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Remarks on gamma semigroups and their generalizations*

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Abstract. The authors in [6] established a bijective and isotone correspondence between positive quasi-orders on a semigroup S and complete 1-sublattices of the lattice of ideals of S. Here we use this correspondence to characterize certain kinds of semigroups through the properties of their ideals. We introduce and study Γ_π-semigroups, where π is a positive quasi-order, whose special cases include Γ-semigroups, introduced by M. S. Putcha in [9], semigroups decomposable into a chain of semilattice indecomposable semigroups, π-archimedean semigroups, introduced by M. S. Putcha in [7] and T. Tamura in [13], semilattice indecomposable semigroups, archimedean semigroups and many other important kinds of semigroups.

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M. S. Putcha in [9] defined and studied Γ-semigroups. Special cases of them include archimedean semigroups, full transformation semigroups and others. One of the purposes of this paper is to give an interesting characterization of Γ-semigroups by means of their ideals, as well as a similar characterization by means of their completely semiprime ideals of semigroups decomposable into a chain of semilattice indecomposable semigroups. But we also consider some generalizations of these semigroups. They are defined using the concept of a positive quasi-order that was shown oneself to be very useful in many investigations carried out by T. Tamura, M. S. Putcha and the authors in the papers quoted in the list of references, especially in studying of semilattice decompositions of semigroups.

Starting from a positive quasi-order π on a semigroup, we introduce the notion of a Γ_π-semigroup as a common generalization of Γ-semigroups and of semigroups decomposable into a chain of semilattice indecomposable semigroups. By Theorem 1 we give a characterization of Γ_π-semigroups through the properties of

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pairs of π-closed ideals. As immediate consequences we obtain the corresponding characterizations of Γ-semigroups and of chains of semilattice indecomposable semigroups.

Another interesting type of semigroups that we investigate here are π-archimedean semigroups which are also defined by means of a positive quasi-order π. These semigroups were first defined and studied by M. S. Putcha in [7] and T. Tamura in [13]. We characterize them by Theorem 2, using a similar methodology. As consequences we obtain characterizations of ordinary archimedean semigroups and of semilattice indecomposable semigroups.

Throughout the paper, $\mathbb{N}$ will denote the set of all nonzero natural numbers. For a semigroup S, S^1 denotes the semigroup obtained from S by adjoining the unity.

If ξ is a binary relation on a set A, then for any $a \in A$ we set

$$a\xi = \{x \in A \,|\, a\,\xi\,x\} \quad \text{and} \quad \xi a = \{x \in A \,|\, x\,\xi\,a\},$$

and for any $X \subseteq A$ we set

$$X\xi = \bigcup_{x\in X} x\xi \quad \text{and} \quad \xi X = \bigcup_{x\in X} \xi x.$$

Let ξ be a binary relation on a semigroup S. We say that ξ is *positive* if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$. If for any $a \in S$ and any $n \in \mathbb{N}$ we have that $a^n\,\xi\,a$, then it is called *lower-potent*. A relation ξ is called *linear* if for any $a, b \in S$ either $a\,\xi\,b$ or $b\,\xi\,a$. By a *quasi-order* we mean a reflexive and transitive binary relation. The partially ordered set of all quasi-orders on a set A is a complete lattice and it is denoted by $\mathcal{Q}(A)$.

For a quasi-order π on a semigroup S, the relation $\widehat{\pi}$ on S defined by

$$a\,\widehat{\pi}\,b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a\,\pi\,b^n$$

is the smallest lower-potent relation on S containing π, and it is called *the lower-potent closure* of π. The lower-potent closure of a quasi-order is not necessary a quasi-order.

The *division relation* $|$ on a semigroup S is defined by

$$a \,|\, b \;\Leftrightarrow\; (\exists x, y \in S^1)\; b = xay.$$

It is the smallest positive quasi-order on S, and moreover, the set of all positive quasi-orders on S is the principal dual ideal of the lattice $\mathcal{Q}(S)$ generated by the division relation on S (see [6] and [7]). In the general case, the division relation is not lower-potent and its lower-potent closure is denoted by $\longrightarrow$ (see [11] and [9]). On the other hand, $\longrightarrow$ is not necessary transitive and its transitive closure is the smallest positive lower-potent quasi-order on S and its natural equivalence is the smallest semilattice congruence on S (see [12]). As in the case of positive quasi-orders, the set of all positive lower-potent quasi-orders on

S is the principal dual ideal of $\mathcal{Q}(S)$ generated by the transitive closure of $\longrightarrow$ (see [6]).

Let S be a semigroup. An ideal I of S is called *completely semiprime* if for any $a \in S$ by $a^2 \in I$ it follows $a \in I$. The complete lattice of all ideals of S is denoted by $\mathcal{I}d(S)$, and the set of all completely semiprime ideals of S, which is a complete 1-sublattice of $\mathcal{I}d(S)$, is denoted by $\mathcal{I}d^{\rm cs}(S)$. Recall that by a *complete 1-sublattice* of a complete lattice L we mean any complete sublattice of L containing its unity. If S has the zero 0 and if for any $a \in S$ there exists $n \in \mathbb{N}$ such that $a^n = 0$, then S is called a *nil-semigroup*.

Let π be a positive quasi-order on a semigroup S. An ideal I of S is called *π-closed* if $I\pi = I$, i.e. if $a \in I$ implies $a\pi \subseteq I$, for any $a \in S$. In other words, I is a π-closed ideal of S if and only if it is an ideal of S and a dual ideal (filter) of the quasi-ordered set (S, π). The set of all π-closed ideals of S is denoted by $\mathcal{I}d^{\pi}(S)$. As was proved in [6], $\mathcal{I}d^{\pi}(S)$ is a complete 1-sublattice of $\mathcal{I}d(S)$ and the mapping defined by the rule $\pi \mapsto \mathcal{I}d^{\pi}(S)$ is a dual isomorphism of the lattice of all positive quasi-orders on S onto the lattice of all complete 1-sublattices of $\mathcal{I}d(S)$. It was also proved in [6] that the rule $\pi \mapsto \mathcal{I}d^{\pi}(S)$ determines also a dual isomorphism of the lattice of positive lower-potent quasi-orders on S onto the lattice of all complete 1-sublattices of $\mathcal{I}d^{\rm cs}(S)$. Hence, if π is the division relation on S, then $\mathcal{I}d^{\pi}(S) = \mathcal{I}d(S)$, and if π is the transitive closure of $\longrightarrow$, i.e. the smallest positive lower-potent quasi-order on S, then $\mathcal{I}d^{\pi}(S) = \mathcal{I}d^{\rm cs}(S)$.

M. S. Pucha defined in [9] a semigroup S to be a *Γ-semigroup* if the relation $\longrightarrow$ is linear, i.e. if for any $a, b \in S$ either $a \,|\, b^n$ for some $n \in \mathbb{N}$, or $b \,|\, a^m$ for some $m \in \mathbb{N}$. Here we generalize this notion replacing the division relation in the Putcha's definition by an arbitrary positive quasi-order. Namely, if π is a positive quasi-order on a semigroup S, then we call S a *Γ_π-semigroup* if the lower-potent closure $\widehat{\pi}$ of π is linear, that is if for any $a, b \in S$ either $a \,\pi\, b^n$, for some $n \in \mathbb{N}$, or $b \,\pi\, a^m$, for some $m \in \mathbb{N}$.

The following theorem gives a characterization of Γ_π-semigroups.

THEOREM 1. *Let π be a positive quasi-order on a semigroup S. Then S is a Γ_π-semigroup if and only if for any $I, J \in \mathcal{I}d^{\pi}(S)$ at least one of semigroups $I/(I \cap J)$ and $J/(I \cap J)$ is a nil-semigroup.*

Proof. Let S be a Γ_π-semigroup and let $I, J \in \mathcal{I}d^{\pi}(S)$. Suppose that neither $I/(I \cap J)$ nor $J/(I \cap J)$ is a nil-semigroup. This means that there exists $a \in I \setminus J$ and $b \in J \setminus I$ such that $a^m \notin J$ and $b^n \notin I$, for all $m, n \in \mathbb{N}$. By the hypothesis we have that $a \,\widehat{\pi}\, b$ or $b \,\widehat{\pi}\, a$ in S, that is $a \,\pi\, b^n$, for some $n \in \mathbb{N}$, or $b \,\pi\, a^m$, for some $m \in \mathbb{N}$. This means that

$$b^n \in a\pi \subseteq I \quad \text{or} \quad a^m \in b\pi \subseteq J,$$

so we get a contradiction. Thus, we conclude that at least one of $I/(I \cap J)$ and $J/(I \cap J)$ must be a nil-semigroup.

Conversely, suppose that for any $I, J \in \mathcal{I}d^{\pi}(S)$ at least one of semigroups $I/(I \cap J)$ and $J/(I \cap J)$ is a nil-semigroup. Assume arbitrary $a, b \in S$ and set

$I = a\pi$ and $J = b\pi$. Then $I, J \in \mathcal{I}d^{\pi}(S)$ and at least one of semigroups $I/(I\cap J)$ and $J/(I\cap J)$ is a nil-semigroup. If $J/(I\cap J)$ is a nil-semigroup then we have that

$$b^n \in I\cap J \subseteq I = a\pi \quad \text{i.e.} \quad a\,\pi\,b^n$$

for some $n \in \mathbb{N}$, so $a\,\widehat{\pi}\,b$. Similarly, if $I/(I\cap J)$ is a nil-semigroup, then $b\,\widehat{\pi}\,a$. Therefore, we have proved that S is a Γ_π-semigroup. □

The next two corollaries concern two particular cases of Γ_π-semigroups.

Corollary 2. *A semigroup S is a Γ-semigroup if and only if for any $I, J \in \mathcal{I}d(S)$ at least one of semigroups $I/(I\cap J)$ and $J/(I\cap J)$ is a nil-semigroup.*

Proof. If we assume π to be the division relation on S, that is the smallest positive quasi-order on S, then $\mathcal{I}d^{\pi}(S) = \mathcal{I}d(S)$, by Theorem 1 of [6], and $\widehat{\pi}$ equals the relation $\longrightarrow$, so Γ_π-semigroups are exactly Γ-semigroups. The rest of the proof is an immediate consequence of Theorem 1. □

Corollary 3. *A semigroup S is a chain of semilattice indecomposable semigroups if and only if for any $I, J \in \mathcal{I}d^{\mathrm{cs}}(S)$ at least one of semigroups $I/(I\cap J)$ and $J/(I\cap J)$ is a nil-semigroup.*

Proof. Let π denote the transitive closure of the relation $\longrightarrow$. Then π is the smallest positive lower-potent quasi-order on S so $\widehat{\pi} = \pi$ and $\mathcal{I}d^{\pi}(S) = \mathcal{I}d^{\mathrm{cs}}(S)$, by Theorem 2 of [6]. Now, by Theorem 7 of [5] we have that S is a Γ_π-semigroup, i.e. π is linear, if and only if S is a chain of semilattice indecomposable semigroups. On the other hand, by Theorem 1 it follows that S is a Γ_π-semigroup if and only if for any $I, J \in \mathcal{I}d^{\mathrm{cs}}(S)$ at least one of semigroups $I/(I\cap J)$ and $J/(I\cap J)$ is a nil-semigroup. This completes the proof of the corollary. □

In the second part of the paper we consider the case in which for $I, J \in \mathcal{I}d^{\pi}(S)$, both of the semigroups $I/(I\cap J)$ and $J/(I\cap J)$ are nil-semigroups, and we prove that semigroups having this property are exactly π-archimedean semigroups.

For a positive quasi-order π on a semigroup S, M. S. Putcha in [7] and T. Tamura in [13] defiined S to be a π-*archimedean* semigroup if for any $a, b \in S$ there exists $n \in \mathbb{N}$ such that $a\,\pi\,b^n$, i.e. if the lower-potent closure $\widehat{\pi}$ of π is the universal relation on S. In the case when π is the division relation on S, then S is an *archimedean semigroup*, i.e. a semigroup in which for any two elements a and b there exists $n \in \mathbb{N}$ such that $a \mid b^n$, i.e. $\longrightarrow$ is the universal relation on S. If π is the smallest positive lower-potent quasi-order on S, i.e. the transitive closure of $\longrightarrow$, then we obtain a semilattice indecomposable semigroup.

The following theorem characterizes π-archimedean semigroups.

Theorem 4. *Let π be a positive quasi-order on a semigroup S. Then S is a π-archimedean semigroup if and only if $I/(I\cap J)$ is a nil-semigroup for any $I, J \in \mathcal{I}d^{\pi}(S)$.*

Proof. Let $I/(I \cap J)$ be a nil-semigroup for any $I, J \in \mathcal{I}d^{\pi}(S)$. Assume arbitrary $a, b \in S$ and set $I = b\pi$ and $J = a\pi$. Then $I, J \in \mathcal{I}d^{\pi}(S)$ so $I/(I \cap J)$ is a nil-semigroup, whence we have that there exists $n \in \mathbb{N}$ such that

$$b^n \in I \cap J \subseteq J = a\pi \quad \text{i.e.} \quad a\,\pi\,b^n.$$

Therefore, $a\,\widehat{\pi}\,b$, so we have proved that S is a π-archimedean semigroup.

Conversely, let S be a π-archimedean semigroup. Also, let $I, J \in \mathcal{I}d^{\pi}(S)$ and fix $a \in I \cap J$. Then for any $b \in I$ we have that $a\,\widehat{\pi}\,b$, i.e. $a\,\pi\,b^n$, for some $n \in \mathbb{N}$. Thus

$$b^n \in a\pi \subseteq I \cap J,$$

since $I \cap J \in \mathcal{I}d^{\pi}(S)$ and $a \in I \cap J$. Hence, we have proved that $I/(I \cap J)$ is a nil-semigroup. □

Two particular cases of π-archimedean semigroups are considered in the next two corollaries.

Corollary 5. *A semigroup S is archimedean if and only if $I/(I \cap J)$ is a nil-semigroup for any $I, J \in \mathcal{I}d(S)$.*

Proof. As in the proof of Corollary 1, if we assume π to be the division relation on S, then $\mathcal{I}d^{\pi}(S) = \mathcal{I}d(S)$ and π-archimedean semigroups are exactly archimedean semigroups. □

Corollary 7. *A semigroup S is semilattice indecomposable if and only if $I/(I \cap J)$ is a nil-semigroup for any $I, J \in \mathcal{I}d^{\mathrm{cs}}(S)$.*

Proof. As in the proof of Corollary 4, if π is the transitive closure of the relation $\longrightarrow$, then $\mathcal{I}d^{\pi}(S) = \mathcal{I}d^{\mathrm{cs}}(S)$ and π-archimedean semigroups are in fact semilattice indecomposable semigroups. □

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