad \dots (2)$$

PROOF : Let every right ideal of a simgroup S be semiprimary. Let $a, b \in S$. Then abS is semiprimary and $ab^2 \in abS$, whence $ab \xrightarrow{r} a$ or $ab \xrightarrow{r} b$. Thus the condition (2) holds.

Conversely, let R be a right ideal of S. Assume that $ab \in R$. Then by hypothesis there exists $n \in \mathbf{Z}^=$ such that $a^n \in abS \subseteq RS \subseteq R$ or $b^n \in abS \subseteq RS \subseteq R$. Therefore, R is semiprimary.□

Lemma 1 — Let R be a right ideal of a semigroup S. If $\sqrt{R}$ is completely prime, then R is semiprimary.

PROOF : Let $a, b \in S$. then

$$ab \in R \subseteq \sqrt{R} \Rightarrow a \in \sqrt{R} \vee b \in \sqrt{R} \Rightarrow (\exists n \in \mathbb{Z}^+)\ a^n \in R \vee b^n \in R \square$$

By the next theorem chains of right Archimedean semigroups will be described by radicals of right idelas.

Theorem 2 — The following conditions are equivalent on a semigroup S :

(i) S is a chain of right Archimedean semigroups;

(ii) $\sqrt{R}$ is a completely prime ideal, for every right ideal R of S;

(iii) S is a semilattice of right Archimedean semigroups and every right ideal of S is semiprimary;

(iv) S is a semilattice of right Archimedean semigroups and the condition (2) holds.

PROOF : (i) $\Rightarrow$ (ii) — By Theorem 5 we have that $\sqrt{R}$ is an ideal of S, for every right ideal R of S. Let $ab \in \sqrt{R}$. Since S is a chain of right Archimedean semigroups, we then have that $ab \overset{r}{\rightarrow} a$ or $ab \overset{r}{\rightarrow} b$, whence

$$a^n \in abS \subseteq \sqrt{R} \text{ or } b^n \in abS \subseteq \sqrt{R}$$

for some $n \in \mathbb{Z}^+$, and therefore, $a \in \sqrt{R}$ or $b \in \sqrt{R}$, i.e. $\sqrt{R}$ is completely prime.

(ii) $\Rightarrow$ (iii). That every right ideal of S is semiprimary follows by Lemma 1. That S is a semilattice of right Archimedean semigroups follows by Theorem 5 of Bogdanović and Ćirić[4].

(iii) $\Rightarrow$ (iv) — It is clear that $a^n \in abS$ or $b^n \in abS$ for some $n \in \mathbb{Z}^+$. Assume that $a^n = abx$, for some $x \in S$. Then by Theorem 5 of Bogdanović and Ćirić[4] we have that $(a^n)^k \in bS$ for some $k \in \mathbb{Z}^+$. Therefore, $b \overset{r}{\rightarrow} a$. Similarly we treat the case $b^n \in abS$. Thus the condition (2) holds.

(iv) $\Rightarrow$ (i) — Let S be a semilattice Y of right Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $\alpha, \beta \in Y$. Assume that $a \in S_\alpha, b \in S_\beta$. Then from $a^n \in bS$ or $b^n \in aS$ for some $n \in \mathbb{Z}^+$, it follows that $a^{n+1} \in abS$ or $b^{n+1} \in baS$, whence $\alpha \le \beta$ or $\beta \le \alpha$, i.e. Y is a chain. $\square$

Chains of t-Archimedean semigroups will be described via bi-ideals by the next theorem.

Theorem 3 — The following conditions are equivalent on a semigroup S :

(i) S is a chain of t-Archimedean semigroups;

(ii) $\sqrt{B}$ is a completely prime ideal, for every bi-ideal B of S;

(iii) S is a semilattice of t-Archimedean semigroups and the condition (2) holds.

PROOF : (i) $\Rightarrow$ (ii) — By Theorem 9 of we have that $\sqrt{B}$ is an ideal of S, for every bi-ideal B of S. Let $ab \in \sqrt{B}$. Since S is a chain of t-Archimedean semigroups, we have that $ab \overset{r}{\rightarrow} a$ or $ab \overset{r}{\rightarrow} b$, whence

$$a^n \in abS \subseteq \sqrt{B} \text{ or } b^n \in abS \subseteq \sqrt{B}$$

for some $n \in \mathbf{Z}^+$. Thus $a \in \sqrt{B}$ or $b \in \sqrt{B}$, i.e. $\sqrt{B}$ is completely prime.

(ii) $\Rightarrow$ (iii) — That S is a semilattice of t-Archimedean semigroups follows by Theorem 9 of Bogdanović and Ćirić[4]. The rest follows by Theorem 2.

(iii) $\Rightarrow$ (i). Let S be semilattice Y of t-Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $\alpha, \beta \in Y$. Assume $a \in S_\alpha, b \in S_\beta$. Then by (2) there exists n $n \in \mathbf{Z}^+$ such that

$$b^n = ax \text{ or } a^n = by$$

for some $x \in S_\gamma$, $y \in S_\delta$, $\gamma, \delta \in Y$, so

$$b^{n+1} = bax \text{ or } a^{n+1} = aby.$$

From this it follows that $\beta = \beta\alpha\gamma$ or $\alpha = \alpha\beta\delta$, whence $\beta = \alpha\beta$ or $\alpha = \alpha\beta$, i.e. $\alpha \leq \beta$ or $\beta \leq \alpha$. Therefore, Y is a chain.□

Finally we characterize semigroups in which radicals of all semigroups are completely prime.

Theorem 4 — The radical of every subsemigroup of semigroup S is completely prime if and only if

$$(\forall a, b \in S)\ (\exists n\ n \in \mathbf{Z}^+)\ a^n \in \langle ab \rangle \vee b^n \in \langle ab \rangle. \quad \dots (3)$$

PROOF : Let $\sqrt{A}$ be completely prime for every subsemigroup A of S. Then for all $a, b \in S$ we have that

$$ab \in \langle ab \rangle \subseteq \sqrt{\langle ab \rangle} \Rightarrow a \in \sqrt{\langle ab \rangle} \vee b \in \sqrt{\langle ab \rangle}.$$

Therefore, condition (3) holds.

Conversely, let A be a subsemigroups of S. Let $ab \in \sqrt{A}$. Then $(ab)^k \in A$, for some $k \in \mathbf{Z}^+$. Since

$$a^n = (ab)^r \vee b^n (ab)^t,$$

for some n, r, $t \in \mathbf{Z}^+$, we then have that

$$a^{nk} = (ab)^{rk} \in A \vee b^{nk} = (ab)^{tk} \in A$$

whence $a \in \sqrt{A}$ or $b \in \sqrt{A}$. Therefore, $\sqrt{A}$ is completely prime□

It is clear that condition (3) is equivalent to

$$(\forall a, b \in S)\ (\exists n \in \mathbf{Z}^+)\ (ab)^n \in \langle a \rangle \cup \langle b \rangle. \quad \dots (4)$$

Semigroups which satisfy (4), called generalized $\mathcal{U}$-semigroups or, shortly, $\mathcal{GU}$-semigroups, are considered by the authors[3, 5]. The structure of periodic $\mathcal{GU}$-semigroups is completely described in Ćirić and Bogdanović[5].

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