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Rings whose multiplicative semigroups are nil–extensions of a union of groups [1]

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Abstract. In this paper we consider one type of extension of rings called a strong extension of rings. Using this concept we describe rings whose multiplicative semigroups are nil–extensions of a union of groups. Moreover, we consider semigroups and rings in which the following two identities hold: $a\cdot(\prod_{i=1}^{n}x_i)=\prod_{i=1}^{n}(ax_i)$ and $(\prod_{i=1}^{n}x_i)\cdot a=\prod_{i=1}^{n}(x_ia)$.

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1. Introduction and preliminaries

In this section we introduce basic notions, notations and results.

A ring R is an *extension* of a ring A by a ring B if R has an ideal I isomorphic to A and the factor ring R/I is isomorphic to B. Usually, we identify A with I and B with R/I. A ring A is a *direct summand* of a ring R if R is isomorphic to a direct sum $A\oplus B$, for some ring B. Two extensions R and R' of a ring A by a ring B are *equivalent* if there exists an isomorphism φ of R onto R' such that $\varphi(a)=a$ for all $a\in A$ and such that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ \nu\searrow & & \swarrow\nu' \\ & B & \end{array}$$

where ν and ν' are natural homomorphisms of R and R' onto B, respectively. The extension problem of rings is as follows: construct all (nonequivalent) extensions of a ring A by a ring B. A solution to this problem has been given by C.J. Everett [7], by the next theorem.

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EVERETT'S THEOREM [7]. *Let A and B be disjoint rings. Let θ be a function of B onto a set of permutable bitranslations of A, in notation*

$$\theta : a \longmapsto \theta^a \in \Omega(A), \quad a \in B\,,$$

and let $[\;,\;], \langle,\rangle \;: B \times B \longrightarrow A$ be functions such that for all $a, b, c \in B$ the following conditions hold:

(E1) $\theta^a + \theta^b - \theta^{a+b} = \pi_{[a,b]}$;

(E2) $\theta^a \cdot \theta^b - \theta^{ab} = \pi_{\langle a,b\rangle}$;

(E3) $\langle ab, c\rangle + \langle a, b\rangle\theta^c = \langle a, bc\rangle + \theta^a\langle b, c\rangle$;

(E4) $[0,0] = 0$;

(E5) $[a,b] = [b,a]$;

(E6) $[a,b] + [a+b,c] = [a,b+c] + [b,c]$;

(E7) $[a,b]\theta^c + \langle a+b, c\rangle = [ac, bc] + \langle a,c\rangle + \langle b,c\rangle$;

(E8) $\theta^a[b,c] + \langle a, b+c\rangle = [ab, ac] + \langle a,b\rangle + \langle a,c\rangle$.

Define an addition and multiplication on $R = A \times B$ by

(E9) $(\alpha, a) + (\beta, b) = (\alpha + \beta + [a,b], a+b)$,

(E10) $(\alpha, a)\cdot(\beta, b) = (\alpha\beta + \langle a,b\rangle + \theta^a\beta + \alpha\theta^b, ab)$.

Then $(R, +, \cdot)$ is a ring isomorphic to an extension of A by B.

Conversely, every extension of A by B can be so constructed.

DEFINITION 1. A ring constructed as in the Everett's theorem we call an *Everett's sum of rings A and B by the triplet of functions* $(\theta; [,], \langle,\rangle)$ and we denote it by $E(A, B; \theta; [,]; \langle,\rangle)$. The representation of a ring R as an Everett's sum of some rings we call an *Everett's representation of R*.

Many informations about the Everett's theorem we can find in [7], [10], [11], [14] and [17]. There we can see that an Everett's representation $E(A, B; \theta; [,]; \langle,\rangle)$ of some ring R is determined by the choice of a set of representatives of the cosets of A in R. Namely, if for every coset $a \in B$ we choose a representative, in notation a', then the set $\{a' | a \in B\}$ determines the triplet $(\theta; [,]; \langle,\rangle)$ in the following way:

(i) $\alpha\theta^a = \alpha \cdot a'$, $\theta^a\alpha = a' \cdot \alpha$, $\alpha \in A$, $a \in B$;

(ii) $[a,b] = a' + b' - (a+b)'$, $a, b \in B$;

(iii) $\langle a,b\rangle = a' \cdot b' - (a \cdot b)'$, $a, b \in B$.

By $\equiv \pmod{I}$ we denote the congruence on a ring R induced by its ideal I. By $\mathcal{U}(R)$ we denote the *annihilator* of a ring R, i.e. the set

$$\mathcal{U}(R) = \{a \in R \mid aR = Ra = 0\}\,.$$

In the Section 2 we consider the special type of ring extensions, called the strong extensions of rings. Using the construction of a strong extension of a ring and the Putcha's result [16], we describe rings whose multiplicative semigroups are nil–extensions of a union of groups (Theorem 1) and a nil–extensions of a band (Theorem 2). Moreover, we give a method for the construction of nilpotent rings (Theorem 3).

M. Petrich [13], described semigroups and rings satisfying the following identities:

$$axy = axay\,, \quad xya = xaya\,,$$

are called distributive semigroups and rings. In the Section 3 of this paper we generalize Petrich's results. Namely, we describe n–distributive semigroups and rings ($n \geq 2$), i.e. semigroups and rings satisfying the following identities:

$$ax_1x_2\dots x_n = ax_1ax_2\dots ax_n\,,$$

$$x_1x_2\dots x_na = x_1ax_2a\dots x_na\,.$$

By $\mathbf{Z}^+$ we denote the set of all positive integers. A semigroup S is *π–regular* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^nSa^n$. A semigroup S is *completely π–regular* if for every $a \in S$ there exist $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^nxa^n$ and $a^nx = xa^n$. These semigroups were studied by Azumaya [1], Drazin [6] and Munn [12]. In [1] these semigroups were called *strongly π–regular*, but we use the name *completely π–regular*, by analogy with completely regular semigroups. By $Reg(S)$ ($Gr(S)$, $E(S)$) we denote the set of all regular (completely regular, idempotent) elements of a semigroup (ring) S. A semigroup S is *archimedean* if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$. A semigroup S is *completely archimedean* if S is archimedean and completely π–regular. A semigroup S is *orthodox* if S is regular and its idempotents form a subsemigroup of S. A semigroup S is *medial* if $abcd = acbd$ for all $a, b, c, d \in S$.

An element a of a semigroup (ring) S with zero 0 is *nilpotent* if there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. A semigroup (ring) S with zero is a *nil–semigroup* (*nil–ring*) if every element of S is nilpotent. A semigroup (ring) S with zero 0 is *n–nilpotent* if $S^n = 0$. By *nil–extension* of a semigroup we mean an ideal extension by a nil–semigroup. An ideal extension S of a semigroup T is a *retract–extension* (or *retractive extension*) if there exists a homomorphism φ of S onto T such that $\varphi(t) = t$ for all $t \in T$. Such a homomorphism we call a *retraction*. A semigroup S is an *n–inflation* of a semigroup T if T is an ideal of S, $S^{n+1} \subseteq T$ and there exists a retraction of S onto T (i.e. this is a retractive extension by a $(n+1)$–nilpotent semigroup).

For undefined notions and notations we refer to [2], [9], [14] or [15].

In next considerations, the following results will be used:

JACOBSON'S THEOREM [8]. *Suppose that for every element a of a ring R there exists $n \in \mathbf{Z}$ such that $a^n = a$. Then R is commutative.*

Bogdanović lemma [3]. *Let ρ be a congruence on a π–regular semigroup S. Then every ρ–class which is a regular element in S/ρ contains a regular element from S, and every ρ–class which is an idempotent in S/ρ contains an idempotent from S.*

Bogdanović–Milić's theorem [4]. *A semigroup S is an n–inflation of a union of groups if and only if*

$$aS^{n-1}b = a^2S^nb^2$$

for every $a, b \in S$.

Putcha's theorem [16]. *Let R be a ring whose multiplicative semigroup is completely π–regular. Then the following conditions are equivalent:*

(i) *$(R, \cdot)$ is a semilattice of archimedean semigroups;*

(ii) *the set of all nilpotents of R is an ideal of the semigroup $(R, \cdot)$;*

(iii) *the set of all nilpotents of R is an ideal of R.*

Munn's lemma [2]. *Let x be an element of a semigroup S such that x^n lies in a subgroup G of S. If e is an identity of G, then*

(i) *$ex = xe \in G_e$;*

(ii) *$x^m \in G_e$ for every $m \in \mathbf{Z}^+$, $m \geq n$.*

2. Strong extensions of rings

In this section we consider Everett's sums for which θ is a zero homomorphism and, using this, we consider rings whose multiplicative semigroups are nil–extensions of a union of groups.

Proposition 1. *If R is a ring whose multiplicative semigroup is a semilattice of completely archimedean semigroups, then R is an extension of a nil–ring by a ring whose multiplicative semigroup is a semilattice of groups.*

Proof. By Putcha's theorem it follows that the set N of all nilpotent elements of R is an ideal of R. Let $x \in R$. Then there exist $e \in R$ and $n \in \mathbf{Z}^+$ such that $x^n \in G_e$. Now we obtain that

$$\begin{aligned}(x-ex)^{n+1} &= (x-ex)^2(x-ex)^{n-1}\\ &= (x^2-xex-exx+exex)(x-ex)^{n-1}\\ &= (x^2-xex-ex^2+ex^2)(x-ex)^{n-1} \quad \text{(by Munn's lemma)}\\ &= (x^2-xex)(x-ex)^{n-1} = x(x-ex)(x-ex)^{n-1}\\ &= x(x-ex)^n = \ldots = x^n(x-ex)\\ &= x^n x - x^n x = 0\,, \qquad \text{(since } x^n e = x^n\text{)}\end{aligned}$$

so

(1) $$x \equiv ex \pmod{N}.$$

Let ν be a natural homomorphism of R onto $Q = R/N$ and let $a \in Q$. Then $a = \nu(x)$ for some $x \in R$ and there exists $e \in E(R)$ such that $x^n \in G_e$ for some $n \in \mathbf{Z}^+$. Moreover, by (1) it follows that

$$a = \nu(x) = \nu(ex).$$

Since ex is a completely regular element and ν is a homomorphism, then we have that a is completely regular. Therefore, the multiplicative semigroup of the ring Q is completely regular.

Let $x \in R$ and $e \in E(R)$. Then

$$(ex - exe)^2 = (xe - exe)^2 = 0\,,$$

so

$$ex \equiv exe \pmod{N} \quad \text{and} \quad xe \equiv exe \pmod{N}.$$

Hence

(2) $$ex \equiv xe \pmod{N}$$

for all $x \in R$, $e \in E(R)$. Now, let $a \in E(Q)$ and let $b \in Q$. By Bogdanović's lemma it follows that $a = \nu(e)$ for some $e \in E(R)$. Let $b = \nu(x)$, $x \in R$. Then

$$\begin{aligned} ab &= \nu(e)\nu(x) = \nu(ex) = \nu(xe) \quad \text{(by (2))}\\ &= \nu(x)\nu(e) = ba\,. \end{aligned}$$

Therefore, idempotents from Q are central, so $(Q,\cdot)$ is a semilattice of groups. □

COROLLARY 1. *Let R be a ring. Then $(R,\cdot)$ is a completely regular semigroup (union of groups) if and only if $(R,\cdot)$ is a semilattice of groups.*

Now we will consider a special case of Everett's sums in which θ is a zero homomorphism.

DEFINITION 2. An Everett's sum $E(A,B;\theta;[,];\langle,\rangle)$ of rings A and B is a *strong Everett's sum of rings A and B* if θ is a zero homomorphism of B into $\Omega(A)$, i.e. if

$$\theta^a = \pi_0$$

for all $a \in B$. In this case, we denote it by $E(A,B;[,];\langle,\rangle)$, and this representation we call a *strong Everett's representation.*

A ring R is a *strong extension of a ring A by a ring B* if there exists some strong Everett's representation $E(A,B;[,];\langle,\rangle)$ of a ring R.

If $R = E(A,B;[,];\langle,\rangle)$ is a strong Everett's sum of rings A and B, then we have that:
(SE1) $\pi_{[a,b]} = \pi_0$;
(SE2) $\pi_{\langle a,b\rangle} = \pi_0$;
(SE3) $\langle ab,c\rangle = \langle a,bc\rangle$;
(SE4) $[0,0] = 0$;
(SE5) $[a,b] = [b,a]$;
(SE6) $[a,b] + [a+b,c] = [a,b+c] + [b,c]$;
(SE7) $\langle a+b,c\rangle = [ac,bc] + \langle a,c\rangle + \langle b,c\rangle$;
(SE8) $\langle a,b+c\rangle = [ab,ac] + \langle a,b\rangle + \langle a,c\rangle$,
for all $a,b,c \in B$, and the multiplication on R is given by:
(SE10) $(\alpha,a)\cdot(\beta,b) = (\alpha\beta + \langle a,b\rangle, ab)$.

Moreover, it is easy to verify that
(SE10') $\prod_{i=1}^n (\alpha_i,a_i) = \left(\prod_{i=1}^n \alpha_i + \langle\prod_{i=1}^k a_i,\ \prod_{j=k+1}^n a_j\rangle,\ \prod_{i=1}^n a_i\right)$ for all $(\alpha_i,a_i) \in R$, $i = 1,2,\dots,n$; $n \in \mathbf{Z}^+$, $1 \le k \le n-1$ (by (SE2) and (SE3)).

THEOREM 1. *The following conditions on a ring R are equivalent:*

(i) *$(R,\cdot)$ is a nil-extension of a union of groups ;*

(ii) *$(R,\cdot)$ is a nil-extension of a semilattice of groups ;*

(iii) *$(R,\cdot)$ is a retractive nil-extension of a semilattice of groups ;*

(iv) *R is a strong extension of a nil-ring by a ring whose multiplicative semigroup is a semilattice of groups.*

Proof. (i) $\Longrightarrow$ (iv). Let $(R,\cdot)$ be a nil-extension of a union of groups T. Then it is clear that $(R,\cdot)$ is a semilattice of completely archimedean semigroups, so by Proposition 1, it follows that R is an extension of a nil ring N by a ring Q whose multiplicative semigroup is a semilattice of groups. By Everett's theorem it follows that R is isomorphic to Everett's sum $E(A,B;\theta;[,];\langle,\rangle)$ of some rings A and B, where A is isomorphic to N, B is isomorphic to Q and the triplet $(\theta;[,];\langle,\rangle)$ is determined by (1)–(3). Let $a \in B$. By the proof of Proposition 1 it follows that we can choose the representation a' of the coset a such that $a' \in T$. Let us denote by S_0 the archimedean component of a semigroup $(R,\cdot)$ containing the zero 0. Since $(S_0 \cap T,\cdot)$ is a completely simple semigroup with zero, then it is clear that $S_0 \cap T = \{0\}$. Moreover, it is clear that $S_0 = N$. Consider an arbitrary element $\alpha \in N$. Then we have

$$\alpha \cdot a', a' \cdot \alpha \in S_0 \cap T,$$

i.e.,

$$\alpha \cdot a' = a' \cdot \alpha = 0,$$

so by (1) we obtain that θ is a zero homomorphism. Therefore, R is a strong extension of a ring N by a ring Q.

(iv) $\Longrightarrow$ (ii). Let $R = E(A,B;[,];\langle,\rangle)$ be a strong Everett's sum of rings A and B, where A is a nil ring and B is a ring whose multiplicative semigroup is a semilattice of groups. Consider an arbitrary element $x = (\alpha, a) \in R$. Then there exists $n \in \mathbf{Z}^+$, $e \in E(B)$ and $b \in B$ such that $\alpha^n = 0$, $a = ea = ae$ and $ab = ba = e$. Then

$$x^n = \left(\alpha^n + \langle a, a^{n-1}\rangle, a^n\right) = \left(\langle a, a^{n-1}\rangle, a^n\right),$$

so

$$\begin{aligned} x^n \cdot (\langle e,e\rangle, e) &= (\langle a, a^{n-1}\rangle, a^n) \cdot (\langle e,e\rangle, e) \\ &= (\langle a^n, e\rangle, a^n e) = (\langle a, a^{n-1}e\rangle, a^n) \\ &= x^n \qquad (\text{ by (SE3)}). \end{aligned}$$

Hence, $x^n \cdot (\langle e,e\rangle, e) = x^n$. In a similar way we prove that $(\langle e,e\rangle,e)\cdot x^n = x^n$. Moreover,

$$\begin{aligned} x^n \cdot (0, b^n) &= (\langle a, a^{n-1}\rangle, a^n) \cdot (0, b^n) \\ &= (\langle a^n, b^n\rangle, a^n b^n) = (\langle ea^n, b^n\rangle, e) \\ &= (\langle e, a^n b^n\rangle, e) = (\langle e,e\rangle, e). \end{aligned}$$

Hence, $x^n \cdot (0, b^n) = (\langle e,e\rangle, e)$. In a similar way we prove that $(0,b^n)\cdot x^n = (\langle e,e\rangle, e)$. Since $(\langle e,e\rangle, e)^2 = (\langle e,e\rangle, e)$, then we obtain that $x^n \in Gr(R)$, so $(R,\cdot)$ is a completely π-regular semigroup.

Let $(\alpha, a) \in Gr(R)$. Then there exists $(\beta, b) \in R$ such that

$$(\alpha,a)\cdot(\beta,b)\cdot(\alpha,a) = (\alpha,a) \quad \text{and} \quad (\alpha,a)\cdot(\beta,b) = (\beta,b)\cdot(\alpha,a),$$

i.e.,

$$(\alpha\beta\alpha + \langle ab, a\rangle, aba) = (\alpha, a) \quad \text{and} \quad (\alpha\beta + \langle a,b\rangle, ab) = (\beta\alpha + \langle b,a\rangle, ba).$$

Therefore, $\alpha\beta\alpha + \langle ab, a\rangle = \alpha$, $aba = a$, $\alpha\beta + \langle a, b\rangle = \beta\alpha + \langle b, a\rangle$ and $ab = ba$, so

$$\alpha\beta\alpha = \alpha\beta(\alpha\beta\alpha + \langle ab, a\rangle) = (\alpha\beta)^2\alpha\,,$$

whence we obtain that

$$\alpha\beta\alpha = (\alpha\beta)^k\alpha = 0\,, \quad (\text{for all } k \in \mathbf{Z}^+)$$

since $(\alpha\beta)^k = 0$ for some $k \in \mathbf{Z}^+$. Hence $\alpha = \langle ab, a\rangle = \langle e, a\rangle$, where a lies in a maximal subgroup of B with the identity e. Therefore,

$$Gr(R) = \{(\langle e, a\rangle, a) | a \in B\,, \quad e \in E(B) \quad \text{such that} \quad a \in G_e\}\,.$$

Also, it is clear that

$$E(R) = \{(\langle e, e\rangle, e) | e \in E(B)\}\,.$$

Let $(\langle e, a\rangle, a) \in Gr(R)$ and $(\beta, b) \in R$. Then

$$(\langle e, a\rangle, a) \cdot (\beta, b) = (\langle a, b\rangle, ab)\,.$$

Assume that $b \in G_f$ for some $f \in E(B)$. Then $ab \in G_{ef}$, so

$$\langle a, b\rangle = \langle ea, b\rangle = \langle e, ab\rangle = \langle e, efab\rangle = \langle eef, ab\rangle = \langle ef, ab\rangle\,,$$

(by (SE3)). Therefore,

$$Gr(R) \cdot R \subseteq Gr(R)\,.$$

In a similar way we prove that

$$R \cdot Gr(R) \subseteq Gr(R)\,,$$

so $(Gr(R), \cdot)$ is an ideal of the semigroup $(R, \cdot)$.

Let $(\langle e, e\rangle, e) \in E(R)$ and let $(\langle f, b\rangle, b) \in Gr(R)$ ($f \in E(B)$, $b \in G_f$). Since idempotents from B are central, then we obtain that

$$\langle e, b\rangle = \langle ee, b\rangle = \langle e, eb\rangle = \langle e, be\rangle = \langle eb, e\rangle = \langle be, e\rangle = \langle b, ee\rangle = \langle b, e\rangle\,,$$

so

$$(\langle e, e\rangle, e) \cdot (\langle f, b\rangle, b) = (\langle e, b\rangle, eb) = (\langle b, e\rangle, be) = (\langle f, b\rangle, b) \cdot (\langle e, e\rangle, e)\,.$$

Therefore, idempotents from $Gr(R)$ are central, so $(Gr(R), \cdot)$ is a semilattice of groups, i.e., $(R, \cdot)$ is a nil–extension of a semilattice of groups.

(ii) $\Longrightarrow$ (i). This follows immediately.

(ii) $\Longleftrightarrow$ (iii). This follows by Theorem 2.3 [5]. □

COROLLARY 2. *Let R be a ring. Then $(R,\cdot)$ is an n-inflation of a union of groups if and only if R is a strong extension of an $(n+1)$-nilpotent ring by a ring whose multiplicative semigroup is a semilattice of groups.*

DEFINITION 3. Let A be a subring of a ring R. A homomorphism $\varphi : R \longrightarrow A$ is a *retraction* of a ring R onto a ring A if $\varphi(a) = a$ for all $a \in A$. A subring A of a ring R is a *retract* of R if there exists a retraction of R onto A. If A is a retract and an ideal of R, then we say that A is a *retractive ideal of* R and R is a *rectract* (or *rectractive*) *extension of the ring* A.

PROPOSITION 2. *A ring R is a retract extension of a ring A if and only if A is a direct summand of R.*

Proof. Let R be a retract extension of a ring A with the retraction φ. Let $B = R/A$ and let ν be a natural homomorphism of R onto B. Let $R' = A \oplus B$. Define a mapping $\psi : R \longrightarrow R'$ by:

$$\psi(a) = (\varphi(a),\ \nu(a - \varphi(a)), \quad a \in R\,.$$

Let $a, b \in R$. Then

$$\begin{aligned}
\psi(a+b) &= (\varphi(a+b)\,,\ \nu(a+b-\varphi(a+b))) \\
&= (\varphi(a)+\varphi(b)\,,\ \nu(a+b-\varphi(a)-\varphi(b))) \\
&= (\varphi(a)+\varphi(b)\,,\ \nu(a-\varphi(a))+\nu(b-\varphi(b))) \\
&= (\varphi(a)\,,\ \nu(a-\varphi(a)))+(\varphi(b)\,,\ \nu(b-\varphi(b))) \\
&= \psi(a)+\psi(b)\,.
\end{aligned}$$

Since

$$\begin{aligned}
(a-\varphi(a))(b-\varphi(b)) &= ab - a\varphi(b) - \varphi(a)b + \varphi(a)\varphi(b) \\
&= ab - \varphi(a)\varphi(b) - \varphi(a)\varphi(b) + \varphi(a)\varphi(b) \\
&= ab - \varphi(a)\varphi(b) = ab \cdot \varphi(ab)\,,
\end{aligned}$$

then we obtain that

$$\begin{aligned}
\psi(ab) &= (\varphi(ab)\,,\ \nu(ab-\varphi(ab))) \\
&= (\varphi(a)\varphi(b)\,,\ \nu((a-\varphi(a)(b-\varphi(b))) \\
&= (\varphi(a)\varphi(b)\,,\ \nu((a-\varphi(a)\nu(b-\varphi(b))) \\
&= (\varphi(a)\,,\ \nu((a-\varphi(a))\cdot(\varphi(b)\,,\ \nu(b-\varphi(b))) \\
&= \psi(a)\cdot\psi(b)\,.
\end{aligned}$$

Therefore, ψ is a homomorphism.

Let $\psi(a) = \psi(b)$, $a, b \in R$. Then $\nu(a - \varphi(a)) = \nu(b - \varphi(b))$ and $\varphi(a) = \varphi(b)$, so $(a - \varphi(a)) \equiv (b - \varphi(b)) \pmod{A}$, i.e., $(a - \varphi(a)) - (b - \varphi(b)) \in A$. Therefore,

$$\begin{aligned}(a - \varphi(a)) - (b - \varphi(b)) &= \varphi((a - \varphi(a)) - (b - \varphi(b))) \\ &= \varphi(a) - \varphi(a) - \varphi(b) + \varphi(b) = 0\,.\end{aligned}$$

Since $\varphi(a) = \varphi(b)$, then we have

$$0 = a - \varphi(a) - b + \varphi(b) = a - b\,,$$

so $a = b$. Therefore, ψ is one–to–one.

Let $(x, y) \in R$. Then $y \in B$, so there exists $b \in R$ such that $y = \nu(b)$. Since $(b - \varphi(b)) \equiv b(\text{mod} A)$, then $\nu(b - \varphi(b)) = y$. Let $a = x + (b - \varphi(b))$. Then $\varphi(a) = \varphi(x + b - \varphi(b)) = \varphi(x) + \varphi(b) - \varphi(b) = \varphi(x) = x$, since $x \in A$. Moreover,

$$a - \varphi(a) = x + b - \varphi(b) - x = b - \varphi(b)\,,$$

so

$$\nu(a - \varphi(a)) = \nu(b - \varphi(b)) = y\,.$$

Therefore, $(x, y) = (\varphi(a),\ \nu(a - \varphi(a))) = \psi(a)$, so ψ is onto. Hence, ψ is an isomorphism of R onto $R' = A \oplus B$, so A is a direct summand of R.

Conversely, let $R = A \oplus B$. Then A can be identified to the ideal $\{(\alpha, 0) \mid \alpha \in A\}$ of a ring R, so R is an extension of A. If we define the mapping $\varphi : R \longrightarrow A$ by

$$\varphi((\alpha, a)) = (\alpha, 0)\,,\ \ (\alpha, a) \in R\,,$$

then it is easy to see that φ is a retraction of R onto A. □

THEOREM 2. *The following conditions on a ring R are equivalent:*

(i) *$(R, \cdot)$ is a nil–extension of a band;*

(ii) *$(R, \cdot)$ is a nil–extension of a semilattice;*

(iii) *$(R, \cdot)$ is a retractive nil–extension of a semilattice;*

(iv) *R is a strong extension of a nil–ring by a Boolean ring;*

(v) *R is a direct sum of a nil–ring and a Boolean ring.*

Proof. (i) $\Longleftrightarrow$(ii) $\Longleftrightarrow$ (iii). This follows by Theorem 1.

(ii) $\Longrightarrow$ (iv). Let $(R, \cdot)$ be a nil– extension of a semilattice E. By Theorem 1, it follows that R is a strong extension of a nil–ring A by a ring B such that $(B, \cdot)$ is a semilattice of groups, where A is isomorphic to the ring N of all nilpotents of R. Since $Reg(R) = E = E(R)$, then by Proposition 1

(2), we obtain that every ($\equiv$ (mod$N$))–class contains an idempotent, so we obtain that every ($\equiv$ (mod$N$))–class is an idempotent in R/N. Therefore, B is a Boolean ring.

(iv) $\Longrightarrow$ (v). Let R be a strong extension of a nil–ring A by a Boolean ring B. Since B is a Boolean ring, then $2x = 0$ for all $x \in B$. By the proof of Theorem 1 we have

$$Reg(R) = E(R) = \{(\langle e,e\rangle, e) | e \in B\}.$$

Let $e, f \in B$. By (SE7) we obtain

$$0 = \langle 0, f\rangle = \langle e+e, f\rangle = [ef, ef] + \langle e, f\rangle + \langle e, f\rangle.$$

Also, by (SE6) we have

$$\begin{aligned} [e+ef, ef+f] + [e, ef] &= [e, ef+ef+f] + [ef, ef+f] \\ &= [e,f] + [ef, ef+f] \end{aligned}$$

and

$$\begin{aligned} [ef, ef+f] + [ef, f] &= [ef, ef] + [ef+ef, f] \\ &= [ef, ef] + [0, f] = [ef, ef]. \end{aligned}$$

Moreover

$$\begin{aligned} \langle f, e\rangle &= \langle ff, ee\rangle = \langle f, feee\rangle = \langle f, eef\rangle = \langle fe, ef\rangle = \langle ef, ef\rangle \\ &= \langle efe, f\rangle = \langle ef, f\rangle = \langle e, ff\rangle = \langle e, f\rangle, \end{aligned}$$

i.e., $\langle e, f\rangle = \langle f, e\rangle = \langle ef, ef\rangle$.

Now by this and by (SE7)

$$\begin{aligned} &\langle e+f, e+f\rangle = [(e+f)e, (e+f)f] + \langle e+f, e\rangle + \langle e+f, f\rangle \\ = \; & [e+fe, ef+f] + [e, fe] + \langle e,e\rangle + \langle f,e\rangle + [ef, f] + \langle e,f\rangle + \langle f,f\rangle \\ = \; & [e,f] + [ef, ef+f] + [ef, f] + \langle e,f\rangle + \langle e,f\rangle + \langle e,e\rangle + \langle f,f\rangle \\ = \; & [e,f] + [ef, ef] + \langle e,f\rangle + \langle e,f\rangle + \langle e,e\rangle + \langle f,f\rangle \\ = \; & [e,f] + \langle e,e\rangle + \langle f,f\rangle. \end{aligned}$$

Therefore,

$$(\langle e,e\rangle, e) \cdot (\langle f,f\rangle, f) = (\langle e,f\rangle, ef) = (\langle ef, ef\rangle, ef)$$

and

$$\begin{aligned} (\langle e,e\rangle, e) + (\langle f,f\rangle, f) &= (\langle e,e\rangle + \langle f,f\rangle + [e,f], e+f) \\ &= (\langle e+f, e+f\rangle, e+f), \end{aligned}$$

so the mapping $\varphi : R \longrightarrow E(R)$ defined by

$$\varphi((\alpha, e)) = (\langle e, e\rangle, e), \quad (\alpha, e) \in R\,,$$

is a retraction of R onto $E(R)$. Hence, by Proposition 1 we have that $E(R)$ is a direct summand of R, and it is clear that $E(R)$ is a Boolean ring. So, R is a direct sum of a nil–ring and a Boolean ring.

(v) $\Longrightarrow$ (i). This follows immediately. □

Theorem 2 implies the following question:

PROBLEM. *Is every strong extension of a ring A by a ring B equivalent to their direct sum $A \oplus B$?*

We have not the general solution for this problem, but we have solutions for some special cases, as in Theorem 2 and as in following lemma.

LEMMA 1. *Every strong extension of a ring A by a ring B with identity is equivalent to their direct sum $A \oplus B$.*

Proof. Let $R = E(A, B; [,]; \langle,\rangle)$ be a strong Everett's sum of rings A and B, and let 1 be the identity of the ring B. Then by (SE3) and (SE7) (if we put $c = 1$) we obtain

$$\langle ab, 1\rangle = \langle a, b\rangle\,,$$

and

$$\langle a + b, 1\rangle = [a, b] + \langle a, 1\rangle + \langle b, 1\rangle\,,$$

whence it follows that the mapping φ defined by

$$\varphi : (\alpha, a) \longmapsto (\alpha + \langle a, 1\rangle, a)$$

is an isomorphism of $A \oplus B$ onto R. □

Example 1. Let $n \in \mathbf{Z}^+$. Then the ring R of all $n \times n$ upper triangular matrices over a field F is a semilattice of archimedean semigroups (see Example 9 [16]). Nilpotents from R are matrices of the form

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and the set N of all nilpotents from R is an ideal of R and it is an n–nilpotent ring. The ring R/N is isomorphic to F^n. By Lemma 1 it follows that R is not a strong extension of a ring N by a ring F^n.

THEOREM 3. *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then R is an $(n+1)$– nilpotent ring if and only if R is a strong extension of a nil–ring by an n–nilpotent ring.*

Proof. Let R be an $(n+1)$–nilpotent ring, i.e., $R^{n+1}=0$. Let $A=\mathcal{U}(R)$ be an annihilator of a ring R. Then it is easy to verify that A is an ideal of R. Therefore R is isomorphic to some Everett's sum $E(A,B;[,];\langle,\rangle)$ of a ring A and a ring $B=R/A$. Since A is an annihilator of R, then by (i) we obtain that this sum must be chosen to be strong. So, R is a strong extension of a nil–ring A by a ring B. Finally, let us prove that B is an n–nilpotent ring. Indeed, for all $x_1,x_2,\ldots,x_n\in R$ we have

$$x_1x_2\ldots x_nR\subseteq R^{n+1}=0 \quad \text{and} \quad Rx_1x_2\ldots x_n\subseteq R^{n+1}=0\,,$$

so

$$x_1x_2\ldots x_n\in A\,,$$

thus B is n–nilpotent.

Conversely, let $R=E(A,B;[,];\langle,\rangle)$ be a strong Everett's sum of a nil–ring A and an n–nilpotent ring B. Let $(\alpha_i,a_i)\in R$, $i=1,2,\ldots,n+1$. Then

$$\begin{aligned}\prod_{i=1}^{n+1}(\alpha_i,a_i) &= \left(\prod_{i=1}^{n+1}\alpha_i+\langle\prod_{i=1}^{n}a_i,a_{n+1}\rangle,\prod_{i=1}^{n+1}a_i\right)\\ &= (0+\langle 0,a_{n+1}\rangle,0)\\ &= (0,0)\,.\end{aligned}$$

Therefore, R is an $(n+1)$–nilpotent ring. □

3. Generalized distributivity in semigroups and rings

In this section we generalize Petrich's results, [13].

Definition 4. Let $n\in\mathbf{Z}^+$, $n\geq 2$. A semigroup S is *n–distributive* if

$$a\cdot\left(\prod_{i=1}^{n}x_i\right)=\prod_{i=1}^{n}(ax_i) \quad \text{and} \quad \left(\prod_{i=1}^{n}x_i\right)\cdot a=\prod_{i=1}^{n}(x_ia)$$

for all $a,x_1,x_2,\ldots,x_n\in S$. A ring R is *n–distributive* if its multiplicative semigroup is n–distributive.

Lemma 2. *Let S be an n–distributive semigroup. Then S is $(n+k(n-1))$–distributive for all $k\in\mathbf{Z}^+$.*

Lemma 3. *A group G is n–distributive if and only if G is commutative and $a^n=a$ for all $a\in G$.*

Proof. Let G be an n–distributive group with the identity element e. Then for all $a \in G$ we have

$$a = ae = ae^n = (ae)^n = a^n .$$

Let $a, b \in G$. By Lemma 2 we can assume that $n \geq 3$. Then $(ba)^n = ba$, so $(ba)^{n-1} = e$. Therefore

$$ab = (ba)^{n-1}abe = (ba)^{n-2}(ba^2)(be) = b(a^{n-2}a^2e) = ba^n = ba ,$$

so G is commutative.

Conversely, let G be a commutative group and let $a^n = a$ for all $a \in G$. Let $a, x_1, x_2, \dots, x_{n+1} \in G$. Then

$$\prod_{i=1}^{n}(x_ia) = \prod_{i=1}^{n}(ax_i) = a^n\left(\prod_{i=1}^{n}x_i\right) = a\cdot\left(\prod_{i=1}^{n}x_i\right) = \left(\prod_{i=1}^{n}x_i\right)\cdot a ,$$

so G is n–distributive. □

THEOREM 4. *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is regular and n-distributive;*

(ii) *S is medial and $a^n = a$ for all $a \in S$;*

(iii) *S is orthodox and a normal band of n– distributive groups.*

Proof. (i) $\Longrightarrow$ (ii). Let S be n– distributive regular semigroup. Let $a \in S$. Then $a = axa$ for some $x \in S$, so

$$a = axa = a(xa)^n = (axa)^n = a^n .$$

Therefore, S is completely regular, i.e., S is a union of n–distributive groups. Let $a, b, c, d \in S$. Then $a \in G_e$ for some $e \in E(S)$, so

$$\begin{aligned} abcd &= \left(ae^{n-2}b\right)cd = ac(ec)^{n-2}bcd = a(ce)^{n-2}(cb)(cd) \\ &= ace^{n-2}bd = a(ee)^{n-2}(ec)(ebd) = ae(e^{n-2}cbd) \\ &= acbd . \end{aligned}$$

Therefore, S is medial.

(ii) $\Longleftrightarrow$ (iii). This follows by Exercise IV 3.10 [15], Theorem IV 2.6 [15] and by Lemma 3.

(ii) $\Longrightarrow$ (i). Let S be a medial semigroup and let $a^n = a$ for all $a \in S$. Then it is not hard to see that

$$ax_1x_2\dots x_kb = ax_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(k)}b$$

for every $k \in \mathbf{Z}^+$, $k \geq 2$, $a, b, x_1, x_2, \ldots, x_k \in S$ and for every permutation σ of a set $\{1, 2, \ldots, k\}$. Therefore,

$$\prod_{i=1}^{n}(ax_1) = a^n \cdot \left(\prod_{i=1}^{n} x_i\right) = a \cdot \left(\prod_{i=1}^{n} x_i\right)$$

and

$$\prod_{i=1}^{n}(x_i a) = \left(\prod_{i=1}^{n} x_i\right) \cdot a^n = \left(\prod_{i=1}^{n} x_i\right) \cdot a\,,$$

so S is n–distributive. □

THEOREM 5. *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an n–distributive semigroup;*

(ii) *S is an n–inflation of a regular n–distributive semigroup;*

(iii) *for all $a, x_1, x_2, \ldots, x_{n-1},\ b \in S$ and for every permutation σ of the set $\{1, 2, \ldots, n-1\}$ we have*

$$\begin{aligned} ax_1x_2\ldots x_{n-1}b &= (ax_1x_2\ldots x_{n-1}b)^n = \\ &= ax_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n-1)}b\,. \end{aligned}$$

Proof. (i) $\Longrightarrow$ (ii). Let S be an n–distributive semigroup and let $a, x_1, x_2, \ldots, x_{n-1},\ b \in S$. Then

$$\begin{aligned} a \cdot \left(\prod_{i=1}^{n-1} x_i\right) b &= \left(\prod_{i=1}^{n-1}(ax_i)\right) ab = ax_1 \left(\prod_{i=2}^{n-1}(ax_i)\right) ab \\ &= a^2x_1a\left(\prod_{i=2}^{n-1}(ax_ia)\right) b = a^2bx_1ab\left(\prod_{i=2}^{n-1}(ax_iab)\right) \\ &= a^2b(x_1ab)\left(\prod_{i=2}^{n-2}(ax_iab)\right)(ax_{n-1}a)b \\ &= a^2(bx_1ab)\left(\prod_{i=2}^{n-2}(bax_iab)\right)(bax_{n-1}a)b^2\,. \end{aligned}$$

Therefore,

$$aS^{n-1}b = a^2S^nb^2\,,$$

so by Bogdanović–Milić's theorem we have that S is an n–inflation of a union of groups T. It is clear that T is a regular n–distributive semigroup.

(ii) $\Longrightarrow$ (i). Let S be an n–inflation of a regular n–distributive semigroup T. Let φ be a retraction of S onto T and let $a, x_1, x_2, \ldots, x_n \in S$. Then $S^{n+1} \subseteq T$, so

$$\begin{aligned} a \cdot \left(\prod_{i=1}^{n} x_i\right) &= \varphi\left(a \cdot \left(\prod_{i=1}^{n} x_i\right)\right) = \varphi(a) \cdot \left(\prod_{i=1}^{n} \varphi(x_i)\right) = \prod_{i=1}^{n} (\varphi(a)\varphi(x_i)) \\ &= \varphi\left(\prod_{i=1}^{n} (ax_i)\right) = \prod_{i=1}^{n} (ax_i), \end{aligned}$$

and similarly,

$$\left(\prod_{i=1}^{n} x_i\right) \cdot a = \prod_{i=1}^{n} (x_i a).$$

Therefore S is n–distributive.

(ii) $\Longrightarrow$ (iii). Let S be an n–inflation of a regular n–distributive semigroup T and let φ be a retraction of S onto T. Let $a, x_1, x_2, \ldots, x_{n-1},\ b \in S$. Then $S^{n+1} = T$, so by Theorem 4 we obtain that

$$ax_1x_2 \ldots x_{n-1}b = (ax_1x_2 \ldots x_{n-1}b)$$

and

$$\begin{aligned} ax_1x_2 \ldots x_{n-1}b &= \varphi(ax_1x_2 \ldots x_{n-1}b) \\ &= \varphi(a)\varphi(x_1)\varphi(x_2) \ldots \varphi(x_{n-1})\varphi(b) \\ &= \varphi(a)\varphi\left(x_{\sigma(1)}\right)\varphi\left(x_{\sigma(2)}\right) \ldots \varphi\left(x_{\sigma(n-1)}\right)\varphi(b) \\ &= \varphi\left(ax_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(n-1)}b\right) \\ &= ax_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(n-1)}b. \end{aligned}$$

(iii) $\Longrightarrow$ (ii). Suppose that (iii) holds. Then it is clear that

$$ax_1x_2 \ldots x_{n-1}b = (ax_1x_2 \ldots x_{n-1}b)^n = a^n x^n_{\sigma(1)} x^n_{\sigma(2)} \ldots x^n_{\sigma(n-1)} b^n$$

for all $a, x_1, x_2, \ldots, x_{n-1},\ b \in S$. Therefore,

$$aS^{n-1}b = a^2 S^n b^2,$$

so by Bogdanović–Milić's theorem we have that S is an n–inflation of a union of groups T. It is clear that $a^n = a$ for every $a \in T$. Moreover, it is not hard to verify that T is medial. Therefore, by Theorem 4 we have that T is a regular n–distributive semigroup. □

THEOREM 6. *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then a ring R is regular and n–distributive if and only if $a^n = a$ for every $a \in R$.*

Proof. Let $a^n = a$ for every $a \in R$. Then it is clear that R is a regular ring. Moreover, by Jacobson's theorem we obtain that R is commutative, so R is medial. Therefore, by Theorem 4 we have that R is n-distributive.

The converse follows by Theorem 4. □

THEOREM 7. *Let $n \in \mathbf{Z}^+$, $n \geq 2$. Then R is an n- distributive ring if and only if R is a strong extension of an $(n+1)$- nilpotent ring by a regular n-distributive ring.*

Proof. Let $R = E(A, B; [,]; \langle,\rangle)$ be a strong Everett's sum of an $(n+1)$-nilpotent ring A by a regular n-distributive ring B. Let $(\alpha, a), (\alpha_i, a_i) \in R$, $i = 1, 2, \ldots, n$. Since $A^{2n} \subseteq A^{n+1} = 0$, then by (SE10') and by the commutativity in B we have

$$\begin{aligned}
\prod_{i=1}^{n}((\alpha,a)(\alpha_i,a_i)) &= \prod_{i=1}^{n}(\alpha\alpha_i + \langle a,a_i\rangle, aa_i) \\
&= \left(\prod_{i=1}^{n}\alpha\alpha_i + \left\langle\prod_{i=1}^{n-1}aa_i, aa_n\right\rangle, \prod_{i=1}^{n}aa_i\right) \\
&= \left(\left\langle\prod_{i=1}^{n-1}aa_i, aa_n\right\rangle, \prod_{i=1}^{n}aa_i\right) \\
&= \left(\left\langle\left(\prod_{i=1}^{n-1}aa_i\right)a, a_n\right\rangle, a^n\cdot\prod_{i=1}^{n}a_i\right) \\
&= \left(\left\langle a^n\cdot\prod_{i=1}^{n-1}a_i, a_n\right\rangle, a\cdot\prod_{i=1}^{n}a_i\right) \\
&= \left(\left\langle a, \prod_{i=1}^{n}a_i\right\rangle, a\cdot\prod_{i=1}^{n}a_i\right) \\
&= \left(\alpha\prod_{i=1}^{n}\alpha_i + \left\langle a, \prod_{i=1}^{n}a_i\right\rangle, a\cdot\prod_{i=1}^{n}a_i\right) \\
&= (\alpha,a)\cdot\left(\prod_{i=1}^{n}(\alpha_i,a_i)\right).
\end{aligned}$$

Therefore, R is n-distributive.

The converse follows by Theorem 5, Theorem 4 and by Corollary 1. □

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