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Decompositions of automata and reversible states

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Abstract. A state a of an automaton A is reversible if A can return to a from any state in which it can come from a. Automata whose every state is reversible were studied by Glushkov, Thierrin, Gécseg and Thierrin, Ćirić and Bogdanović, Ćirić, Bogdanović and Petković and others. Here we study some general properties of reversible states, certain generalizations of reversibility and decompositions of automata connected to reversible states. More exactly, we study various extensions of reversible automata by trap-connected automata, subdirect product decompositions of automata without reversible states and some related direct sum decompositions.

1. Introduction and preliminaries

In this paper we investigate some general properties of reversible states, certain generalizations of reversibility and decompositions of automata connected to reversible states. In Section 2 we study extensions and retractive extensions of reversible automata by trap-connected automata, as well as extensions of strongly connected automata by trap-connected automata. We also characterize automata without reversible states as automata which do not have a trap and are subdirect products of trap-connected automata. In Section 3 we introduce the notions of a π-reversible state and a π-connected automaton, and in the case when the input alphabet is countable we prove that an automaton is π-connected without π-reversible states if and only if it does not have a trap and it is a subdirect product of countably many trap-connected automata. In Section 4 we study quasi-orders on an automaton satisfying the so-called quadrangle property. We show

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how such quasi-orders determine direct sum decompositions and using this result we describe direct sums whose components are among the automata considered in the previous sections. The last theorem of the paper describes direct sums of connected automata.

Automata considered throughout this paper will be automata without outputs in the sense of the definition from the book by GÉCSEG and PEÁK [6]. It is well known that automata without outputs, with the input alphabet X, can be considered as unary algebras of type indexed by X, so the notions such as a *congruence*, *homomorphism*, *generating set* etc., will have their usual algebraic meanings. In order to simplify notation, an automaton with the state set A will be also denoted by the same letter A. For any considered automaton A, its input alphabet is denoted by X, and the free monoid over X, the input monoid of A, is denoted by X^*. Under action of an input word $u \in X^*$, the automaton A goes from a state a into the state that is denoted by au.

A subset H of an automaton A is a *subautomaton* of A if for $a \in A$ and $u \in X^*$, $a \in H$ implies $au \in H$, and it is a *dual subautomaton* of A if for $a \in A$ and $u \in X^*$, $au \in H$ implies $a \in H$. For a subset H of A, the set $S(H) = \{b \in A \,|\, (\exists a \in H)(\exists u \in X^*)\ au = b\}$ is the least subautomaton of A containing H, called the *subautomaton generated by* H, whereas $D(H) = \{b \in A \,|\, (\exists a \in H)(\exists u \in X^*)\ bu = a\}$ is the least dual subautomaton of A containing H, called the *dual subautomaton generated by* H. In the case $H = \{a\}$ we write simply $S(a)$ and $D(a)$, and $S(a)$ is called the *monogenic* subautomaton generated by the state a.

Let L be a lattice with the least element 0. An element $a \in L \setminus \{0\}$ is an *atom* in L if there is no $x \in L$ such that $0 < x < a$. We say that L is *atomic* if for any $b \in L$ there exists an atom a such that $0 < a \leq b$, and that it is *atomistic* if any element $b \in L \setminus \{0\}$ is the join of some set of atoms of L. The lattice of all subautomata of an automaton A, including the empty subautomaton, is denoted by $\mathrm{Sub}(A)$. The atoms in $\mathrm{Sub}(A)$ are called *minimal subautomata* of A. If there is the least nonempty subautomaton of A, then it is called the *kernel* of A.

In analogy with the corresponding notions in number theory and semigroup theory, the relation $|$ on an automaton A defined by $a \,|\, b$ if and only if $b = au$, for some $u \in X^*$, is called the *division relation* on A [4]. It is easy to check that it is a quasi-order on A, where by a *quasi-order* we mean a reflexive and transitive relation.

A state a of an automaton A is called a *trap* if $au = a$, for every $u \in X^*$. An automaton A is *connected* if for all $a, b \in A$ there exist $u, v \in X^*$ such that $au = bv$, and a connected automaton with a trap is called *trap-connected*. In other words, A is trap-connected if and only if it has a trap a_0 and for every $a \in A$ there exists $u \in X^*$ such that $au = a_0$. If A satisfies a more restrictive condition that it has a trap a_0 and there exists $u \in X^*$ such that $au = a_0$, for every $a \in A$, then it is called *trap-directable*. An automaton A is *strongly connected* if it has no a subautomaton different than the whole A and the empty subautomaton, i.e. if for all $a, b \in A$ there exists $u \in X^*$ such that $au = b$. If any monogenic subautomaton of A is connected, then A is called *locally connected*.

If B is a subautomaton of an automaton A, then the congruence relation ϱ_B on A, defined by $(a, b) \in \varrho_B$ if and only if $a = b$ or $a, b \in B$, is called the *Rees congruence* on A determined by B. The factor automaton A/ϱ_B is usually denoted by A/B. We say that an automaton A is an *extension* of an automaton B by an automaton C if B is a subautomaton of A and the factor automaton A/B is isomorphic to C.

An automaton A is the *direct sum* of its subautomata A_α, $\alpha \in Y$, in notation $A = \sum_{\alpha \in Y} A_\alpha$, if $A = \bigcup_{\alpha \in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \varnothing$, for $\alpha \neq \beta$. The equivalence relation on A whose classes are different A_α, $\alpha \in Y$, is a congruence relation on A and it is called a *direct sum congruence* on A. If the universal (full) relation on A is the only direct sum congruence on A, then A is said to be *direct sum indecomposable*.

For undefined notions and notation we refer to the books by Gécseg and Peák [6], Birkhoff [1] and Saliĭ [9].

2. The reversible part of an automaton

A state a of an automaton A will be called *reversible* if for any input word $u \in X^*$ there exists an input word $v \in X^*$ such that $auv = a$, that is if the automaton can return to a from any state in which it can come from a. In this section we study the main properties of reversible states of an automaton. First we prove the following lemma.

Lemma 1. *The following conditions for a state a of an automaton A are equivalent:*

(i) *a is a reversible state of A;*

(ii) *$S(a)$ is a strongly connected subautomaton of A;*

(iii) *$S(a)$ is a minimal subautomaton of A;*

(iv) *$S(a) \subseteq D(a)$;*

(v) *$au \,|\, a$, for any $u \in X^*$.*

Proof. The equivalences (i)$\Leftrightarrow$(v) and (ii)$\Leftrightarrow$(iii) are evident.

(i)$\Rightarrow$(iv). If $b \in S(a)$, then $b = au$, for some $u \in X^*$, and by reversibility of a it follows that $bv = auv = a$, for some $v \in X^*$. Thus, $b \in D(a)$, which is to be proved.

(iv)$\Rightarrow$(ii). Let B be a subautomaton of $S(a)$ and $b \in B$. Then $b \in D(a)$, so $bv = a$, for some $v \in X^*$. But now we have that $a = bv \in B$, so we conclude that $S(a) = B$. Hence, $S(a)$ is a strongly connected automaton.

(ii)$\Rightarrow$(v). For any $u \in X^*$, $S(au)$ is a subautomaton of $S(a)$, and since $S(a)$ is strongly connected, then $S(a) = S(au)$. Therefore, $a \in S(au)$, i.e. $au \mid a$. □

The set of all reversible states of an automaton A will be denoted by $R(A)$ and called the *reversible part* of A. An automaton A is called *reversible* if any its state is reversible, i.e. $A = R(A)$. The following is true.

Lemma 2. *If the set $R(A)$ of all reversible states of an automaton A is nonempty, then it is a reversible subautomaton of A.*

Proof. For any $a \in R(A)$ and $u, u' \in X^*$, there is $v \in X^*$ such that $a(uu')v = a$ and so $(au)u'vu = au$. Thus $au \in R(A)$ for every $u \in X^*$. Hence $R(A)$ is a subautomaton of A. It is clear that $R(A)$ is a reversible automaton. □

Note that reversible automata have been studied under various names. Glushkov in [8] called them *invertible*, and Thierrin in [11] and Gécseg and Thierrin in [7] called them *locally transitive*. Various characterizations of these automata were given by Thierrin in [11], Gécseg and Thierrin in [7], Ćirić and Bogdanović in [3], Ćirić, Bogdanović and Petković in [5] and others. We quote here the most interesting characterizations of reversible automata.

Theorem 1. *The following conditions on an automaton A are equivalent:*

(i) *A is a reversible automaton;*

(ii) *A is a direct sum of strongly connected automata;*

(iii) *γ is a direct sum congruence on A;*

(iv) *$S(a) = D(a)$, for every $a \in A$;*

(v) Sub(A) *is a Boolean algebra;*

(vi) Sub(A) *is an atomistic lattice;*

(vii) *the stabilizer* stab(a) *of every state $a \in A$ is generated by a maximal prefix code.*

Note that the equivalence relation γ is defined by $(a,b)\in\gamma$ if an only if $S(a)=S(b)$, which is also equivalent to $D(a)=D(b)$.

The equivalence of the conditions (i) and (ii) was proved by THIERRIN in [11] and by GLUSHKOV in [8], the condition (vii) due to GÉCSEG and THIERRIN [7], whereas the remaining conditions due to ĆIRIĆ and BOGDANOVIĆ [3] and ĆIRIĆ, BOGDANOVIĆ and PETKOVIĆ [5].

If an automaton A has reversible states, then by Lemma 2, it is an extension of a reversible automaton, but in the general case we do not know anything about the structure of the factor automaton $A/R(A)$. In the next theorem we study a very interesting case when the factor automaton $A/R(A)$ is trap-connected.

Theorem 2. *The following conditions on an automaton A are equivalent:*

(i) *A is an extension of a reversible automaton by a trap-connected automaton;*

(ii) *any monogenic subautomaton of A has a reversible state;*

(iii) $\mathrm{Sub}(A)$ *is an atomic lattice;*

(iv) $(\forall a\in A)(\exists u\in X^*)(\forall v\in X^*)(\exists w\in X^*)\ auvw=au$.

PROOF. (i)⇒(iv). Let A be an extension of a reversible automaton B by a trap-connected automaton and consider any state $a\in A$. Seeing that A/B is trap-connected, then $au\in B$, for some word $u\in X^*$. This means that au is a reversible state, whence it follows that for each $v\in X^*$ there exists $w\in X^*$ such that $auvw=au$.

(iv)⇒(ii). Let $a\in A$ be an arbitrary state. By (iv) we have that there is a word $u\in X^*$ such that au is a reversible state from $S(a)$.

(ii)⇒(i). By Lemma 2, A is an extension of a reversible automaton $R(A)$. Consider an arbitrary state $a\in A$. Then $S(a)$ contains a reversible state au, for some $u\in X^*$, which means that $au\in R(A)$. Therefore, $A/R(A)$ is a trap-connected automaton.

(ii)⇒(iii). Let B be any subautomaton of A. Then B has a reversible state a and by Lemma 1, $S(a)$ is a strongly connected subautomaton of B, that is every subautomaton of A contains a strongly connected subautomaton of A. According to SALIĬ [9], the atoms in $\mathrm{Sub}(A)$ are exactly all strongly connected subautomata of A, so we conclude that $\mathrm{Sub}(A)$ is an atomic lattice.

(iii)⇒(ii). If $\mathrm{Sub}(A)$ is an atomic lattice, then every subautomaton B of A contains a strongly connected subautomaton C, and by Lemma 1 we have that any state $c\in C$ is a reversible state of B. □

The previous theorem gives an important consequence concerning finite automata.

Theorem 3. *Every finite automaton A can be uniquely represented as an extension of a reversible automaton by a trap-directable automaton.*

Proof. It is well known that any finite lattice is atomic. Therefore, if A is a finite automaton, then $\mathrm{Sub}(A)$ is atomic, so by Theorem 2, A is an extension of a reversible automaton B by a trap-connected automaton. According to Starke [10], any finite trap-connected automaton is trap-directable. Finally, since $B = R(A)$, then A can be uniquely represented as an extension of a reversible automaton by a trap-directable automaton. □

Extensions of strongly connected by trap-connected automata are described by the following theorem.

Theorem 4. *The following conditions on an automaton A are equivalent:*

(i) *A is an extension of a strongly connected automaton by a trap-connected automaton;*

(ii) *A has a kernel;*

(iii) *A is connected and $R(A) \neq \varnothing$.*

Proof. (i)⇒(ii). Let A be an extension of a strongly connected automaton B by a trap-connected automaton. By Theorem 2, the lattice $\mathrm{Sub}(A)$ is atomic and its atoms are all strongly connected subautomata of A. But, B is the unique strongly connected subautomaton of A, so we conclude that $\mathrm{Sub}(A)$ has the unique atom B, i.e. B is the kernel of A.

(ii)⇒(iii). Let B be the kernel of A. Then B is strongly connected and $B \subseteq R(A)$. Hence, $R(A) \neq \varnothing$. On the other hand, for arbitrary $a, b \in A$ we have that $B \subseteq S(a) \cap S(b)$, which means that $S(a) \cap S(b) \neq \varnothing$, i.e. $au = bv$, for some $u, v \in X^*$. Therefore, A is connected.

(iii)⇒(i). Let (iii) hold. Any subautomaton of a connected automaton is also connected, so we have that $R(A)$ is a connected automaton, and hence, it is direct sum indecomposable. But, being a reversible automaton, $R(A)$ is a direct sum of strongly connected automata, whence we conclude that $R(A)$ is a strongly connected automaton. Let $a \in A$ and $b \in R(A)$. Then there exist $u, v \in X^*$ such that $au = bv$, and $bv \in R(A)$ yields $au \in R(A)$. Thus, we have proved that $A/R(A)$ is a trap-connected automaton, as required. □

An extension A of an automaton B is called a *retractive extension* of B if there exists a homomorphism φ of A onto B such that $b\varphi = b$, for every $b \in B$, and in this case φ is called a *retraction* of A onto B. It was proved by Bogdanović, Ćirić, Petković, Imreh and Steinby in [2] that if an automaton A is a retractive extension of an automaton B by an automaton C, then A is a subdirect product of B and C. This result imposes the following question: Under which conditions the converse of that result holds, i.e. a subdirect product of an automaton B and an automaton with a trap can be represented as a retractive extension of B? One case in which we can give an answer to this question is described by the next theorem.

Theorem 5. *An automaton A is a retractive extension of a reversible automaton B by a trap-connected automaton if and only if it is a subdirect product of B and a trap-connected automaton.*

Proof. The direct part of the theorem follows by the mentioned result of Bogdanović, Ćirić, Petković, Imreh and Steinby [2].

Conversely, let $A \subseteq B \times C$ be a subdirect product of a reversible automaton B and a trap-connected automaton C. Let c_0 be the unique trap of C. We shall prove that $B' = B \times \{c_0\} \subseteq A$. Consider an arbitrary $a \in B$. Since A is a subdirect product of B and C, then there exists $c \in C$ such that $(a,c) \in A$. Because C is trap-connected, we have that $cu = c_0$, for some $u \in X^*$, and because B is reversible, then there exists $v \in X^*$ such that $auv = a$, whence it follows that

$$(a, c_0) = (auv, cuv) = (a, c)uv \in A.$$

Therefore, B' is a subautomaton of A isomorphic to B and the Rees factor automaton A/B' is clearly trap-connected. Finally, the mapping $(a,c) \mapsto (a,c_0)$ is a retraction of A onto B'. □

It is also interesting to investigate automata without reversible states, which is done in the last theorem of this section.

Theorem 6. *The following conditions on an automaton A are equivalent:*

(i) *A does not have any reversible state;*

(ii) *For any $a \in A$, $\bigcap_{u \in X^*} S(au) = \varnothing$;*

(iii) *A does not have a trap and it is a subdirect product of trap-connected automata.*

Proof. (i)⇒(ii). Suppose that (ii) does not hold, i.e that there exists $a \in A$ such that $\bigcap_{u \in X^*} S(au) \neq \varnothing$, and consider an arbitrary $b \in \bigcap_{u \in X^*} S(au)$. By $b \in S(a)$ it follows that $b = au$, for some $u \in X^*$, and for any $v \in X^*$ we have that $b \in S(auv) = S(bv)$, i.e. $bv \,|\, b$. Now, according to Lemma 1, b is a reversible state, which contradicts (i). Hence, we conclude that (ii) holds.

(ii)$\Rightarrow$(iii). If A has a trap a, then $\bigcap_{u\in X^*} S(au) = \{a\}$, which contradicts (ii). Therefore, A does not have a trap.

For any $a \in A$ let $A_a = \{b \in A \,|\, a \notin S(b)\}$. If $b \in A$ and $u \in X^*$, then $S(bu) \subseteq S(b)$, so $a \notin S(b)$ yields $a \notin S(bu)$. Thus, $bu \in A_a$ and we have proved that A_a is a subautomaton of A. By $a \notin A_a$, for any $a \in A$, it follows that $\bigcap_{a\in A} A_a = \varnothing$, so we have that A is a subdirect product of automata A/A_a, $a \in A$. It remains to prove that any A/A_a is trap-connected. For this reason consider an arbitrary $b \in A$. Because $a \notin \bigcap_{u\in X^*} S(bu)$, then there exists $u \in X^*$ such that $a \notin S(bu)$, i.e. $bu \in A_a$, and thus we conclude that A/A_a is indeed a trap-connected automaton.

(iii)$\Rightarrow$(i). Let A do not have a trap and let it be a subdirect product of trap-connected automata A_α, $\alpha \in Y$. Without loss of generality we can assume that $A \subseteq \prod_{\alpha\in Y} A_\alpha$. For any $\alpha \in Y$, let φ_α be the projection homomorphism of A onto A_α. If b is a reversible state of A, then any $b\varphi_\alpha$ is a reversible state of A_α, and since the unique trap of A_α is also the unique reversible state of A_α, then we have that any $b\varphi_\alpha$ is a trap. But, by this it follows that b is a trap of A, which contradicts our starting hypothesis. Therefore, we conclude that A does not have reversible states. □

3. Generalizations of reversibility and connectivity

In the previous section we characterized connected automata with reversible states as extensions of strongly connected automata by trap-connected automata. On the other hand, by Theorem 6, connected automata without reversible states are represented as subdirect products of trap-connected automata. But, a subdirect product of trap-connected automata is not necessarily trap-connected, nor connected, as the following example shows.

Example 1. Let A be an automaton whose set of states is the set $\mathbb{N}^0$ of natural numbers with the zero adjoined, the input alphabet is $X = \{x\}$ and the transition function is defined by

$$kx = \begin{cases} k-1 & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases} .$$

Then A is trap-connected, but it is not trap-directable.

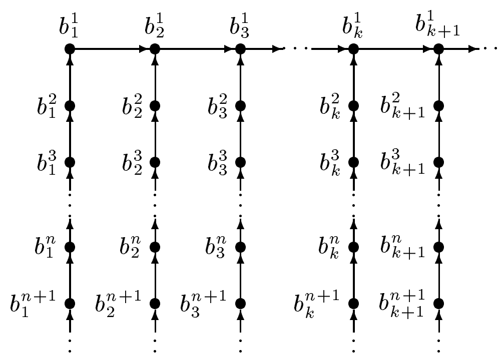
Further, for any $i \in \mathbb{N}$ let $A_i = A$ and let $P = \prod_{i\in\mathbb{N}} A_i$. Then P has a trap, but it is neither trap-connected nor connected. Therefore, the classes of trap-connected automata and connected automata are not closed under direct products and direct powers. Note that these classes are closed under subautomata, homomorphic images and finite direct and subdirect products.

The first of the next two examples demonstrates that there are subdirect products of countably many trap-connected automata which do not have traps and are connected, whereas the second one shows that there are such subdirect products which are not connected.

Example 2. Let P be the automaton constructed in Example 1, and let $B = \{b_k^n \mid k, n \in \mathbb{N}\}$ with $b_k^n \in P$ determined by

$$b_k^n \varphi_i = \begin{cases} 0 & \text{if } i < k \\ n + i - k & \text{if } i \geq k\,, \end{cases}$$

where for any $i \in \mathbb{N}$, φ_i denotes the projection homomorphism of P onto A_i. For any $k, n \in \mathbb{N}$ we have that $b_k^n x = b_k^{n-1}$, for $n \geq 2$, and $b_k^1 x = b_{k+1}^1$, so B is a subautomaton of P represented by the following transition graph



The automaton B does not have a trap and it is a subdirect product of trap-connected automata A_i, $i \in \mathbb{N}$. If $k_1, k_2, n_1, n_2 \in \mathbb{N}$, and, for example, $k_1 \leq k_2$, then $b_{k_1}^{n_1} x^{n_1+k_2-k_1} = b_{k_2+1} = b_{k_2}^{n_2} x^{n_2}$. Therefore, B is a connected automaton.

Example 3. Let P be the automaton constructed in Example 1, and let $C = \{c_k^n \mid k, n \in \mathbb{N}\}$ with $c_k^n \in P$ determined by

$$c_k^n \varphi_i = \begin{cases} 0 & \text{if } i < k \text{ or } i = k + 2j + 1,\ j \in \mathbb{N}^0 \\ n + j & \text{if } i = k + 2j,\ j \in \mathbb{N}^0\,. \end{cases}$$

For any $k, n \in \mathbb{N}$ we have that $c_k^n x = c_k^{n-1}$, for $n \geq 2$, and $c_k^1 x = c_{k+2}^1$. Therefore, C is the direct sum of automata

$$C_1 = \{c_{2j+1}^n \mid n \in \mathbb{N},\ j \in \mathbb{N}^0\} \quad \text{and} \quad C_0 = \{c_{2j}^n \mid n \in \mathbb{N},\ j \in \mathbb{N}\},$$

any of whose is an isomorphic copy of the automaton B from Example 2. Therefore, the automaton C is not connected, but it also does not have a trap and it is a subdirect product of trap-connected automata A_i, $i \in \mathbb{N}$.

Therefore, a subdirect product of countably many trap-connected automata is not necessarily connected. But, under the assumption that the input alphabet X is countable, we prove that subdirect products of countably many trap-connected automata satisfy a condition which can be viewed as a generalization of the connectivity.

As a generalization of the division relation on an automaton, Ćirić, Bogdanović and Petković [4] defined a notion of a *positive quasi-order* on an automaton A as a quasi-order π on A satisfying the condition $a\,\pi\,au$, for every $a \in A$ and $u \in X^*$. On the other hand, if π is a positive quasi-order on an automaton A, the condition (v) of Lemma 1 motivates us to define a state $a \in A$ to be *π-reversible* if $au\,\pi\,a$, for every $u \in X^*$. Finally, for a positive quasi-order π on an automaton A, A is called *π-connected* if for any two states $a, b \in A$ there exists a word $v \in X^*$ such that $a\,\pi\,bv$. It is not hard to check that if π is the division relation on A, then π-connected automata are exactly the connected ones. Therefore, this concept is a generalization of connectivity. Now we are ready to prove the following theorem.

Theorem 7. *Let A be an automaton with a countable input alphabet X. Then the following conditions are equivalent:*

(i) *There exists a positive quasi-order π on A such that A is π-connected and it does not have any π-reversible state.*

(ii) *A does not have a trap and it is a subdirect product of countably many trap-connected automata.*

Proof. (i)⇒(ii). If A had a trap, then it would be also a π-reversible state, so we conclude that A can not have a trap. Fix an arbitrary $a \in A$. Since X is countable, then the free monoid X^* is countable, so the monogenic subautomaton $S(a)$ is also countable. For any $b \in S(a)$ let $A_b = \{c \in A \mid b\,\pi\,c\}$. By the positivity of π it follows that A_b is a subautomaton of A, for each $b \in S(a)$. Let $b \in S(a)$, i.e. $b = au$, for some $u \in X^*$, and consider an arbitrary state $c \in A$. Because A is π-connected, we have that there exists $v \in X^*$ such that $b\,\pi\,cv$, that is $cv \in A_b$. Therefore, A/A_b is a trap-connected automaton, for every $b \in S(a)$.

In order to prove that A is a subdirect product of trap-connected automata A/A_b, $b \in S(a)$, it is enough to prove that $\bigcap_{b\in S(a)} A_b = \varnothing$. For that reason suppose that $c \in \bigcap_{b\in S(a)} A_b$, i.e. $b\,\pi\,c$, for each $b \in S(a)$.

Let $v \in X^*$ be an arbitrary word. Since A is π-connected, then there exists $u \in X^*$ such that $cv\,\pi\, au$, whereas by $au \in S(a)$ it follows that $au\,\pi\, c$, so we obtain that $cv\,\pi\, c$. But this means that c is π-reversible, which contradicts the starting hypothesis. Therefore, we conclude that $\bigcap_{b\in S(a)} A_b = \varnothing$, which is to be proved. Hence, (ii) holds.

(ii)$\Rightarrow$(i). Let $A \subseteq \prod_{i\in\mathbb{N}} A_i$ be a subdirect product of trap-connected automata A_i, $i \in \mathbb{N}$, and let it do not have a trap. For any $i \in \mathbb{N}$ let φ_i denote the projection homomorphism of A onto A_i and let t_i denote the unique trap of A_i. Define a mapping $\eta : A \to \mathbb{N}$ by:

$$a\eta = \min\{i \in \mathbb{N} \,|\, a\varphi_i \neq t_i\}.$$

Because A does not have a trap, then for any $a \in A$ the above set is nonempty and it has the minimal element, so the mapping η is well-defined.

Define now a relation π on A as follows

$$a\,\pi\, b \quad\Leftrightarrow\quad a\eta \leq b\eta \ \text{ and } \ a\varphi_{a\eta} \,|\, b\varphi_{a\eta} \ \text{ in } \ A_{a\eta}.$$

The relation π is evidently reflexive. In order to prove the transitivity, assume that $a\,\pi\, b$ and $b\,\pi\, c$. First we observe that $a\eta \leq b\eta \leq c\eta$. If $a\eta < b\eta$, then also $a\eta < c\eta$ and $c\varphi_{a\eta} = t_{a\eta}$, whence $a\varphi_{a\eta} \,|\, t_{a\eta} = c\varphi_{a\eta}$, so we have that $a\,\pi\, c$. If $a\eta = b\eta$, then $a\varphi_{a\eta} \,|\, b\varphi_{a\eta} = b\varphi_{b\eta} \,|\, c\varphi_{b\eta} = c\varphi_{a\eta}$ in $A_{a\eta} = A_{b\eta}$, so $a\,\pi\, c$. Therefore, π is transitive.

Consider arbitrary $a \in A$ and $u \in X^*$. Because $a\varphi_i = t_i$ implies $(au)\varphi_i = t_i u = t_i$, then $\{i \in \mathbb{N} \,|\, (au)\varphi_i \neq t_i\} \subseteq \{i \in \mathbb{N} \,|\, a\varphi_i \neq t_i\}$ and $a\eta \leq (au)\eta$. Moreover, $a\varphi_{a\eta} \,|\, (a\varphi_{a\eta})u = (au)\varphi_{a\eta}$, so we have that $a\,\pi\, au$. Thus, π is a positive quasi-order.

Let $a, b \in A$. Since the class of all trap-connected automata is closed under finite direct products, we have that there exists a word $v \in X^*$ such that $(bv)\varphi_i = (b\varphi_i)v = t_i$, for each $i \leq a\eta$, so $a\eta < (bv)\eta$ and $a\varphi_{a\eta} \,|\, t_{a\eta} = (bv)\varphi_{a\eta}$, whence $a\,\pi\, bv$. By this it follows that A is a π-connected automaton.

It remains to prove that A does not have any π-reversible state. Suppose that a is a π-reversible state of A. Using again the fact that the class of trap-connected automata is closed under finite direct products, we obtain that there exists a word $u \in X^*$ such that $(au)\varphi_i = (a\varphi_i)u = t_i$, for each $i \leq a\eta$, whence it follows that $a\eta < (au)\eta$. But, by the π-reversibility of a we have that $au\,\pi\, a$ and $(au)\eta \leq a\eta$, so we have obtained a contradiction. Hence, we conclude that A does not have any π-reversible state. $\square$

The next theorem unites Theorems 2 and 7.

Theorem 8. *Let A be an automaton with a countable input alphabet X. Then the following conditions are equivalent:*

(i) *There exists a positive quasi-order π on A such that A is π-connected and every π-reversible state of A is an ordinary reversible state.*

(ii) *A satisfies one of the following two conditions:*
  (1) *A is an extension of a reversible automaton by a trap-connected automaton.*
  (2) *A does not have a trap and it is a subdirect product of countably many trap-connected automata.*

Proof. (i)⇒(ii). Let π be a positive quasi-order on A such that A is π-connected and any π-reversible state of A is an ordinary reversible state. If A does not have any π-reversible state, then by Theorem 7 we have that (2) holds. Suppose now that A have π-reversible states. Let B denote the set of all π-reversible states of A. By the hypothesis, B is also the set of all reversible states of A, so by Lemma 2 we have that B is a reversible subautomaton of A.

It remains to prove that A/B is a trap-connected automaton. Let $a \in A$ and $b \in B$. Since A is π-connected, we have that $b \,\pi\, au$, for some $u \in X^*$. On the other hand, for an arbitrary word $w \in X^*$, in the same way we conclude that there exists $v \in X^*$ such that $(au)w \,\pi\, bv$. But, b is π-reversible, so $bv \,\pi\, b$. Therefore, $(au)w \,\pi\, bv \,\pi\, b \,\pi\, au$, that is $(au)w \,\pi\, au$, whence it follows that au is a π-reversible state, i.e. $au \in B$. This means that A/B is a trap-connected automaton.

(ii)⇒(i). If (2) is satisfied, then by Theorem 7 it follows that (i) holds. Suppose that (1) is satisfied, i.e. A is an extension of a reversible automaton B by a trap-connected automaton. Define a relation π on A by

$$a \,\pi\, b \;\Leftrightarrow\; b \in B \;\text{ or }\; a \,|\, b \text{ in } A.$$

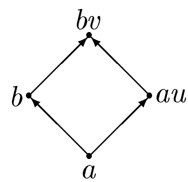
Evidently, π is a reflexive relation. Let $a, b, c \in A$ such that $a \,\pi\, b$ and $b \,\pi\, c$. If $c \in B$, then $a \,\pi\, c$, which is to be proved. Suppose that $c \notin B$. Then $b \,|\, c$ in A, i.e. $c = bv$, for some $v \in X^*$, whence we conclude that $b \notin B$, and hence, $a \,|\, b$ in A. Now we have that $a \,|\, b$ and $b \,|\, c$, so $a \,|\, c$, which yields $a \,\pi\, c$. Hence, we have proved that π is a quasi-order on A. Because $a \,|\, au$, for all $a \in A$ and $u \in X^*$, we have that $a \,\pi\, au$, so π is positive.

Let $a, b \in A$. By the trap-connectivity of A/B it follows that there exists $v \in X^*$ such that $bv \in B$, so we have that $a \,\pi\, bv$. Thus, A is a π-connected automaton.

Finally, let a be an arbitrary π-reversible state of A. Consider an arbitrary word $u \in X^*$. Then $au \, \pi \, a$, because a is π-reversible. If $a \in B$, then it is a reversible state of A, because $B = R(A)$, so $au \mid a$. Otherwise we also have that $au \mid a$. Therefore, we have that $au \mid a$, for any $u \in X^*$, so by Lemma 1 it follows that a is a reversible state of A. Hence, we have proved that any π-reversible state of A is an ordinary reversible state. This completes the proof of the theorem. □

4. The quadrangle property

In this section we study direct sum decompositions whose components are automata satisfying the conditions of Theorem 8. We use one new notion defined as follows. A positive quasi-order π on an automaton A is said to satisfy the *quadrangle property* if for any $a, b \in A$, $a \, \pi \, b$ implies that for every $u \in X^*$ there exists $v \in X^*$ such that $au \, \pi \, bv$. This is demonstrated by the following figure:



First we prove the following theorem.

Theorem 9. *Let A be an arbitrary automaton. If π is a positive quasi-order on A having the quadrangle property, then A can be represented as the direct sum of automata A_α, $\alpha \in Y$, such that any A_α is direct sum indecomposable and π_α-connected, where π_α is the restriction of π onto A_α.*

Conversely, if A is the direct sum of automata A_α, and for any $\alpha \in Y$, A_α is π_α-connected, then $\pi = \bigcup_{\alpha \in Y} \pi_\alpha$ is a positive quasi-order on A with the quadrangle property.

Proof. Let π be a positive quasi-order π on A having the quadrangle property. It is evident that A can be represented as the direct sum of direct sum indecomposable automata A_α, $\alpha \in Y$.

Fix $\alpha \in Y$ and consider arbitrary $a, b \in A_\alpha$. It is clear that there exists a sequence $\{c_i\}_{i=0}^n \subseteq A$ such that $a = c_0$, $b = c_n$ and $S(c_i) \cap S(c_{i+1}) \neq \varnothing$, for each $i \in [0, n-1]$. In other words, for any $i \in [0, n-1]$ there exist $u_i, v_i \in X^*$ such that $c_i u_i = c_{i+1} v_i$. First we observe that $a \, \pi \, au_0 = c_0 u_0 = c_1 v_0$. Suppose now that for some $i \in [0, n-1]$ there exists $w_i \in X^*$ such that $a \, \pi \, c_i w_i$. By the positivity of π it follows that $c_i \, \pi \, c_i u_i = c_{i+1} v_i$, whereas by the quadrangle property we have that there exists $w'_{i+1} \in X^*$

such that $c_i w_i \,\pi\, c_{i+1} v_i w'_{i+1}$. Therefore, if we set $w_{i+1} = v_i w'_{i+1}$, then we have that $a \,\pi\, c_i w_i \,\pi\, c_{i+1} w_{i+1}$, which is to be proved. Hence, we conclude that for each $i \in [0, n]$ there exists $w_i \in X^*$ such that $a \,\pi\, c_i w_i$, so $a \,\pi\, b w_n$. This means that A_α is a π_α-connected automaton.

Conversely, let A be the direct sum of automata A_α, $\alpha \in Y$, and for each $\alpha \in Y$ let A_α be a π_α-connected automaton, where π_α is a positive quasi-order on A_α. It is easy to verify that $\pi = \bigcup_{\alpha \in Y} \pi_\alpha$ is a positive quasi-order on A. Consider $u \in X^*$ and $a, b \in A$ such that $a \,\pi\, b$. Then $a \,\pi_\alpha\, b$ and hence $a, b, au \in A_\alpha$, for some $\alpha \in Y$. Because A_α is π_α-connected, we have that there exists $v \in X^*$ such that $au \,\pi_\alpha\, bv$. Thus, $au \,\pi\, bv$, so we have proved that π has the quadrangle property. □

Now we are ready to prove the following theorem.

Theorem 10. *Let A be an automaton with a countable input alphabet X. Then the following conditions are equivalent:*

(i) *There exists a positive quasi-order π on A having the quadrangle property such that every π-reversible state of A is an ordinary reversible state.*

(ii) *A is the direct sum of automata A_α, $\alpha \in Y$, such that any A_α is either an extension of a reversible automaton by a trap-connected automaton or it does not have a trap and it is a subdirect product of countably many trap-connected automata.*

(iii) *A is the direct sum of direct sum indecomposable automata A_α, $\alpha \in Y$, such that any A_α is either an extension of a reversible automaton by a trap-connected automaton or it does not have a trap and it is a subdirect product of countably many trap-connected automata.*

Proof. (i)⇒(iii). Let π be a positive quasi-order on A with the quadrangle property and let every π-reversible state of A be an ordinary reversible state. By Theorem 9 we have that A is represented as the direct sum of automata A_α, $\alpha \in Y$, such that any A_α is direct sum indecomposable and π_α-connected, where π_α is the restriction of π onto A_α.

Consider an arbitrary $\alpha \in Y$. By the hypothesis, every π_α-reversible state of A_α is an ordinary reversible state, so by Theorem 8, either A_α is an extension of a reversible automaton by a trap-connected automaton, or A_α does not have a trap and it is a subdirect product of countably many trap-connected automata.

(iii)⇒(ii). This implication is evident.

(ii)⇒(i). Let A be the direct sum of automata A_α, $\alpha \in Y$, such that any A_α is either an extension of a reversible automaton by a trap-connected automaton or it does not have a trap and it is a subdirect product of countably many trap-connected automata. By Theorem 8, for any $\alpha \in Y$ there exists a positive quasi-order π_α on A_α such that A_α is π_α-connected and any π_α-reversible state of A is an ordinary reversible state. On the other hand, by Theorem 9 we have that $\pi = \bigcup_{\alpha \in Y} \pi_\alpha$ is a positive quasi-order on A having the quadrangle property, and it is not hard to verify that any π-reversible state is an ordinary reversible state. □

Finally, we characterize automata on which the division relation has the quadrangle property.

Theorem 11. *The following conditions on an automaton A are equivalent:*

(i) *The division relation on A has the quadrangle property;*

(ii) $(\forall a \in A)(\forall p, q \in X^*)(\exists u, v \in X^*)$ $apu = aqv$;

(iii) *A is locally connected;*

(iv) *A is a direct sum of connected automata;*

(v) *$D(H)$ is a subautomaton of A, for any subautomaton H of A.*

Proof. (i)⇒(iv). This follows by Theorem 9.

(iv)⇒(iii) and (iii)⇒(ii). These implications are evident.

(ii)⇒(i). Let (ii) hold and let $a, b \in A$ such that $a \,|\, b$, i.e. $b = aq$, for some $q \in X^*$. If we are given an arbitrary word $p \in X^*$, then by (ii) it follows that there exist $u, v \in X^*$ such that $apu = aqv$, i.e. $apu = bv$, whence $ap \,|\, bv$.

(i)⇒(v). Let (i) hold and let H be an arbitrary subautomaton H of A. Let $a \in D(H)$ and $u \in X^*$. Then there exists $b \in H$ such that $a \,|\, b$, so by the quadrangle property of the division relation we have that $au \,|\, bv$, for some $v \in X^*$. But $bv \in H$, whence it follows that $au \in D(H)$. Hence, $D(H)$ is a subautomaton of A.

(v)⇒(i). Consider $a, b \in A$ such that $a \,|\, b$. Then $a \in D(S(b))$, and for any $u \in X^*$ we have that $au \in D(S(b))$, because $D(S(b))$ is a subautomaton of A. This means that $au \,|\, c$, for some $c \in S(b)$, and since then $c = bv$, for some $v \in X^*$, we have that $au \,|\, bv$, which is to be proved. □

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