Semigroups, Proceedings of the International Conference
in Semigrroups and its Related Topics, Kunming, China, 1995,
(K. P. Shum, Y. Guo, M. Ito and Y. Fong, eds.) Springer-Verlag, 1998, pp. 27–56

Quasi-orders and Semilattice Decompositions of Semigroups (A Survey)

Stojan Bogdanović*
University of Niš, Faculty of Economics, Trg JNA 11, 18000 Niš, Yugoslavia
e-mail: root@eknux.eknfak.ni.ac.yu

Miroslav Ćirić*
University of Niš, Faculty of Philosophy, Ćirila i Metodija 2, 18000 Niš, Yugoslavia
e-mail: mciric@archimed.filfak.ni.ac.yu

Contents

Introduction

It is well-known that quasi-orders have significant applications in many areas of mathematics. In Theory of semigroups their role is especially emphasized

* Supported by Grant 0401B of RFNS through Math. Inst. SANU.

in studying of semilattice decompositions of semigroups. Such a role is predominantly based on the well-known theorem proved in the book of G. Birkhoff [1]. By this theorem, to any quasi-order on a nonempty set A, in a very natural way, it can be associated an equivalence relation, called a natural equivalence of this quasi-order. Moreover, on the corresponding factor set of A this quasi-order induces a partial order. Applied to theory of semilattice decompositions of semigroups this leads to the following problem: Under which conditions on a quasi-order on a semigroup, its natural equivalence is a semilattice congruence on this semigroup? Such a problem was first treated and successfully solved by T. Tamura in [55], 1974. In the quoted paper T. Tamura used several very interesting concepts concerning quasi-orders on semigroups. At first, he used the concept of positivity, introduced by B. M. Schein in [42]. Also, T. Tamura introduced in [55] the notions of a lower-potent quasi-order and the notion of a half-congruence, and using this he established a bijective and isotone mapping between the poset of positive lower-potent half-congruences and the poset of semilattice congruences on a semigroup.

A further progress in studying semilattice decompositions through quasi-orders was made by M. Ćirić and S. Bogdanović in [14]. They first made certain modifications of above quoted Tamura's theorem. At first, in the proof of another theorem from the same paper of T. Tamura, they observed that the notion "lower-potent half-congruence" in the Tamura's theorem can be replaced by the notion "quasi-order satisfying the *cm*-property", which has shown oneself to be very useful for the further development of the theory. Note that the notion of a quasi-order on a semigroup satisfying the *cm*-property was also introduced by T. Tamura, in [51]. Another modification of the Tamura's theorem consists in its translation to the language of lattices and lattice isomorphisms.

The paper of the authors [14] has been based on the idea of studying quasi-orders through its left and right cosets. Following this idea, the authors in [14] and [7] has connected positive quasi-orders satisfying the *cm*-property with some sublattices of the lattice of consistent subsets, and on the other hand with some sublattices of the lattices of ideals of a semigroup. Using these connections, several characterizations of the lattice of semilattice decompositions and of semilattice homomorphic images of a semigroup have been given. Especially interesting are the ones by means of sublattices of the lattice of ideals.

All above quoted results will be presented in Section 3. Before that, in Section 1 we will introduce notions and notations which will be used in the further text, and in Section 2 we give definitions of several types of quasi-orders and we describe its main features.

Applications of quasi-orders in semilattice decompositions of semigroups are not exhausted by the above given applications. M. S. Putcha in [34] used the division relation on a semigroup (this is the smallest positive quasi-order on a semigroup) in studying decompositions into a semilattice of Archimedean semigroups. After that, by the division relation T. Tamura in [52] has generated the smallest positive lower-potent half-congruence on a semigroup, whose natural equivalence is the smallest semilattice congruence on a semigroup. This and some other results concerning the generation of positive lower-potent half-congruences and of semilattice congruences on a semigroup, obtained in papers of T. Tamura, M. S. Putcha and S. Bogdanović and M. Ćirić, will be presented in Section 4.

Section 5 is devoted to the notion of an Archimedean semigroup, to its several generalizations and to semilattices of these "generalized Archimedean semigroups". Presented definitions and results are from the papers of M. Ćirić and S. Bogdanović, T. Tamura and M. S. Putcha.

In Section 6 we present some results concerning the role of quasi-orders in studying of chain and ordinal decompositions of semigroups, obtained by the authors in [14] and [16].

Finally, in the last section we talk about the quasi-semilattice decompositions of semigroups with zero. Such decompositions were introduced and studied by M. Ćirić and S. Bogdanović in [17]. These decompositions are carried by a partially ordered set and we will see that these are generalizations of semilattice decompositions. We will present characterizations of the lattice of quasi-semilattice decompositions of a semigroup with zero in through quasi-orders and through some sublattices of the lattice of ideals. These characterizations are similar to the related characterizations of the lattice of semilattice decompositions.

1. Preliminaries

Throughout this paper, $\mathbb{Z}^+$ will denote the set of all positive integers. Further, $S = S^0$ means that S is a semigroup with zero 0. If $S = S^0$, we will write 0 instead $\{0\}$, and if A is a subset of S, then $A^\bullet = A - 0$, $A^0 = A \cup 0$ and $A' = (S - A)^0$. By S^1 we denote the semigroup obtained from a semigroup S by adjoining an identity. If A is a subset of a semigroup S, then by a *radical of* A we mean the set $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{Z}^+)\, x^n \in A\}$.

Let ξ be a binary relation on a set A. For $n \in \mathbb{Z}^+$, ξ^n will denote the n-th power of ξ in the semigroup of binary relations on A, ξ^∞ will denote the transitive closure of ξ and ξ^{-1} will denote the relation defined by: $a\, \xi^{-1}\, b \iff b\, \xi\, a$. For $a \in A$, the set $a\xi = \{x \in A \mid a\, \xi\, x\}$ will be called the *left coset of* A *determined by* a, and the set $\xi a = \{x \in A \mid x\, \xi\, a\}$ will be called the *right coset of* A *determined by*

a. Similarly, for $X \subseteq A$, the sets

$$X\xi = \bigcup_{a\in X} a\xi \qquad \text{and} \qquad \xi X = \bigcup_{a\in X} \xi a$$

will be called the *left coset* and the *right coset of* A *determined by* X, respectively. By a *quasi-order* we mean a reflexive and transitive binary relation. By Δ_A and ∇_A we denote the identity and the universal relation on A, respectively. If it is clear on which set these relations are considered, then we write simply Δ and ∇.

A relation ξ on a semigroup $S = S^0$ is called *left 0-restricted* if $0\xi = 0$. A *right 0-restricted* relation on S is defined dually. We say that ξ is *0-restricted* if it is both left and right 0-restricted, i.e. if $0\xi = \xi 0 = 0$.

Let K be a subset of a lattice L (not necessary complete). If K contains the meet of any its nonempty subset having the meet in L, then K is called a *complete meet-subsemilattice* of L. A *complete join-subsemilattice* is defined dually. If K is both complete meet-subsemilattice and complete join-subsemilattice of L, then it is called a *complete sublattice* of L. If L is a lattice with unity, then any sublattice of L containing its unity is called a *1-sublattice* of L. Dually we define a *0-sublattice* of a lattice with zero, and we define a sublattice of a lattice L with zero and unity to be a *0,1-sublattice* if it is both 0-sublattice and 1-sublattice of L. If any element of L is the meet of some nonempty subset of K, then K is called *meet-dense* in L.

A mapping φ of a poset P into a poset Q is *isotone* (*antitone*) if for $x, y \in P$, $x \le y$ implies $x\varphi \le y\varphi$ ($x \le y$ implies $y\varphi \le x\varphi$), and φ is an *order isomorphism* (*dual order isomorphism*) if it is an isotone (antitone) bijection with isotone (antitone) inverse. Note that a poset isomorphic or dually isomorphic to a (complete) lattice is also a (complete) lattice, and by Lemma II 3.2 [1] and its dual, any (dual) order isomorphism between lattices is a (dual) lattice isomorphism.

A mapping φ of a lattice L into itself is: *extensive*, if $x \le x\varphi$, for any $x \in L$, *contractive*, if $x\varphi \le x$, for any $x \in L$, and *idempotent*, if $(x\varphi)\varphi = x\varphi$, for any $x \in L$. An extensive, idempotent and isotone mapping of a lattice L into itself is called a *closure operation* on L, and all elements $x \in L$ for which $x\varphi = x$ are called *closed elements* of L (with respect to φ). Similarly, a contractive, idempotent and isotone mapping of a lattice L into itself is called an *interior operation* on L, and all elements $x \in L$ for which $x\varphi = x$ are called *open elements* of L (with respect to φ).

For a nonempty set A, $\mathcal{P}(A)$ will denote the *lattice of subsets* of A. Let A be a nonempty set and let L be a sublattice of $\mathcal{P}(A)$ containing its unity and having the property that any nonempty intersection of elements of L is also in A. Then for any $a \in A$ there exists the smallest element of L containing a (it is the intersection of all elements of L containing a), which will be called the *principal element* of L

generated by a. The set of all principal elements of L is called the *principal part* of L.

Let A be a subset of a semigroup S. If for any $x \in S$ and any $n \in \mathbb{Z}^+$, $x^n \in A$ implies $x \in A$, or equivalently, if $\sqrt{A} \subseteq A$, then A is called a *completely semiprime* subset of S. If for all $x, y \in S$, $xy \in A$ implies the either $x \in A$ or $y \in A$, then A is called a *completely prime* subset of A. We say that A is a *consistent* subset of S if for all $x, y \in S$, $xy \in A$ implies $x \in A$ and $y \in A$. It is easy to verify that the set of all consistent subsets of S is a complete 0,1-sublattice of $\mathcal{P}(S)$. A consistent subsemigroup of a semigroup S will be called a *filter* of S. The empty set will be also defined to be a filter. By $\mathcal{F}(S)$ we denote the *lattice of filters* of S, which is a complete meet-subsemilattice of $\mathcal{P}(S)$, and therefore a complete lattice, but it is not necessary a sublattice of $\mathcal{P}(S)$. It is well known that a subset A of a semigroup S is a filter of S if and only its complement is either empty or a completely prime ideal of S.

Let S be a semigroup. By $\mathcal{I}d(S)$ we denote the *lattice of ideals* of S. This lattice is a sublattice of $\mathcal{P}(S)$ and also a complete join-subsemilattice of $\mathcal{P}(S)$, but it is not necessary a complete meet-subsemilattice, since the empty set is not included in $\mathcal{I}d(S)$ and the intersection of an infinite family of ideals may be empty. The principal element of $\mathcal{I}d(S)$, called the *principal ideal*, generated by $a \in S$ is denoted by $J(a)$. By $\mathcal{I}d^{\mathbf{cs}}(S)$ we denote the *lattice of completely semprime ideals* of S. This lattice is a complete 1-subsemilattice of $\mathcal{I}d(S)$.

The *division* relation $\mid$ on a semigroup S is defined by

$$a \mid b \iff (\exists x, y \in S^1)\ b = xay.$$

On S we also define the relation $\longrightarrow$ by

$$a \longrightarrow b \iff (\exists n \in \mathbb{Z}^+)\ a \mid b^n.$$

An element $a \in S$ is called *intra-regular* if $a^2 \mid a$.

If $S = S^0$, then an element $a \in S$ is called a *nilpotent element* or a *nilpotent* if $a^n = 0$, for some $n \in \mathbb{Z}^+$. A semigroup with zero whose all elements are nilpotent is called a *nil-semigroup*.

A semigroup S is called *semilattice indecomposable* if the universal relation on S is the unique semilattice congruence on S.

For undefined notions and notations we refer to the following books: G. Birkhoff [1], S. Bogdanović [2], S. Bogdanović and M. Ćirić [4], S. Burris and H. P. Sankappanavar [9], A. H. Clifford and G. B. Preston [18], [19], G. Grätzer [20], J. M. Howie [21], E. S. Lyapin [26], M. Petrich [32], [33], L. N. Shevrin [44] and G. Szász [47].

2. Lattices of quasi-orders

In this section we will present some properties of the lattice of quasi-orders on a semigroup and of its certain subsets and sublattices. Namely, we will consider several types of quasi-orders significant from the aspect of its usage in studying of semilattice decompositions of semigroups and quasi-semilattice-decompositions of semigroups with zero.

The set $\mathcal{Q}(A)$ of quasi-orders on an nonempty set A, partially ordered by inclusion of relations, is a complete lattice, in which the meet and the join of a subset X of $\mathcal{Q}(A)$ are defined as follows. The meet of X equals the set-theoretical intersection of all elements of X. If $\mathcal{B}(A)$ denote the semigroup of all binary relations on A (with respect to usual product of relations), then the join of X equals the set-theoretical union of all elements of the subsemigroup of $\mathcal{B}(A)$ generated by X.

An important complete sublattice of $\mathcal{Q}(A)$ is the lattice $\mathcal{E}(A)$ of all equivalence relations on A. If A is any algebra, then the set $\mathrm{Con}(A)$ of all congruence relations on A is a complete sublattice both of $\mathcal{Q}(A)$ and $\mathcal{E}(A)$.

Among the other special types of quasi-orders on a semigroup, we will first talk about the positive quasi-orders. We say that a relation ξ on a semigroup S is *positive* if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$. This concept has been introduced by B. M. Schein in [42]. After that, positive quasi-orders have been studied from different points of view by many authors, mainly by T. Tamura [51]–[55], M. S. Putcha [35], [37]–[40], and the authors [7], [8], [11], [14], [17]. One remark on positive quasi-orders given by M. S. Putcha in [35], can be formulated in the following way:

Theorem 2.1. *The set of positive quasi-orders on a semigroup S is the principal dual ideal of $\mathcal{Q}(S)$ generated by the division relation $\mid$ on S.*

Another important concept is the lower-potency, introduced by T. Tamura in [55]. A relation ξ on a semigroup S will be called *lower-potent* if $a^n\,\xi\,a$, for any $a \in S$ and any $n \in \mathbb{Z}^+$. When ξ is a quasi-order, then it is lower-potent if and only if $a^2\,\xi\,a$, for any $a \in S$. The place of such quasi-orders inside the lattice $\mathcal{Q}(S)$ is described by:

Theorem 2.1. *The set of lower-potent quasi-orders on a semigroup S is the principal dual ideal of $\mathcal{Q}(S)$ generated by the relation $\sqrt{}$ on S defined by*

$$a \sqrt{}\, b \iff a \in \sqrt{b}.$$

The following theorem, taken from the proof of Theorem 2 from [14], can be also obtained as a combination of the previous two theorems.

Theorem 2.3. *The set of lower-potent positive quasi-orders on a semigroup S is the principal dual ideal of $\mathcal{Q}(S)$ generated by $\longrightarrow^{\infty}$.*

Relations satisfying the *cm*-property were introduced by T. Tamura in [51], by the following definition: A relation ξ on a semigroup S satisfies the *common multiple property*, or shortly the *cm-property*, if for all $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$. It can be easily proved that the quasi-orders on S with this property form a complete meet subsemilattice of $\mathcal{Q}(S)$, and hence a complete lattice, but it is not known does they form a (complete) sublattice of $\mathcal{Q}(S)$. However, the following theorem holds:

Theorem 2.4. (S. Bogdanović and M. Ćirić [7]) *The poset of quasi-orders on a semigroup S satisfying the cm-property is a complete lattice. The smallest element of this lattice is the relation $\sqrt{\ }$.*

T. Tamura in [55] also defined *half-congruences* as compatible quasi-orders. The standard proof of the fact that the congruences on any algebra form a complete sublattice of the lattice of equivalence relations can be easily translated to half-congruences and the following theorem can be obtained:

Theorem 2.5. *The set of half-congruences on a semigroup S is a complete sublattice of $\mathcal{Q}(S)$.*

By combination of the previous results concerning positive and lower-potent quasi-orders on a semigroup, the following theorem has been obtained:

Theorem 2.6. (S. Bogdanović and M. Ćirić [7]) *The set of lower-potent positive half-congruences on a semigroup S is the principal dual ideal of the lattice of half-congruences on S generated by $\longrightarrow^{\infty}$.*

It is important to note that, whenever it is needed, instead of the lower-potent positive half-congruences we can use positive quasi-orders satisfying the *cm*-property. This follows by the following theorem, which is taken from the proof of Theorem 4.9 of [55].

Theorem 2.7. *A positive quasi-order ξ on a semigroup S satisfies the cm-property if and only if it is a lower-potent half-congruence.*

The second part of this section is devoted to quasi-orders on a semigroup with zero. In studying of semigroups with zero it is often of interest to use relations and subsets with some "restrictions" and "weakenings" on the zero. For a relation ξ on a semigroup $S = S^0$, the authors in [11] have defined the operations:

$$^{\bullet}\xi = \xi - (0 \times S^{\bullet}), \quad \xi^{\bullet} = \xi - (S^{\bullet} \times 0), \quad ^{\bullet}\xi^{\bullet} = \xi - (0 \times S^{\bullet} \cup S^{\bullet} \times 0)$$

and

$$\xi^0 = \xi \cup (S^\bullet \times 0),$$

and they have introduced the notations:

$$\gamma_l = {}^\bullet\nabla, \quad \gamma_r = \nabla^\bullet \quad \text{and} \quad \gamma = {}^\bullet\nabla^\bullet,$$

where ∇ denotes the universal relation on S. For any semigroup $S = S^0$, the mappings

$$\xi \mapsto {}^\bullet\xi, \quad \xi \mapsto \xi^\bullet \quad \text{and} \quad \xi \mapsto {}^\bullet\xi^\bullet$$

are interior operations on the lattice $\mathcal{Q}(S)$, and the related sets of open elements are the sets of left 0-restricted, right 0-restricted and 0-restricted quasi-orders on S, respectively. Moreover, the following theorem has been proved:

Theorem 2.8. (M. Ćirić and S. Bogdanović [11]) *The sets of left 0-restricted quasi-orders, right 0-restricted quasi-orders and 0-restricted quasi-orders on a semigroup $S = S^0$ are the principal ideals of $\mathcal{Q}(S)$ generated by γ_l, γ_r and γ, respectively.*

On the other hand, it has been proved in [11] that the mapping $\xi \mapsto \xi^0$ is a closure operation on the lattice of left 0-restricted quasi-orders on S.

Jointly with positive relations, in studying of semigroups with zero we also use 0-positive relations defined in the following way: a relation ξ on a semigroup $S = S^0$ is called *0-positive* if for all $a, b \in S$, $ab \neq 0$ implies $a\,\xi\,ab$ and $b\,\xi\,ab$. Some properties of the set of 0-positive quasi-orders inside the lattice $\mathcal{Q}(S)$ have been described in [11], by the following theorem:

Theorem 2.9. (M. Ćirić and S. Bogdanović [11]) *The sets of 0-positive quasi-orders on a semigroup $S = S^0$ is the principal dual ideal of $\mathcal{Q}(S)$ generated by the relation $\|$ on S defined by*

$$a \parallel b \iff a = b = 0 \ \ \textit{or} \ \ \big((\exists x, y \in S^1)\ b = xay \neq 0\big).$$

Combining Theorems 2.8 and 2.9, it has been proved in [11] that the set of left 0-restricted positive quasi-orders on a semigroup $S = S^0$ and the set of 0-restricted 0-positive quasi-orders on S equal the intervals $[|, \gamma_l]$ and $[\|, \gamma]$ of $\mathcal{Q}(S)$, respectively, so these are complete sublattices of $\mathcal{Q}(S)$. Moreover, it has been observed that the mappings $\xi \mapsto \xi^\bullet$ and $\eta \mapsto \eta^0$ are mutually inverse isomorphisms between these lattices, whence the following theorem has been obtained:

Theorem 2.10. (M. Ćirić and S. Bogdanović [11]) *The lattice of left 0-restricted positive quasi-orders on a semigroup $S = S^0$ is isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S.*

Except the lower-potent relations, on semigroups with zero we consider also 0-lower-potent relations defined as follows: a relation ξ on a semigroup $S = S^0$ is called *0-lower-potent* if for any $a \in S$ and any $n \in \mathbb{Z}^+$, $a^n \neq 0$ implies $a^n \,\xi\, a$. The sets of left 0-restricted positive 0-lower-potent quasi-orders and of 0-restricted 0-positive 0-lower-potent quasi-orders have been also studied in [11] and the following two results have been obtained:

Theorem 2.11. (M. Ćirić and S. Bogdanović [11]) *The set of left 0-restricted positive 0-lower-potent quasi-orders on a semigroup $S = S^0$ is the principal dual ideal of $\mathcal{Q}(S)$ generated by the relation $\rightsquigarrow$ on S defined by*

$$a \rightsquigarrow b \iff b = 0 \;\; \textit{or} \;\; \big((\exists n \in \mathbb{Z}^+)(\exists x, y \in S^1)\; b^n = xay \neq 0\big).$$

Theorem 2.12. (M. Ćirić and S. Bogdanović [11]) *The set of 0-restricted 0-positive 0-lower-potent quasi-orders on a semigroup $S = S^0$ is the principal dual ideal of $\mathcal{Q}(S)$ generated by the relation $\twoheadrightarrow$ on S defined by*

$$a \twoheadrightarrow b \iff a = b = 0 \;\; \textit{or} \;\; \big((\exists n \in \mathbb{Z}^+)(\exists x, y \in S^1)\; b^n = xay \neq 0\big).$$

Seeing that the mapping $\xi \mapsto \xi^\bullet$ preserves the 0-lower-potentcy, then the following also holds:

Theorem 2.13. (M. Ćirić and S. Bogdanović [11]) *The lattice of left 0-restricted positive 0-lower-potent quasi-orders on a semigroup $S = S^0$ is isomorphic to the lattice of 0-restricted 0-positive 0-lower-potent quasi-orders on S.*

The *cm*-property has been also modified for semigroups with zero in the following way: we say that a relation ξ on a semigroup $S = S^0$ satisfies the *0-cm-property* if for all $a, b, c \in S$, $a \,\xi\, c$, $b \,\xi\, c$ and $ab \neq 0$ implies $ab \,\xi\, c$. The authors in [11] proved the following:

Theorem 2.14. (M. Ćirić and S. Bogdanović [11]) *The poset of left 0-restricted positive quasi-orders on a semigroup $S = S^0$ satisfying the 0-cm-property and the poset of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property are isomorphic complete lattices.*

The sense of such defined concepts will be explained later, in Section 7.

3. Semilattice decompositions viewed from quasi-orders

In studying of semilattice decompositions of semigroups, which has been started by T. Tamura in [55], a starting point has been the following result given by G. Birkhoff in his book [1].

Theorem 3.1. (G. Birkhoff [1]) *Let ξ be a quasi-order on a set X. Then*

1. $\widetilde{\xi} = \xi \cap \xi^{-1}$ *is an equivalence relation on X.*
2. *If E and F are two $\widetilde{\xi}$-classes, then $a\,\xi\,b$ either for no $a \in E$, $b \in F$, or for all $a \in E$, $b \in F$.*
3. *The quotient set $X/\widetilde{\xi}$ is a poset if $E \leq F$ is defined to mean that $a\,\xi\,b$ for some (hence all) $a \in E$, $b \in F$.*

In other words, the previous theorem says that to any quasi-order ξ on a set X in a natural way it can be associated an equivalence relation $\widetilde{\xi}$, called the *natural equivalence* of ξ. On the other hand, to ξ it can be also naturally associated a partially ordered set $X/\widetilde{\xi}$ with the partial order $\leq$ which will be further denoted by $\xi/\widetilde{\xi}$. It can be also showed that the mapping $\xi \mapsto \widetilde{\xi}$ is an interior operation on the lattice $\mathcal{Q}(X)$, whose related set of open elements is exactly the set of all equivalence relations on S.

If we consider a quasi-order ξ on a semigroup S, then several important questions concerning ξ and its natural equivalence naturally arise. The first of them is: Under what conditions $\widetilde{\xi}$ is a congruence on S? One answer on this question has been given in above mentioned paper of T. Tamura:

Theorem 3.2. (T. Tamura [55]) *If ξ is a half-congruence on a semigroup S, then $\widetilde{\xi}$ is a congruence on S and $S/\widetilde{\xi}$ is a partially ordered semigroup with respect to $\xi/\widetilde{\xi}$.*

Conversely, if $\varphi : S \to S'$ is a homomorphism of a semigroup S onto a partially ordered semigroup (S', ξ'), and if we define a relation ξ on S by:

$$a\,\xi\,b \iff (a\varphi)\,\xi'\,(b\varphi),$$

then ξ is a half-congruence on S and $(S/\widetilde{\xi}, \xi/\widetilde{\xi})$ and (S', ξ') are isomorphic partially ordered semigroups.

Another interesting question treated in the same paper of T. Tamura has been the following: Under what conditions on ξ, $\widetilde{\xi}$ becomes a semilattice congruence? One answer to this question has been given by T. Tamura in [55], by means of the concepts of positivity and lower-potency, that have been treated in the preceding section. Namely, T. Tamura has proved the following theorem

Theorem 3.3. (T. Tamura [55]) *If ξ is a lower-potent positive half-congruence on a semigroup S, then $\widetilde{\xi}$ is a semilattice congruence on S. Every semilattice congruence is obtained in this manner, and the correspondence $\xi \mapsto \widetilde{\xi}$ is one-to-one and isotone.*

In fact, the previous theorem gives somewhat more than was required in the above question. First of all, Theorem 3.3 says that the correspondence $\xi \mapsto \widetilde{\xi}$ is one-to-one, which does not hold when ξ is simply a half-congruence, An example for this has been given by T. Tamura in [55]. On the other hand, Theorem 3.3 says that the mapping $\xi \mapsto \widetilde{\xi}$ of the poset of lower-potent positive half-congruences into the poset of semilattice congruences on S is also onto and isotone. But, Tamura has not investigated the inverse of this mapping, although it can be easily proved that this inverse is also isotone. Moreover, Tamura has not treated these posets as complete lattices. Namely, recall that by Theorem 2.6 it follows that the poset of lower-potent positive half-congruences on a semigroup S is a complete sublattice of $\mathcal{Q}(S)$. On the other hand, by a general result obtained by T. Tamura and N. Kimura in [56], the poset of semilattice congruences on S is also a complete lattice. Moreover, by another general result given by M. Ćirić and S. Bogdanović in [15], semilattice congruences on S form a principal dual ideal of $\mathrm{Con}\,(S)$.

If we take into consideration all the above mentioned facts, Theorem 3.3 can be formulated in the following way:

Theorem 3.4. *The lattice of semilattice congruences on a semigroup S is isomorphic to the lattice of lower-potent positive half-congruences on S.*

On the other hand, in view of Theorem 2.7, Theorem 3.3 can be also given in the following version:

Theorem 3.5. *The lattice of semilattice congruences on a semigroup S is isomorphic to the lattice of positive quasi-orders satisfying the cm-property on S.*

This version of Theorem 3.3 has been given by M. Ćirić and S. Bogdanović in [14], where it has been shown that it has a great importance for further studying of semilattice decompositions through quasi-orders.

A new way in studying of quasi-orders on a semigroup has been discovered by the authors in [14]. In this paper, the authors started an investigation of quasi-orders through its left and right cosets. In order to explain this approach to studying of quasi-orders, we will start from a quasi-order ξ on a nonempty set X. To this quasi-order we can associate the subset K_ξ of $\mathcal{P}\,(X)$ defined by:

$$K_\xi = \{A \in \mathcal{P}\,(X) \mid A\xi = A\}. \tag{3.1}$$

Significant properties of this set are the following: K_ξ is a complete 0,1-sublattice of $\mathcal{P}\,(X)$, whose principal elements are exactly the left cosets of ξ determined by the elements of X, and moreover, the mapping $\xi \mapsto K_\xi$ is a dual isomorphism of the lattice $\mathcal{Q}(X)$ onto the lattice of complete 0,1-sublattices of $\mathcal{P}(X)$. This will be summarized as

Theorem 3.6. *The lattice $\mathcal{Q}(X)$ of quasi-orders on a nonempty set X is dually isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{P}(X)$.*

The inverse of the given dual isomorphism is defined in the following way: for a given complete 0,1-sublattice K of $\mathcal{P}(X)$, a quasi-order ξ_K on S related to K is defined by:

$$a\,\xi_K\,b \iff K(b) \subseteq K(a), \tag{3.2}$$

where for any $a \in X$, $K(a)$ denotes the principal element of K generated by a, and then the mapping $K \mapsto \xi_K$ is the inverse of $\xi \mapsto K_\xi$.

Another isomorphism between the lattice $\mathcal{Q}(X)$ and the lattice of complete 0,1-sublattices of $\mathcal{P}(X)$ can be obtained in the following way, by means of the right cosets of quasi-orders. Namely, the mapping $\xi \mapsto K'_\xi$, where

$$K'_\xi = \{A \in \mathcal{P}(X) \mid \xi A = A\}, \tag{3.3}$$

is also a dual isomorphism between of the first onto the second quoted lattice. The inverse of this mapping is the mapping $K \mapsto \xi'_K$, where ξ'_K is defined by

$$a\,\xi'_K\,b \iff K(a) \subseteq K(b), \tag{3.4}$$

The principal elements of K'_ξ are the right cosets determined by the elements of X.

However, when we deal with some special quasi-orders, then using of left and right cosets leads to different results. For example, positive quasi-orders have been described in [14] in the following way:

Theorem 3.7. (M. Ćirić and S. Bogdanović [14]) *The following conditions for a quasi-order ξ on a semigroup S are equivalent:*

(i) *ξ is positive;*
(ii) $(\forall a, b \in S)\ (ab)\xi \subseteq a\xi \cap b\xi$;
(iii) $(\forall a, b \in S)\ \xi a \cup \xi b \subseteq \xi(ab)$;
(iv) *$a\xi$ is an ideal of S, for each $a \in S$;*
(v) *ξa is a consistent subset of S, for each $a \in S$.*

By this theorem, positive quasi-orders can be characterized in two different ways. The first one is by means of consistent subsets of a semigroup:

Theorem 3.8. (S. Bogdanović and M. Ćirić [7]) *The lattice of positive quasi-orders on a semigroup S is dually isomorphic to the lattice of complete 0,1-sublattices of the lattice of consistent subsets of S.*

Another characterization has been given in [14] by means of ideals:

Theorem 3.9. (M. Ćirić and S. Bogdanović [14]) *The lattice of positive quasi-orders on a semigroup S is dually isomorphic to the lattice of complete 1-sublattices of $\mathcal{I}d(S)$.*

By the same methodology, lower-potent positive quasi-orders have been described by:

Theorem 3.10. (M. Ćirić and S. Bogdanović [14]) *A quasi-order ξ on a semigroup S is lower-potent and positive if and only if $a\xi$ is a completely semiprime ideal of S, for any $a \in S$.*

Applying this to Theorem 3.9 it has been obtained

Theorem 3.11. (M. Ćirić and S. Bogdanović [14]) *The lattice of lower-potent positive quasi-orders on a semigroup S is dually isomorphic to the lattice of complete 1-sublattices of $\mathcal{I}d^{\mathbf{cs}}(S)$.*

Finally, positive quasi-orders with the *cm*-property have been characterized in the following way:

Theorem 3.12. (M. Ćirić and S. Bogdanović [14]) *The following conditions for a quasi-order ξ on a semigroup S are equivalent:*

(i) *ξ is positive and it satisfies the cm-property;*
(ii) *ξa is a filter of S, for each $a \in S$;*
(iii) $(\forall a, b \in S)\ a\xi \cap b\xi = (ab)\xi$.

Using (i) $\iff$ (ii) of the previous theorem, the authors in [7] obtained the following result:

Theorem 3.13. (S. Bogdanović and M. Ćirić [14]) *The lattice of positive quasi-orders on a semigroup S satisfying the cm-property is dually isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{P}(S)$ whose principal elements are filters of S.*

A more interesting characterization of the lattice of positive quasi-orders with the *cm*-property has been obtained in [14] by means of (i) $\iff$ (iii) of Theorem 3.12. In order to present this result, we introduce several new notions. Given a sublattice K of the lattice $\mathcal{I}d^{\mathbf{cs}}(S)$ of completely semiprime ideals of a semigroup

S. If any element from K can be represented as an intersection of some family of completely prime ideals from K, then we say that K satisfies the *completely prime ideal property*, or shortly the *cpi-property*. In other words, K satisfies the *cpi*-property if and only if the set of all completely prime ideals from K is meet-dense in K. Using this concept, the authors in [14] obtained the following result:

Theorem 3.14. (M. Ćirić and S. Bogdanović [14]) *The lattice of positive quasi-orders on a semigroup S satisfying the cm-property is dually isomorphic to the lattice of complete 1-sublattices of $\mathcal{I}d^{\mathbf{cs}}(S)$ satisfying the cpi-property.*

Theorem 3.12 also says that for any positive quasi-order ξ on a semigroup S satisfying the *cm*-property, the principal part of the lattice K_ξ is a meet-subsemilattice of K_ξ. Clearly, this semilattice is isomorphic to the semilattice homomorphic image of S corresponding to the semilattice congruence $\widetilde{\xi}$. Using these facts the authors in [14] gave the following characterization of all semilattice homomorphic images of a semigroup:

Theorem 3.15. (M. Ćirić and S. Bogdanović [14]) *A semilattice Y is a semilattice homomorphic image of a semigroup S if and only if it is isomorphic to the principal part of some complete 1-sublattice of $\mathcal{I}d^{\mathbf{cs}}(S)$ satisfying the cpi-property.*

Especially, the greatest semilattice homomorphic image of a semigroup S has been described in an earlier paper [13] of the authors. In this paper, the principal element of the lattice $\mathcal{I}d^{\mathbf{cs}}(S)$ generated by an element a of a semigroup S has been called the *principal radical* of S generated by a, and it has been denoted by $\Sigma(a)$. Using this notion, the authors have proved the following theorem

Theorem 3.16. (M. Ćirić and S. Bogdanović [13]) *If a, b is any pair of elements of a semigroup S, then*

$$\Sigma(a) \cap \Sigma(b) = \Sigma(ab).$$

Furthermore, the set Σ_S of all principal radicals of S, partially ordered by inclusion, is a semilattice and it is the greatest semilattice homomorphic image of S.

Also, the following algorithm for computation of principal radicals of a semigroup has been given:

Theorem 3.17. (M. Ćirić and S. Bogdanović [13]) *The principal radical $\Sigma(a)$ of a semigroup S generated by an element $a \in S$ can be computed using the following formulas:*

$$\Sigma_1(a) = \sqrt{SaS}, \qquad \Sigma_{n+1}(a) = \sqrt{S\Sigma_n(a)S}, \quad n \in \mathbb{Z}^+,$$

and

$$\Sigma(a) = \bigcup_{n\in\mathbb{Z}^+} \Sigma_n(a).$$

Clearly, for all $n \in \mathbb{Z}^+$, $a \in S$, $\Sigma_n(a) = \{x \in S \mid a \longrightarrow^n x\}$ and $\Sigma(a) = \{x \in S \mid a \longrightarrow^\infty x\}$.

Note finally that the concepts similar to the *cpi*-property play an important role in certain other algebraic theories. For example, the theorem that asserts similar property of the lattice of ideals of a distributive lattice is known as the Prime Ideal Theorem. For related concepts in Ring theory, the reader is referred to the paper of W. Krul [25] or the book of N. H. Mc Coy [28]. It is also known that the lattice $\mathcal{I}d^{\mathbf{cs}}(S)$ of completely semiprime ideals of an arbitrary semigroup S satisfies the *cpi*-property. This has been proved in some special cases by Š. Schwarz [43] and K. Iséki [22], an in the general case by M. Petrich in [32] (see also Y. S. Park, J. P. Kim and M. G. Sohn [29]). In the book [20] of G. Grätzer it has been mentioned that the Prime Ideal Theorem for distributive lattices, lattices and rings is a consequence of the Axiom of Choice. In the Petrich's book [32], the *cpi*-property for the lattice of completely semiprime ideals of a semigroup has been also proved my means of the Axiom of Choice (or more precisely, by the Zorn's lemma arguments). Using Theorem 3.18, M. Ćirić and S. Bogdanović in [13] shown that the same result can be obtained without use of argumens of the Zorn's lemma or the Axiom of Choice.

4. Semilattice congruences generation

One of the most important problems in the theory of semilattice decompositions of semigroups is given in the question: how to generate semilattice congruences starting from other relations that can be more easily constructed? The most significant result from this area is probably the result given by T. Tamura in [52], which gives a way for construction of the smallest semilattice congruence starting from the division relation on a semigroup. This result is given by the following

Theorem 4.1. (T. Tamura [52]) *The smallest semilattice congruence on a semigroup S equals the natural equivalence of the quasi-order* $\longrightarrow^\infty$.

Seeing that $\longrightarrow^\infty$ is the smallest lower-potent positive quasi-order on a semigroup, the previous theorem can be formulated in the following way, as done in [53]:

Theorem 4.2. *The smallest semilattice congruence on a semigroup S equals the natural equivalence of the smallest lower-potent positive quasi-order on S.*

In the same paper, T. Tamura generalized this result by the following theorem:

Theorem 4.3. (T. Tamura [54]) *For any compatible relation ξ on a semigroup S, the smallest semilattice congruence on S containing ξ equals the natural equivalence of the smallest lower-potent positive quasi-order on S containing ξ.*

The statement of Theorem 4.2 can be obtained from the previous theorem, in the case when ξ is the identity relation on S.

Interesting characterizations of the smallest semilattice congruence on a semigroup have been obtained by M. S. Putcha in [36]. In this paper he first proved the following:

Theorem 4.4. (M. S. Putcha [36]) *The smallest semilattice congruence on a semigroup S equals the equivalence relation on S generated by the relation $xy \equiv xyx \equiv yx$, for all $x, y \in S^1$.*

Afterwards, using the previous theorem and the relation — defined by the rule: $— = \longrightarrow \cap (\longrightarrow)^{-1}$, M. S. Putcha in [36] obtained also the following theorem:

Theorem 4.5. (M. S. Putcha [36]) *The smallest semilattice congruence on a semigroup S equals the relation $—^{\infty}$.*

Each of Theorems 4.1 and 4.4 has been proved without using the other, but T. Tamura in [54] shown that each one can be directly derived from the other.

Some new results concerning the question of lower-potent positive half-congruences generating have been obtained in a recent paper of the authors. Recall that the smallest lower-potent positive half-congruence on a semigroup is obtained from the division relation using the construction of the relation $\longrightarrow$ and its transitive closure. A similar construction can be applied to any relation π on a semigroup S. Namely, S we can define the relation $\xrightarrow{\pi}$ on S by:

$$a \xrightarrow{\pi} b \iff (\exists n \in \mathbb{Z}^+)\ a \, \pi \, b^n,$$

and to consider its transitive closure $\xrightarrow{\pi} {}^{\infty}$. In order to simplify notations, the relation $\xrightarrow{\pi} {}^{\infty}$ will be denoted by $\overline{\pi}$. The authors in [8] considered the following problem: Under what conditions the relation $\overline{\pi}$ is a lower-potent positive half-congruence on S? It is not hard to provide the lower-potency and the positivity of $\overline{\pi}$, since the following two theorems hold:

Theorem 4.6. *The smallest lower-potent quasi-order on a semigroup S containing a relation ξ on S equals $\overline{\pi}$, where $\pi = \xi \cup \Delta$.*

Theorem 4.7. *The smallest lower-potent positive quasi-order on a semigroup S containing a relation ξ on S equals $\overline{\pi}$, where $\pi = \xi \cup \mid$.*

Applying Theorem 4.7 on quasi-orders the following can be obtained:

Theorem 4.8. *The mapping $\pi \mapsto \overline{\pi}$ is a closure operation on the lattice $\mathcal{Q}(S)$ of quasi-orders on a semigroup S. The corresponding set of closed elements is the set of all lower-potent quasi-orders on S.*

On the other hand, an immediate consequence of Theorem 4.6 is the following theorem:

Theorem 4.9. *Let π be a reflexive positive relation on a semigroup S. Then $\overline{\pi}$ is the smallest lower-potent positive quasi-order on S containing π.*

Hence, if we want to prove that $\overline{\pi}$ is a lower-potent positive half-congruence, where π is a reflexive positive relation on a semigroup S, the main problem is to prove the compatibility of π. A necessary and sufficient condition for π to $\overline{\pi}$ be compatible has been established by the authors in [8] by the following theorem:

Theorem 4.10. (S. Bogdanović and M. Ćirić) [8] *Let π be a reflexive positive relation on a semigroup S. Then $\overline{\pi}$ is compatible if and only if for all $a, b, c, d \in S$ the following condition holds*

$$a\,\pi\,c \;\&\; b\,\pi\,d \implies (\exists u \in S)\ a \mid u \;\&\; b \mid u \;\&\; u\overline{\pi}cd. \tag{5}$$

Especially, for positive quasi-orders the authors obtained:

Theorem 4.11. (S. Bogdanović and M. Ćirić) [8] *Let π be a positive quasi-order on a semigroup S. Then $\overline{\pi}$ is compatible if and only if for all $a, b, c \in S$ the following condition holds*

$$a\,\pi\,c \;\&\; b\,\pi\,c \implies (\exists u \in S)\ a \mid u \;\&\; b \mid u \;\&\; u\overline{\pi}c. \tag{6}$$

A key role in the proofs of the preceding two theorems plays the following theorem:

Theorem 4.12. *Let ξ be a lower-potent positive quasi-order on a semigroup S and $a_1, a_2, \dots, a_n \in S$. Then*

$$a_1 a_2 \cdots a_n \,\xi\, a_{a\varphi} a_{2\varphi} \cdots a_{n\varphi},$$

for any permutation φ of the set $\{1, 2, \dots, n\}$.

This theorem has been first explicitly formulated and proved by the authors in [8], although a proof of this feature of lower-potent positive quasi-orders can be also found in the proof of Theorem 2.3 of [53]. Notice that this feature of lower-potent positive quasi-orders is closely related to a property of completely semiprime ideals of a semigroup given by Lemma II 3.5 from the book of M. Petrich [32]. In fact, this is an immediate consequence of Theorem 3.10.

If in Theorem 4.11 we put $\pi =|$, then we immediately obtain Theorem 4.1, since the relation $|$ satisfies the condition (6). Therefore, Theorem 4.11 generalizes Theorem 4.1. On the other hand, if a compatible relation ξ on a semigroup S is given and if we put $\pi = \xi\cup\,|$, then π satisfies (5), so the assertion of Theorem 4.3 follows by Theorem 4.10.

This section will be finished by two theorems given by T. Tamura that characterize the smallest lower-potent positive half-congruence and the smallest semilattice congruence on a semigroup.

Theorem 4.13. (T. Tamura [54]) *The smallest lower-potent positive half-congruence on a semigroup S equals the transitive closure of the relation*

$$\Delta \cup \{(xa^2y, xay) \mid a \in S,\ x, y \in S^1\} \cup \{(xay, xaby) \mid a, b \in S,\ x, y \in S^1\}\cup \\ \cup \{(xay, xbay) \mid a, b \in S,\ x, y \in S^1\}.$$

Theorem 4.14. (T. Tamura [53]) *The smallest semilattice congruence on a semigroup S containing a compatible relation ξ on S equals the symmetric transitive closure of the relation*

$$\xi \cup \Delta \cup \{(xa^2y, xay) \mid a \in S,\ x, y \in S^1\} \cup \{(xaby, xbay) \mid a, b \in S,\ x, y \in S^1\}.$$

5. Archimedeaness and its generalizations

One of the most significant results of the theory of semilattice decompositions of semigroups is the following theorem of T. Tamura, proved first in [50]:

Theorem 5.1. (T. Tamura [50]) *In the greatest semilattice decomposition of a semigroup, any component is a semilattice indecomposable semigroup.*

This theorem has initiated intensive studying of semilattice indecomposable semigroups. In the general case, these semigroups have been characterized by T. Tamura and M. Petrich:

Theorem 5.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is semilattice indecomposable;*

(ii) $(\forall a, b \in S)\ a \longrightarrow^{\infty} b$;
(iii) *S has no proper completely semiprime ideals;*
(iv) *S has no proper completely prime ideals.*

The equivalence of conditions (i) and (ii) has been established by T. Tamura in [52], (i) $\iff$ (iii) has been proved by M. Petrich in [32], and (i) $\iff$ (iv) also by M. Petrich in [31].

Among the various special types of semilattice indecomposable semigroups, an outstanding place is taken by semigroups in which $a \longrightarrow b$, for all its elements a and b, called *Archimedean semigroups*. Also, in the theory of semilattice decompositions, an outstanding place is taken by semigroups which are semilattices of Archimedean semigroups. These semigroups have been first completely described by M. S. Putcha in [34]. Various other characterizations of semilattices of Archimedean semigroups have been given by many authors, and the most interesting ones will be presented in the next theorem. But first we introduce the following notion: we say that a relation ξ on a semigroup S is *powerfull* if for all $a, b \in S$, $a\,\xi\,b$ implies $a^2\,\xi\,b$.

Theorem 5.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is semilattice of Archimedean semigroups;*
(ii) $\longrightarrow$ *is powerfull;*
(iii) $\longrightarrow$ *satisfies the cm-property;*
(iv) $\longrightarrow$ *is transitive;*
(v) $(\forall a, b \in S)\ a^2 \longrightarrow ab$;
(vi) *the radical of any ideal of S is also an ideal of S;*
(vii) *in any homomorphic image of S having a zero, the set of all nilpotents is an ideal;*
(viii) $(\forall a, b \in S)\ \Sigma_1(ab) = \Sigma_1(a) \cap \Sigma_1(b)$.

The equivalence of conditions (i) and (ii) has been established by M. S. Putcha in [34], and of (ii), (iii) and (iv) by T. Tamura in [51]. M. Ćirić and S. Bogdanović proved (i) $\iff$ (v) in [10], (i) $\iff$ (vii) in [3], and (i) $\iff$ (viii) in [13]. The equivalence of conditions (i) and (vi) has been proved by F. Kmeť in [24], and independently by M. Ćirić and S. Bogdanović in [10]. For more informations on semilattices of Archimedean semigroups the reader is referred to the survey paper of S. Bogdanović and M. Ćirić [5] or its book [4]. For corresponding results concerning π-regular and completely π-regular semigroups we also refer to the papers of L. N. Shevrin [45] and [46], and M. L. Veronesi [57].

The concept of archimedeaness has been generalized many a times. One of its generalizations is the notion of a σ_n-simple semigroup, defined for an arbitrary $n \in \mathbb{Z}^+$ in the following way: the relation σ_n on a semigroup S is defined by

$$a \, \sigma_n \, b \iff \Sigma_n(a) = \Sigma_n(b),$$

and we say that S is *σ_n-simple* if σ_n equals the universal relation on S, or equivalently, if $a \longrightarrow^n b$, for all $a, b \in S$. Clearly, σ_n-simple semigroups are both a generalization of Archimedean semigroups (which are obtained for $n = 1$) and a specialization of semilattice indecomposable semigroups. Semilattices of σ_n-simple semigroups have been investigated by M. Ćirić and S. Bogdanović in [13], where they gave the following result:

Theorem 5.4. (M. Ćirić and S. Bogdanović [13]) *Let $n \in \mathbb{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of σ_n-simple semigroups;*
(ii) *every σ_n-class of S is a subsemigroup;*
(iii) $(\forall a \in S)\ a \, \sigma_n \, a^2$*;*
(iv) $\longrightarrow^n$ *is powerfull;*
(v) $\longrightarrow^n$ *satisfies the cm-property;*
(vi) $\longrightarrow^n$ *is transitive;*
(vii) *for any $a \in S$, $\Sigma_n(a)$ is an ideal of S;*
(viii) $(\forall a, b \in S)\ \Sigma_n(ab) = \Sigma_n(a) \cap \Sigma_n(b)$*;*
(ix) $\sigma_n = \longrightarrow^n \cap (\longrightarrow^n)^{-1}$ *on S.*

Another generalization of Archimedean semigroups has been given by T. Tamura in [55]. Given a nonempty class $\mathfrak{C}$ of semigroups closed under ideals and filters, i.e. which contains all ideals and filters of any semigroup from $\mathfrak{C}$. Further, given a mapping $S \mapsto \pi_S$ which to any semigroup $S \in \mathfrak{C}$ associates a positive quasi-order π_S on S which satisfies the following condition: if $S, T \in \mathfrak{C}$ and $T \subseteq S$, then

$$a \, \pi_T \, b \implies a \, \pi_S \, b.$$

In this case the set $\Pi_{\mathfrak{C}} = \{\pi_S \mid S \in \mathfrak{C}\}$ is called a *positive quasi-order system* on $\mathfrak{C}$. For $S \in \mathfrak{C}$, by π_S^* we denote the relation $\xrightarrow{\pi_S}$, and the set $\Pi_{\mathfrak{C}}^* = \{\pi_S^* \mid S \in \mathfrak{C}\}$ is called a *root system* of $\Pi_{\mathfrak{C}}$. Finally, a semigroup $S \in \mathfrak{C}$ is called $\Pi_{\mathfrak{C}}$*-Archimedean* if π_S^* equals the universal relation on S.

In studying of semilattices of such semigroups, T. Tamura has considered root systems $\Pi_{\mathfrak{C}}^*$ satisfying the condition that any π_S^*, $S \in \mathfrak{C}$, has the *cm*-property. He proved the following theorem:

Theorem 5.5. (T. Tamura [55]) *Let $\Pi^*_{\mathfrak{C}}$ be a root system and $S \in \mathfrak{C}$. Then π^*_S satisfies the cm-property if and only if π^*_S is a half-congruence.*

He also proved the following

Theorem 5.6. (T. Tamura [55]) *Let $\Pi^*_{\mathfrak{C}}$ be a root system and $S \in \mathfrak{C}$. Then π^*_S is powerfull if and only if π^*_S is transitive.*

T. Tamura also introduced the following notions: for a root system $\Pi^*_{\mathfrak{C}}$ we say that it is *upper-supporting* if the following condition holds: if $a \in S$ and $I = a\pi^*_S$, then

$$x, y \in I \;\&\; x\,\pi^*_S\,y \implies x\,\pi^*_I\,y.$$

Similarly, we say that $\Pi^*_{\mathfrak{C}}$ is *lower-supporting* if the following condition holds: if $a \in S$ and $F = \pi^*_S a$, then

$$x, y \in F \;\&\; x\,\pi^*_S\,y \implies x\,\pi^*_F\,y.$$

Using these notions, T. Tamura has proved the following theorem:

Theorem 5.7. (T. Tamura [55]) *Let $\Pi^*_{\mathfrak{C}}$ be a root system such that π^*_S is a half-congruence on S for any $S \in \mathfrak{C}$. Then the following conditions are equivalent:*

(i) $\Pi^*_{\mathfrak{C}}$ *is upper-supporting and lower-supporting;*
(ii) *each $\widetilde{\pi^*_S}$-class is a maximal $\Pi_{\mathfrak{C}}$-Archimedean subsemigroup of S, for any $S \in \mathfrak{C}$;*
(iii) *for all $S \in \mathfrak{C}$, $\widetilde{\pi^*_S}$ is the greatest semilattice congruence on S having the property that each its class is a $\Pi_{\mathfrak{C}}$-Archimedean semigroup.*

In a similar way the archimedeaness has been generalized by M. S. Putcha in [35]. Namely, he introduced the following definition: if π is a quasi-order on a semigroup S, then S is called *π-Archimedean* if $a \xrightarrow{\pi} b$, for all $a, b \in S$. Also, an element $a \in S$ is called a *π-idempotent* if $a^n\,\pi\,a\,\pi\,a^n$, for any $n \in \mathbb{Z}^+$. Clearly, if π is positive, then it is enough to require that $a^2\,\pi\,a$.

M. S. Putcha proved in [35] that such notions play a significant role in studying of certain subdirect decompositions of semigroups:

Theorem 5.8. (M. S. Putcha [35]) *A semigroup S does not have a zero and is a subdirect product of countably many nil-semigroups if and only if there exists a positive quasi-order π on S such that S has no π-idempotents and is π-Archimedean.*

As a consequence of this theorem, M. S. Putcha obtained the following:

Theorem 5.9. (M. S. Putcha [35]) *Let S be a subsemigroup of an Archimedean semigroup without intra-regular elements. Then S is a subdirect product of countably many nil-semigroups.*

Parallelly with positive quasi-orders, M. S. Putcha has investigated positive mappings into a poset, defined in the following way: a mapping $\varphi : S \to P$, where S is a semigroup and P is a poset, is called *positive* if $(ab)\varphi \geq a\varphi$ and $(ab)\varphi \geq b\varphi$, for all $a, b \in S$. A connection between these mappings and positive quasi-orders has been given by

Theorem 5.10. (M. S. Putcha [38]) *If φ is a positive mapping of a semigroup S into a poset P, then the relation π on S defined by*

$$a\,\pi\,b \iff a\varphi \leq b\varphi$$

is a positive quasi-order on S.

Moreover, any positive quasi-order on S is obtained in this manner.

In fact, the previous theorem is a consequence of a Birkhoff's theorem given here as Theorem 3.1.

Using such connection between quasi-orders and mappings of a semigroup into a poset, various notions concerning quasi-orders can be translated to the notions concerning the corresponding mappings. So, for a mapping φ of a semigroup S into a poset P we define the following notions: an element $a \in S$ is called a *φ-idempotent* if $a^n\varphi = a\varphi$, for any $n \in \mathbb{Z}^+$, S is called *φ-Archimedean* if for all $a, b \in S$ there exists $n \in \mathbb{Z}^+$ such that $a\varphi \leq a^n\varphi$, and we say that φ is *powerfull* if for all $a, b \in S$,

$$a\varphi \leq b\varphi \implies (\exists n \in \mathbb{Z}^+)\; a^2\varphi \leq b^n\varphi.$$

Using these notions, M. S. Putcha in [35] has proved the following:

Theorem 5.11. (M. S. Putcha [35]) *Let S be a semigroup and φ a positive powerfull mapping of S into a poset P. Let T be a subsemigroup of S contained in a semilattice indecomposable subsemigroup of S. Then the following hold:*

(1) *T is φ-Archimedean.*
(2) *If T contains a φ-idempotent a, then φ attains a maximum on T at a. Moreover the set $I = \{x \in T \mid x\varphi = a\varphi\}$ is an ideal of T and T/I is a nil-semigroup. Also, I consists exactly of all the φ-idempotents of T.*
(3) *If T does not contain a φ-idempotent, then φ does not attain a maximum on T. Moreover, T can be expressed as a subdirect product of countably many nil-semigroups.*

Using this, M. S. Putcha also proved

Theorem 5.12. (M. S. Putcha [35]) *Let S be a semigroup. Then the following are equivalent:*

(i) *there exists a positive powerfull mapping φ of S into a poset P and every φ-idempotent is an ordinary idempotent;*
(ii) *S is a semilattice Y of semigroups S_α, $\alpha \in Y$, such that each S_α is either idempotent-free and a subdirect product of countably many nil-semigroups or else is an ideal extension of a band by a nil-semigroup;*
(iii) *S is a semilattice Y of semilattice indecomposable semigroups S_α, $\alpha \in Y$, such that each S_α is either idempotent-free and a subdirect product of countably many nil-semigroups or else is an ideal extension of a rectangular band by a nil-semigroup.*

6. Chain decompositions viewed from quasi-orders

Chain congruences and chain decompositions can be also studied through quasi-orders. This has been shown by M. Ćirić and S. Bogdanović in [14]. A central place in such a studying is taken by linear quasi-orders, used also by M. S. Putcha in [38]. By a *linear quasi-order* on a set X we mean a quasi-order ξ for which for all $a, b \in X$, $a \,\xi\, b$ or $b \,\xi\, a$. Combining the linearity with the positivity and the *cm*-property, we go to the quasi-orders characterized by the following theorem

Theorem 6.1. (M. Ćirić and S. Bogdanović [14]) *The following conditions for a quasi-order ξ on a semigroup S are equivalent:*

(i) *ξ is positive, linear and it satisfies the cm-property;*
(ii) *ξ is positive and for all $a, b \in S$, $ab \,\xi\, a$ or $ab \,\xi\, b$;*
(iii) *$a\xi$ is a completely prime ideal of S, for each $a \in S$;*
(iv) *$(\forall a, b \in S)\ \xi a \cup \xi b = \xi(ab)$.*

Using this theorem, the authors in [14] obtained the following:

Theorem 6.2. (M. Ćirić and S. Bogdanović [14]) *The poset of positive linear quasi-orders on a semigroup S satisfying the cm-property is isomorphic to the poset of chain congruences on S.*

In terms of lattices K_ξ defined in Section 3, positive linear quasi-orders satisfying the *cm*-property can be also described as follows:

Theorem 6.3. (M. Ćirić and S. Bogdanović [14]) *The following conditions for a positive quasi-order ξ on a semigroup S are equivalent:*

(i) *ξ is linear and it satisfies the cm-property;*
(ii) *ξ satisfies the cm-property and the poset of all completely prime ideals from K_ξ is a chain;*
(iii) *K_ξ consists of completely prime ideals.*

By the previous theorem and Theorem 3.11, the following results follow:

Theorem 6.4. (M. Ćirić and S. Bogdanović [14]) *The poset linear positive quasi-orders on a semigroups S satisfying the cm-property is dually isomorphic to the poset of complete 1-sublattices of $\mathcal{I}d^{\mathbf{cs}}(S)$ consisting of completely prime ideals of S.*

Theorem 6.5. (M. Ćirić and S. Bogdanović [14]) *The poset of chain decompositions of a semigroup S is isomorphic to the poset of complete 1-sublattices of $\mathcal{I}d^{\mathbf{cs}}(S)$ consisting of completely prime ideals of S.*

In view of Theorem 6.3, for any positive linear quasi-order ξ on a semigroup S satisfying the *cm*-property, K_ξ is a chain, and hence its principal elements also form a chain. This gives the following characterization of chain homomorphic images of a semigroup:

Theorem 6.6. (M. Ćirić and S. Bogdanović [14]) *A chain Y is a chain homomorphic image of a semigroup S if and only if it is isomorphic to the principal part of some complete 1-sublattice of $\mathcal{I}d^{\mathbf{cs}}(S)$ consisting of completely prime ideals.*

Using the right cosets of quasi-orders, another characterization of posets of chain congruences has been obtained in [7]. Assume that K'_ξ is a complete 1-sublattice of $\mathcal{P}(S)$ corresponding to a positive linear quasi-order ξ with the *cm*-property on a semigroup S, defined as in (3) of Section 3. By Theorem 3.8, K'_ξ consists of consistent subsets of S. Seeing that for any $A \in K'_\xi$, $A = \cup_{a\in A}\xi a$, then using Theorem 6.1 it is not hard to verify that A is a subsemigroup, and hence a filter of S. In this manner, the authors in [7] have obtained the following theorem:

Theorem 6.7. (S. Bogdanović and M. Ćirić [7]) *The poset of positive linear quasi-orders on a semigroup S satisfying the cm-property is dually isomorphic to the poset of complete 0,1-sublattices of $\mathcal{P}(S)$ consisting of filters of S.*

Corresponding characterizations can be given for posets of chain congruences and chain decompositions, and for chain homomorphic images of a semigroup.

The same methodology has been also applied in studying of so-called ordinal decompositions of a semigroup. Recall that a semigroup S is called an *ordinal sum* of semigroups S_α, $\alpha \in Y$, if Y is a chain and for any $a \in S_\alpha$, $b \in S_\beta$, $\alpha < \beta$ implies $ab = ba = a$. In such a case, the related chain congruence is called an *ordinal sum congruence* on S, the related partition is called an *ordinal decomposition* of S, and the components $S_\alpha, \alpha \in Y$, are called *ordinal components* of S. A semigroup S is called *ordinally indecomposable* if it has no an ordinal decomposition with more than one component. Note that the sum of any two components in an ordinal decomposition of a semigroup can be considered as the ordinal sum of posets (see G. Birkhoff [1, p. 198]), with respect to its partial orders defined by: $a \leq b \iff a = b$ or $ab = ba = a$. Ordinal decompositions of semigroups were first defined and studied by A. M. Kaufman [23], in connection with studying of linearly ordered groups, where they have been called "successively-annihilating sums". After that, they have been studied by a many authors, in connection with various important problems of the theory of semigroups, and they were obtained the name "ordinal sums" (for more informations we refer to [12], [16], [23], [26], [33] and [44]).

E. S. Lyapin proved in [26] that ordinal decompositions of any semigroup S form a complete sublattice of the partition lattice of S, and the components of the greatest ordinal decomposition of S are ordinally indecomposable. A characterization of this lattice has been given by M. Ćirić and S. Bogdanović in [16], using the following notion introduced in this paper: a *strongly prime* ideal of a semigroup S is defined as an ideal P of S having the property that for all $x, y \in S$, $xy = p \in P$ implies that either $x = p$ or $y = p$ or else $x, y \in P$. The set of all strongly prime ideals of a semigroup S, denoted by $\mathcal{I}d^{\mathbf{sp}}(S)$, is a complete 1-sublattice of the lattice $\mathcal{I}d(S)$ of ideals of S. In terms of this lattice, the lattice ordinal decompositions of a semigroup has been described as follows:

Theorem 6.8. (M. Ćirić and S. Bogdanović [16]) *The lattice of ordinal decompositions of a semigroup S is isomorphic to the lattice of complete 1-sublattices of $\mathcal{I}d^{\mathbf{sp}}(S)$.*

7. Quasi-semilattice decompositions of semigroups with zero viewed from quasi-orders

Quasi-semilattice decompositions of semigroups with zero are an analogue of semilattice decompositions. These decompositions were introduced in a recent paper of the authors [17].

Given a partially ordered set Y. For $\alpha, \beta \in Y$, let $\alpha\beta$ denote the meet of α and β, if it exists. A semigroup $S = S^0$ is called a *quasi-semilattice sum* of semigroups S_α, $\alpha \in Y$, if

$$S = \bigcup_{\alpha \in Y} S_\alpha, \qquad S_\alpha \cap S_\beta = 0, \quad \text{for } \alpha \neq \beta,\ \alpha, \beta \in Y,$$

and for all $\alpha, \beta \in Y$ the following holds:

$$\begin{aligned} S_\alpha S_\beta &\subseteq S_{\alpha\beta}, &&\text{if } \alpha\beta \text{ exists} \\ S_\alpha S_\beta &= 0 &&\text{otherwise.} \end{aligned}$$

The related partition of S whose components are 0 and the sets $S_\alpha^\bullet$, $\alpha \in Y$, is called a *quasi-semilattice decomposition* of S.

M. Ćirić and S. Bogdanović investigated quasi-semilattice decompositions through quasi-orders and they proved the following:

Theorem 7.1. (M. Ćirić and S. Bogdanović [17]) *The poset of quasi-semilattice decompositions of a semigroup $S = S^0$ is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property.*

In view of Theorem 2.14, another characterization of the lattice of quasi-semilattice decompositions can be given as follows:

Theorem 7.2. (M. Ćirić and S. Bogdanović [17]) *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ is dually isomorphic to the lattice of left 0-restricted positive quasi-orders on S satisfying the 0-cm-property.*

The idea of studying of positive quasi-orders on a semigroup through certain sublattices of the lattice of its ideals, coming from [14], has been also used in studying of quasi-orders on a semigroup with zero. The authors in [11] defined a *completely 0-semiprime* ideal of a semigroup $S = S^0$ as an ideal A of S for which the set $A^\bullet$ is completely semiprime. The set of all completely 0-semiprime ideals of $S = S^0$, denoted by $\mathcal{I}d^{\mathbf{c0s}}(S)$ is a complete sublattice of $\mathcal{I}d(S)$. A connection similar to the one which was established in [14] between the lower-potent positive quasi-orders and the lattice of completely semiprime ideals, has been also established by the authors in [11] between the left 0-restricted 0-lower-potent positive quasi-orders on a semigroup $S = S^0$ and the lattice $\mathcal{I}d^{\mathbf{c0s}}(S)$. Namely, the following theorem has been proved:

Theorem 7.3. (M. Ćirić and S. Bogdanović [11]) *The lattice of left 0-restricted 0-lower-potent positive quasi-orders on a semigroup $S = S^0$ is dually isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$.*

Of course, the previous theorem can be also stated in terms of 0-restricted 0-lower-potent 0-positive quasi-orders.

The 0-*cm*-property has been connected with the following property of some sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$: We say that an ideal A of a semigroup $S = S^0$ is *completely 0-prime* if the set $A^\bullet$ is completely prime. We say also that a sublattice

K of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfies the *c-0-pi-property* (*completely 0-prime ideal property*) if any element of K can be written as an intersection of some family of completely 0-prime ideals from K, or equivalently, if the set of completely 0-prime ideals from K is meet dense in K. The authors in [11] proved a theorem similar to the one concerning positive quasi-orders satisfying the *cm*-property:

Theorem 7.4. (M. Ćirić and S. Bogdanović [11]) *For any semigroup $S = S^0$, the poset of complete 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi property is a complete lattice and it is dually isomorphic to the lattice of 0-restricted 0-positive quasi-orders on S satisfying the 0-cm-property.*

The corresponding theorem has been stated also for left 0-restricted positive quasi-orders satisfying the 0-*cm*-property.

In view of the preceding results, the lattice of quasi-semilattice decompositions of a semigroup with zero has been alternatively characterized by

Theorem 7.5. (M. Ćirić and S. Bogdanović [17]) *The lattice of quasi-semilattice decompositions of a semigroup $S = S^0$ is isomorphic to the lattice of complete 0,1-sublattices of $\mathcal{I}d^{\mathbf{c0s}}(S)$ satisfying the c-0-pi property.*

At the beginning of this section we said that quasi-semilattice decompositions of semigroups with zero are an analogue of semilattice decompositions. Moreover, quasi-semilattice decompositions can be also treated as a generalization of semilattice decompositions. This is illustrated by the following theorem:

Theorem 7.6. (M. Ćirić and S. Bogdanović [17]) *The lattice of semilattice decompositions of a semigroup S is isomorphic to the lattice of quasi-semilattice decompositions of the semigroup T arising from S by adjoining the zero.*

The previous theorem shows that many results concerning semilattice decompositions can be deduced from the ones concerning quasi-semilattice decompositions.

References

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc, Coll. Publ. Vol. 25, (3rd. edition, 3rd. printing), Providence, 1979.

[2] S. Bogdanović, *Semigroups with a system of subsemigroups*, Inst. of Math., Novi Sad, 1985.

[3] S. Bogdanović and M. Ćirić, *Semigroups in which the radical of every ideal is a subsemigroup*, Zb. rad. Fil. fak. Niš, Ser. Mat. **6** (1992), 129–135.

[4] S. Bogdanović and M. Ćirić, *Semigroups*, Prosveta, Niš, 1993. (in Serbian)

[5] S. Bogdanović and M. Ćirić, *Semilattices of Archimedean semigroups and (completely) π-regular semigroups* I (*A survey*), Filomat (Niš) **7** (1993), 1–40.

[6] S. Bogdanović and M. Ćirić, *A new approach to some greatest decompositions of semigroups* (*A survey*), Southeast Asian Bull. Math. **18** (1994), no. 3, 27–42.

[7] S. Bogdanović and M. Ćirić, *Positive quasi-orders with the common multiple property on a semigroup*, Proc. Math. Conf. in Priština 1994, Lj. Kočinac, Ed., Priština, 1995, pp. 1–6.

[8] S. Bogdanović and M. Ćirić, *Generation of positive lower-potent half-congruences*, SEA Bull. Math. (to appear).

[9] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, 1983.

[10] M. Ćirić and S. Bogdanović, *Decompositions of semigroups induced by identities*, Semigroup Forum **46** (1993), 329–346.

[11] M. Ćirić and S. Bogdanović, *The lattice of positive quasi-orders on a semigroup* II, Facta Univ. (Niš), Ser. Math. Inform. **9** (1994), 7–17.

[12] M. Ćirić and S. Bogdanović, *Theory of greatest decompositions of semigroups* (*A survey*), FILOMAT (Niš) **9:3** (1995), 385–426.

[13] M. Ćirić and S. Bogdanović, *Semilattice decompositions of semigroups*, Semigroup Forum **52** (1996), 119–132.

[14] M. Ćirić and S. Bogdanović, *The lattice of positive quasi-orders on a semigroup*, Israel J. Math. (to appear).

[15] M. Ćirić and S. Bogdanović, *Posets of $\mathfrak{C}$-congruences*, Algebra Universalis (to appear).

[16] M. Ćirić and S. Bogdanović, *Ordinal sums of semigroups*, Univ. Beograd. Publ. Elektr. Fak, Ser. Mat. **7** (1996), 30–33.

[17] M. Ćirić and S. Bogdanović, *Quasi-semilattice decompositions of semigroups with zero* (to appear).

[18] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol 1, Amer. Math. Soc., 1961.

[19] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol 2, Amer. Math. Soc., 1967.

[20] G. Grätzer, *Universal Algebra*, Van Nostrand, Princeton, 1968.

[21] J. M. Howie, *An introduction to semigroup theory*, Acad. Press, New York, 1976.

[22] K. Iséki, *Contribution to the theory of semigroups* IV, Proc. Japan Acad. **32** (1956), 430–435.

[23] A. M. Kaufman, *Successively-annihilating sums of associative systems*, Uch. Zap. Leningrad. Gos. Ped. Inst. **86** (1949), 149–165. (in Russian)

[24] F. Kmeť, *radicals and their left ideal analogues in a semigroup*, Math. Slovaca **38** (1988), 139–145.

[25] W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Math. Ann. **101** (1929), 729–744.

[26] E. S. Lyapin, *Semigroups*, Fizmatgiz, Moscow, 1960, English transl. Amer. Math. Soc. 1968 (2nd edition). (in Russian)

[27] E. S. Lyapin, *Identities of successively-annihilating bands of semigroups*, Izv. Vysh. Uchebn. Zav. Mat. **1 (200)** (1979), 38–45. (in Russian)

[28] N. H. Mc Coy, *Theory of rings*, (7th printing), Mc Millan, New York, 1970.

[29] Y. S. Park, J. P. Kim and M. G. Sohn, *Semiprime ideals in semigroups*, Math. Japonica **33** (1988), no. 2, 269–273.

[30] M. Petrich, *The maximal semilattice decomposition of a semigroup*, Bull. Amer. Math. Soc. **69** (1963), 342–344..

[31] M. Petrich, *The maximal semilattice decomposition of a semigroup*, Math. Zeitschr. **85** (1964), 68–82.

[32] M. Petrich, *Introduction to semigroups*, Merill, Ohio, 1973.

[33] M. Petrich, *Lectures in semigroups*, Akad. Verlag, Berlin, 1977.

[34] M. S. Putcha, *Semilattice decompositions of semigroups*, Semigroup Forum **6** (1973), 12–34.

[35] M. S. Putcha, *Positive quasi-orders on semigroups*, Duke Math. J. **40** (1973), 857–869.

[36] M. S. Putcha, *Minimal sequences in semigroups*, Trans. Amer. Math. Soc. **189** (1974), 93–106.

[37] M. S. Putcha, *The Archimedean graph of the free semigroup*, J. Pure Appl. Algebra **15** (1974), 307–319.

[38] M. S. Putcha, *Paths in graphs and minimal π-sequences in semigroups*, Discrete Math. **11** (1975), 173–185.

[39] M. S. Putcha, *Positive functions from $\mathcal{S}$-indecomposable semigroups into partially ordered sets*, Check. Math. J. **26 (101)** (1976), 161–170.

[40] M. S. Putcha, *The Archimedean graph of a positive function on a semigroup*, Semigroup Forum **12** (1976), 221–232.

[41] M. S. Putcha and A. H. Schoenfeld, *Applications of algebraic and combinatoric techniques to a problem in geometry*, J. Pure Appl. Algebra **7** (1976), 235–237.

[42] B. M. Schein, *On certain classes of semigroups of binary relations*, Sibirsk. Mat. Zhurn. **6** (1965), 616–635. (in Russian)

[43] Š. Schwarz, *On Gallois connections in the theory of characters of commutative semigroups*, Check. Math. J. **4** (1956), 296–313. (in Russian)

[44] L. N. Shevrin, *Semigroups*, in: general Algebra, L. A. Skornyakov (editor), Nauka, Moscow, 1991, pp. 11–191. (in Russian)

[45] L. N. Shevrin, *Theory of epigroups* I, Mat. Sbornik **185** (1994), no. 8, 129–160. (in Russian)

[46] L. N. Shevrin, *Theory of epigroups* II, Mat. Sbornik **185** (1994), no. 9, 153–176. (in Russian)

[47] G. Szász, *Théorie des treillis*, Akad. Kiadó, Budapest, et Dunod, Paris, 1971.

[48] T. Tamura, *The theory of constructions of finite semigroups* I, Osaka Math. J. **8** (1956), 243–261.

[49] T. Tamura, *Operations on binary relations and their applications*, Bull. Amer. Math. Soc. **70** (1964), 113–120.

[50] T. Tamura, *The theory of operations on binary relations*, Trans. Amer. Math. Soc. **120** (1965), 343–358.

[51] T. Tamura, *On Putcha's theorem concerning semilattice of Archimedean semigroups*, Semigroup Forum **4** (1972), 83–86.

[52] T. Tamura, *Note on the greatest semilattice decomposition of semigroups*, Semigroup Forum **4** (1972), 255–261.

[53] T. Tamura, *Semilattice congruences viewed from quasi-orders*, Proc. Amer. Math. Soc. **41** (1973), 75–79.

[54] T. Tamura, *Remark on the smallest semilattice congruence*, Semigroup Forum **5** (1973), 277–282.

[55] T. Tamura, *Quasi-orders, generalized Archimedeaness, semilattice decompositions*, Math. Nachr. **68** (1975), 201–220.

[56] T. Tamura and N. Kimura, *Existence of greatest decomposition of a semigroup*, Kodai Math. Sem. Rep. **7** (1955), 83–84.

[57] M. L. Veronesi, *Sui semigruppi quasi fortemente regolari*, Riv. Mat. Univ. Parma **(4) 10** (1984), 319–329.