

BANDS OF MONOIDS

Stojan Bogdanović and Miroslav Ćirić

In this paper a construction of a band of monoids in general case is given.

A.H. Clifford, [1] gave a construction for a semilattice of groups by a transitive system of homomorphisms (called also strong semilattice of groups). B.M. Schein, [3] generalized Clifford's result to the band of unipotent monoids. For some related results see also [4, p.100]. One generalization of Clifford's result is given by M. Yamada, [5]. He considered a construction of a systematic normal band of P-monoids. In this paper we give a construction of a band of monoids in general case.

Let S be a band I of monoids S_i with identities $e_i, i \in I$, and let the following condition holds:

$$ij = i \implies e_i e_j = e_i \wedge ij = j \implies e_i e_j = e_j.$$

Then S is a systematic band I of monoids $S_i, i \in I$, [5].

For the notions and notations which are not defined here we refer to [2] and [4].

The main result of this paper is the following theorem.

Theorem 1. Let I be a band. To each $i \in I$ we associate a monoid S_i with the identity e_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$. Let \leq_1 and \leq_2 be quasiorders on I defined in the following way:

$$i \leq_1 j \iff ji = i, \quad i \leq_2 j \iff ij = i.$$

Let ϕ_{ij} and ψ_{ij} be homomorphisms of S_j into S_i over \leq_1 and \leq_2 , respectively, for which the following properties hold:

(1) for every $i \in I$, ϕ_{ii} and ψ_{ii} are the identical automorphisms of S_i ,

$$(2) \phi_{ij} \circ \phi_{jk}(s_k) = \phi_{ik}(s_k) \phi_{ij}(e_j), \quad i \leq_1 j \leq_1 k,$$

$$(3) \psi_{ij} \circ \psi_{jk}(s_k) = \psi_{ij}(e_j) \psi_{ik}(s_k), \quad i \leq_2 j \leq_2 k.$$

Let (a_{ij}) be an $(I \times I)$ -matrix over $S = \bigcup_{i \in I} S_i$ such that $a_{ij} \in S_{ij}$, $a_{ii} = e_i$ and

$$(4) \phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k} = a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk})$$

for all $i, j, k \in I$. Define a multiplication $*$ on S by:

$$(5) s_i * s_j = \phi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j), \quad s_i \in S_i, s_j \in S_j.$$

Then $(S, *)$ is a band of monoids.

Conversely, every band of monoids can be so constructed.

Proof. Let $s_i \in S_i$, $s_j \in S_j$ and $s_k \in S_k$. Then

$$\begin{aligned} (s_i * s_j) * s_k &= (\phi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j)) * s_k \\ &= \phi_{ijk,ij}(\phi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,ij}(\phi_{ij,i}(s_i))\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,i}(s_i)\phi_{ijk,ij}(e_{ij})\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,i}(s_i)\phi_{ijk,ij}(e_{ij}a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,i}(s_i)(\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k})\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,i}(s_i)(a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk}))\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk}e_{jk})\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk})\psi_{ijk,jk}(e_{jk})\psi_{ijk,k}(s_k) \\ &= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk})\psi_{ijk,jk}(\psi_{jk,k}(s_k)) \\ &= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk}\psi_{jk,k}(s_k)) \\ &= s_i * (\phi_{jk,j}(s_j)a_{jk}\psi_{jk,k}(s_k)) \\ &= s_i * (s_j * s_k). \end{aligned}$$

Hence, $(S, *)$ is a semigroup. It is clear that S is a band I of monoids S_i , $i \in I$.

Conversely, let S be a band I of monoids S_i , $i \in I$. By e_i we denote the identity of S_i , $i \in I$. We define mappings

$$\phi_{ij}(s_j) = s_j e_i, \quad i \leq_1 j, \quad \psi_{ij}(s_j) = e_i s_j, \quad i \leq_2 j.$$

Then

$$\begin{aligned} \phi_{ij}(s_j t_j) &= s_j t_j e_i = s_j t_j e_{ji}, \quad (\text{since } i = j_1) \\ &= s_j e_{ji} t_j e_{ji} = s_j e_i t_j e_i = \phi_{ij}(s_j) \phi_{ij}(t_j). \end{aligned}$$

Thus ϕ_{ij} is a homomorphism of S_j into S_i . Let $i \leq j \leq k$. Then

$$\begin{aligned}\phi_{ij} \circ \phi_{jk}(s_k) &= \phi_{ij}(s_k e_j) = (s_k e_j) e_i = s_k (e_j e_i) = \\ &= s_k (e_i (e_j e_i)) = (s_k e_i) (e_j e_i) = \\ &= \phi_{ik}(s_k) \phi_{ij}(e_j).\end{aligned}$$

Therefore, the condition (2) holds. In a similar way we prove that ψ_{ij} is a homomorphism and that (3) holds. It is clear that $a_{ij} = e_i e_j e S_{ij}$ and that $a_{ii} = e_i$, for all $i, j \in I$. Now, we prove that the condition (4) holds. Indeed,

$$\begin{aligned}\phi_{ijk,ij}(a_{ij} \psi_{ij,j}(s_j)) a_{ij,k} &= \phi_{ijk,ij}(a_{ij} e_{ij} s_j) a_{ij,k} = \\ &= a_{ij} e_{ij} s_j e_{ijk} a_{ij,k} = a_{ij} s_j a_{ij,k} = e_i e_j s_j e_{ij} e_k = e_i s_j e_k = \\ &= e_i e_j k s_j e_k = a_{i,jk} s_j e_k = a_{i,jk} s_j e_j e_k = a_{i,jk} s_j a_{jk} = \\ &= a_{i,jk} e_{ijk} s_j a_{jk} = a_{i,jk} e_{ijk} s_j e_{jk} a_{jk} = a_{i,jk} \psi_{ijk,jk}(s_j e_{jk} a_{jk}) = \\ &= a_{i,jk} \psi_{ijk,jk}(\phi_{jk,j}(s_j) a_{jk}).\end{aligned}$$

Finally,

$$\begin{aligned}s_i s_j &= s_i e_i e_j s_j = s_i e_{ij} e_i e_j e_{ij} s_j = s_i e_{ij} a_{ij} e_{ij} s_j = \\ &= \phi_{ij,i}(s_i) a_{ij} \psi_{ij,j}(s_j) (=s_i * s_j). \quad ||\end{aligned}$$

Theorem 2. Let I be a semilattice. To each $i \in I$ we associate a monoid S_i with the identity e_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$. For each pair i, j of elements of I such that $i \leq j$, let $\phi_{ij}: S_j \rightarrow S_i$ be a homomorphism for which:

(6) ϕ_{ii} is the identical automorphism of S_i for each $i \in I$,

(7) $\phi_{ij} \circ \phi_{jk}(s_k) = \phi_{ik}(s_k) \phi_{ij}(e_j) = \phi_{ij}(e_j) \phi_{ik}(s_k)$,

for every $i, j, k \in I$ such that $i \leq j \leq k$. Let (a_{ij}) be an $(I \times I)$ -matrix over $S = \bigcup_{i \in I} S_i$ such that $a_{ij} \in S_{ij}$, $a_{ii} = e_i$ and

(8) $\phi_{ijk,ij}(a_{ij} \phi_{ij,j}(s_j)) a_{ij,k} = a_{i,jk} \phi_{ijk,jk}(\phi_{jk,j}(s_j) a_{jk})$.

Define a multiplication $*$ on S by:

$$s_i * s_j = \phi_{ij,i}(s_i) a_{ij} \phi_{ij,j}(s_j), \quad s_i \in S_i, \quad s_j \in S_j).$$

Then $(S, *)$ is a semilattice of monoids.

Conversely, every semilattice of monoids can be so constructed.

Proof. Let S be a semilattice I of monoids S_i and let e_i be the identity of S_i , $i \in I$. For $i, j \in I$, $i \leq j$ we define the following mappings:

$$\phi_{ij}(s_j) = s_j e_i, \quad \psi_{ij}(s_j) = e_i s_j.$$

Then

$$\begin{aligned} \phi_{ij}(s_j) &= \phi_{ij,j}(s_j) = s_j e_{ij} = e_{ij} s_j e_{ij} = \\ &= e_{ij} s_j = \psi_{ij,j}(s_j) = \psi_{ij}(s_j) \end{aligned}$$

if $i \leq j$. Thus

$$(9) \quad \phi_{ij}(s_j) = \psi_{ij}(s_j), \quad i \leq j.$$

By Theorem 1. we have that the conditions (2) and (3) hold and by (9) it follows that

$$\begin{aligned} \phi_{ik}(s_k) \phi_{ij}(e_j) &= \phi_{ij} \circ \phi_{jk}(s_k) = \psi_{ij} \circ \psi_{jk}(s_k) = \\ &= \psi_{ij}(e_j) \psi_{ik}(s_k) = \phi_{ij}(e_j) \phi_{ik}(s_k). \end{aligned}$$

Therefore, the condition (7) holds. The remains of the proof follows by Theorem 1. and by (9).

The converse is similar as in the proof of Theorem 1. ||

Theorem 3. Let I be a band. To each $i \in I$ we associate a monoid S_i with the identity e_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$. Let ϕ_{ij} and ψ_{ij} be homomorphisms for which the conditions (1)-(4) holds and

$$\phi_{ij}(e_j) = e_i, \quad \psi_{ij}(e_j) = e_i.$$

Let (a_{ij}) be an $(I \times I)$ -matrix over $S = \bigcup_{i \in I} S_i$ such that $a_{ij} \in S_{ij}$, $a_{ii} = e_i$ and

$$ij = i \implies a_{ij} = a_{ii} \wedge ij = j \implies a_{ij} = a_{jj}.$$

Define a multiplication $*$ on S by (5). Then $(S, *)$ is a systematic band.

Conversely, every systematic band can be so constructed. ||

R E F E R E N C E S

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TRAKE MONOIDA

S. Bogdanović, M. Ćirić

R e z i m e

U radu je dat opis trake monoida u opštem slučaju.

Faculty of Economics
18000 Niš
Trg JNA 11
Yugoslavia

Faculty of Mechanical Engineering
18000 Niš
Beogradska 14
Yugoslavia