Mathematica
Balkanica

New Series Vol. 10, 1996, Fasc. 1

Direct Sums of Nil-Rings and of Rings with Clifford's Multiplicative Semigroups[1]

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In this paper we consider rings whose multiplicative semigroups are nil-extensions of a unions of groups, and we prove that such a ring is a complete direct sum of a nil-ring and a Clifford's ring (i.e. a ring with Clifford's multiplicative semigroup). Some interesting corollaries whenever ring is periodic are also obtained.

1. Introduction and preliminaries

Throughout this paper $\mathbf{Z}^+$ will denote the set of all positive integers. A semigroup S is *π-regular* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^n S a^n$. A semigroup S is *Archimedean* if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$. A semigroup S is *completely Archimedean* if S is Archimedean and has a primitive idempotent.

By $E(S)$ we denote the set of all idempotents of a semigroup (ring) S. If e is an idempotent of a semigroup S, then G_e will denote the maximal subgroup of S with e as its identity and T_e will denote the set $T_e = \{x \in S \mid (\exists n \in \mathbf{Z}^+)\ x^n \in G_e\}$. The same notation we will use in rings (i.e. in multiplicative semigroups of rings).

An element a of a semigroup (ring) S with the zero 0 is *nilpotent* if there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. A semigroup (ring) S is a *nil-semigroup* (*nil-ring*) if all of its elements are nilpotents. If $n \in \mathbf{Z}^+$, then a semigroup (ring) S is *n-nilpotent* if $S^n = \{0\}$. An ideal extension S of a semigroup K is a *nil-extension* (*n-nilpotent extension*) of K if S/K is a nil-semigroup (n-nilpotent semigroup). A subsemigroup K of a semigroup S is a *retract* of S if there exists a homomorphism φ of S onto K such that $a\varphi = a$, for all $a \in K$. Such

[1]Supported by Grant 04M03 of RFNS through Math. Inst. SANU

a homomorphism will be called a *retraction*. An ideal extension S of K is a *retract extension* (or *retractive extension*) of K if K is a retract of S.

By $\mathcal{UG}\circ\mathcal{N}$ we denote the class of all semigroups that are nil-extensions of a union of groups. A semigroup identity $u = v$ is a $\mathcal{UG}\circ\mathcal{N}$*-identity* if every semigroup that satisfies $u = v$ is in $\mathcal{UG}\circ\mathcal{N}$, i.e. if the semigroup variety $[u = v]$ is a subclass of $\mathcal{UG}\circ\mathcal{N}$. All of $\mathcal{UG}\circ\mathcal{N}$-identities were described by Theorem 1 [6].

If R is a ring, $\mathcal{M}R$ will denote the multiplicative semigroup of R. A semigroup S is a *Clifford's semigroup* if it is regular and idempotents of S are central (or, equivalently, if S is a semilattice of groups). A ring R is a *Clifford's ring* if $\mathcal{M}R$ is a Clifford's semigroup. A ring R is a *J-ring* if it satisfies the Jacobson's property, i.e. if for every $a \in R$ there exists $n \in \mathbf{Z}^+, n \geq 2$, such that $a^n = a$.

It is known [8] that a ring R is a *p-ring*, where p is a prime, iff R is isomorphic to a subdirect product of fields of order p. A. Abian and W. A. Mc Worter [1] proved that a commutative ring R whose characteristic is p and $xy^p = x^p y$ holds for all $x, y \in R$ is isomorphic to a direct sum of a p-ring and a nil-ring. M. Petrich [9] described rings in which the identities $axy = axay$ and $xya = xaya$ hold. These rings are direct sums of a Boolean ring and a 3- nilpotent ring. Here we describe rings in which $\mathcal{M}R$ is a nil-extension of a union of groups and rings that satisfies $\mathcal{UG}\circ\mathcal{N}$-identities, which generalize results of [1], [9] and [5].

For undefined notions and notations we refer to [2], [7] and [5].

In the next considerations the following results will be used.

Lemma 1. [3] *Let ρ be a congruence on a π-regular semigroup S. Then every ρ-class of S that is a regular element in S/ρ contains a regular element from S and every ρ-class of S that is an idempotent in S/ρ contains an idempotent from S.*

Lemma 2. [4] *Let S be a nil-extension of a union of groups K. Then every retraction φ of S onto K has the following representation:*

$$x\varphi = xe \qquad \text{if } x \in T_e,\ e \in E(S).$$

Veronesi's theorem. [10] *A semigroup S is a semilattice of completely Archimedean semigroups if and only if S is π-regular and every regular element of S is completely regular.*

Proposition 1. [5] *If R is a ring such that $\mathcal{M}R$ is a semilattice of completely Archimedean semigroups, then R is an extension of a nil-ring by a Clifford's ring.*

2. The main results

Lemma 3. *If R is a ring such that $\mathcal{M}R$ is a nil-extension of a Clifford's semigroup K, then K is a subring of R.*

Proof. Clearly, K is closed under multiplication. Assume that $x, y \in K$. Then $x \in G_e,\ y \in G_f$, for some $e, f \in E(R)$. Assume that $x-y \in T_g$, for some $g \in E(R)$. Since K is an ideal of $\mathcal{M}R$, then

$$u(x-y) = u[(x-y)\varphi],$$

for $u \in \{e, f, ef\}$, and $(x-y)\varphi = (x-y)g$, by Lemma 2. Thus

$$u(x-y) = u(x-y)g,$$

for $u \in \{e, f, ef\}$, so

$$x - ey = xg - eyg, \quad fx - y = fxg - yg, \quad fx - ey = fxg - eyg,$$

since $E(R)$ is a semilattice. Therefore

$$\begin{aligned} x - y &= xg - eyg + ey + fxg - yg - fx \\ &= xg - yg + ey - fx + fx - ey \\ &= xg - yg = (x-y)g \in K. \end{aligned}$$

Thus, K is a subring of R. ■

Theorem 1. *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a nil-extension of a union of groups;*

(ii) *$\mathcal{M}R$ is a nil-extension of a Clifford's semigroup;*

(iii) *R is a direct sum of a nil-ring and a Clifford's ring;*

(iv) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a Clifford's semigroup.*

Proof. (i) ⇒ (ii). This follows by Theorem 1 [5].

(ii) ⇒ (iii). Let $\mathcal{M}R$ be a nil-extension of a Clifford's semigroup K. By Theorem 2.3 [4] we obtain that there exists a retraction φ of $(R, \cdot)$ onto $(K, \cdot)$. By Veronesi's theorem and by Proposition 1 it follows that the set N

of all nilpotents of R is a ring ideal of R and that the multiplicative semigroup of the factor ring $B = R/N$ is a Clifford's semigroup. Let ν be the natural homomorphism of R onto B. Since $\mathcal{M}R$ is π-regular, then by Lemma 1 it follows that for every coset $a \in B$ we can choose a representative, in notation a', such that $a' \in K$ (i.e. we can choose $a' \in K$ such that $(a')\nu = a$). By Everett's theorem (see [5]) we obtain that R is isomorphic to the Everett's sum $E(N; B; \theta; [,]; \langle,\rangle)$, where the triplet $(\theta; [,]; \langle,\rangle)$ is determined by

$$(1)\ \alpha\theta^a = \alpha \cdot a',\ \theta^a\alpha = a' \cdot \alpha,\ \alpha \in N,\ a \in B,$$

$$(2)\ [a,b] = a' + b' - (a+b)',\ a,b \in B,$$

$$(3)\ \langle a,b\rangle = a' \cdot b' - (a \cdot b)',\ a,b \in B,$$

and the addition and the multiplication on $N \times B$ are defined by

$$(\alpha, a) + (\beta, b) = (\alpha + \beta + [a,b], a+b),$$
$$(\alpha, a) \cdot (\beta, b) = (\alpha \cdot \beta + \langle a,b\rangle + \theta^a\beta + \alpha\theta^b, a \cdot b).$$

By Proposition 1 and Lemma 3 it follows that N and K are ideals of R, so for all $a, b \in B,\ \alpha \in N$, we have that

$$\alpha\theta^a = \alpha \cdot a' \in N \cap K = \{0\}, \quad \theta^a\alpha = a' \cdot \alpha \in N \cap K = \{0\},$$
$$[a,b] = a' + b' - (a+b)' \in N \cap K = \{0\},$$
$$\langle a,b\rangle = a' \cdot b' - (a \cdot b)' \in N \cap K = \{0\},$$

so $\theta, [,]$ and $\langle,\rangle$ are zero functions. Thus, R is a direct sum of rings N and B.

(iii) ⇒ (iv) ⇒ (i). This follows immediately. ■

Corollary 1. *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a nil-extension of a union of periodic groups;*

(ii) *$\mathcal{M}R$ is a nil-extension of a semilattice of periodic groups;*

(iii) *R is a direct sum of a nil-ring and a J-ring;*

(iv) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a semilattice of periodic groups.*

Proof. (i) ⇒ (ii). This follows immediately.

(ii) ⇒ (iii). Let (ii) hold. Then by Theorem 1 we obtain that R is a direct sum of a nil-ring N and a Clifford's ring B. Clearly, $\mathcal{M}B$ is a union of periodic groups, so B is a J-ring.

(iii) $\Rightarrow$ (iv). Let R be a direct sum of a nil-ring N and a J-ring B. Then by the Jacobson's "$a^n = a$ theorem" it follows that B is commutative and it is clear that $\mathcal{M}B$ is a union of periodic groups, so $\mathcal{M}B$ is a semilattice of periodic groups.

(iv) $\Rightarrow$ (i). This follows immediately. ■

Corollary 2. [5] *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a nil-extension of a band;*

(ii) *$\mathcal{M}R$ is a nil-extension of a semilattice;*

(iii) *R is a direct sum of a nil-ring and a Boolean ring;*

(iv) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a semilattice.* ■

Corollary 3. *Let R be a ring. Then $\mathcal{M}R$ is an n-nilpotent extension of a union of groups if and only if R is a direct sum of an n-nilpotent ring and a Clifford's ring.* ■

Let

$$u = v \tag{4}$$

be a semigroup identity that contain letters $x_1, x_2, \dots, x_n$. For $i \in \{1, 2, \dots, n\}$ by $|x_i|_u$ ($|x_i|_v$) we denote the number of appearances of the letter x_i in the word u (v), and by p_i we denote the number $p_i = ||x_i|_u - |x_i|_v|$. The identity (4) is *periodic* if some of numbers $p_1, p_2, \dots, p_n$ is greater than 0 [6]. In this case the number

$$p = \gcd(p_1, p_2, \dots, p_n)$$

is the *period* of an identity (4). Every semigroup that satisfies a periodic identity is periodic. By Theorem 1 [6] it follows that every $\mathcal{UG} \circ \mathcal{N}$-identity is periodic.

Lemma 4. (i) *Every group that satisfies the identity of the period p satisfies the identity $x = x^{p+1}$.*

(ii) *Every commutative semigroup that satisfies the identity $x = x^{p+1}$ satisfies every identity of the period p.*

Proof. (i). This follows immediately.

(ii). Let S be a commutative semigroup that satisfies the identity $x = x^{p+1}$, let $u = v$ be an identity as in (4) of the period p. Then it is clear that S is a union of groups, so S satisfies all of identities $x^{l_i} = x^{r_i}$, where $l_i = |x_i|_u$ and $r_i = |x_i|_v$, $i \in \{1, 2, \dots, n\}$, whence S satisfies the identity

$$x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n},$$

so by the commutativity in S it follows that S satisfies $u = v$. ■

Theorem 2. *A ring R satisfies the $\mathcal{UG} \circ \mathcal{N}$-identity* (4) *of the period p if and only if R is a direct sum of a nil-ring that satisfies* (4) *and a nil-ring that satisfies the identity $x = x^{p+1}$.*

P r o o f. Let R satisfy (4). Then $\mathcal{M}R$ is a nil-extension of a union of groups, and by Theorem 1 [6] it follows that subgroups of $\mathcal{M}R$ are periodic. Thus, by Corollary 1 we obtain that R is a direct sum of a nil-ring N and a J-ring B. Clearly N and B satisfy (4). Since $\mathcal{M}B$ is a union of groups and since (4) implies the identity $x = x^{p+1}$ in subgroups of $\mathcal{M}B$, we then have that B satisfies the identity $x = x^{p+1}$.

Conversely, let R be a direct sum of a nil-ring N that satisfies (4) and of a ring B that satisfies the identity $x = x^{p+1}$. By the Jacobson's "$a^n = a$ theorem" it follows that B is commutative, so by Lemma 4, B satisfies (4). Therefore, R satisfies (4). ■

By A_2^+ we denote the free semigroup over an alphabet $A_2 = \{x, y\}$. By the next result we describe one class of identities that implies commutativity in rings.

Corollary 3. *Every ring that satisfies the identity*

$$xy = w,$$

where $w \in A_2^+$ is a word such that $w \notin \{xy^m \mid m \in \mathbf{Z}^+\} \cup \{x^m y \mid m \in \mathbf{Z}^+\}$, is commutative.

P r o o f. This follows since every nil-ring that satisfies the identity $xy = w$ is a null ring and since this identity is either the identity $xy = yx$ or it is a $\mathcal{UG} \circ \mathcal{N}$-identity (by Theorem 1 [6]). ■

Example. Identities of the form $xy = x^m y$ or $xy = xy^m$, $m \in \mathbf{Z}^+$, does not imply commutativity in rings. For example, the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \;\middle|\; a, b \in \mathbf{Z}_2 \right\}$$

is not commutative and it satisfies all of identities $xy = x^m y$, $m \in \mathbf{Z}^+$.

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Received 18.01.1994

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