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# A note on $\pi$-regular rings*

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*Dedicated to Professor S. Lajos on his 60th birthday*

**Abstract**. In this paper the following result is obtained: If $R$ is a $\pi$-regular ring, then $(R, \cdot)$ is a semilattice of Archimedean semigroups if and only if the nilpotents of $R$ form an ideal of $R$.

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Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. A semigroup (ring) $S$ is *$\pi$-regular* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^n S a^n$. A semigroup $S$ is *Archimedean* if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$. A semigroup $S$ is *completely Archimedean* if $S$ is Archimedean and has a primitive idempotent.

Let $a$ and $b$ be arbitrary elements of a semigroup $S$. Then $a \longrightarrow b \iff (\exists i \in \mathbf{Z}^+)\ b^i \in SaS$, $a \longrightarrow^{n+1} b \iff (\exists x \in S)\ a \longrightarrow^n x \longrightarrow b$, [7]. Let $\Sigma_n(a) = \{x \in S \mid a \longrightarrow^n x\}$, $n \in \mathbf{Z}^+$. On $S$ we define $\sigma_n$ by $a\ \sigma_n\ b \iff \Sigma_n(a) = \Sigma_n(b)$, [7]. We call a semigroup $S$ *$\sigma_n$-simple* if and only if $\sigma_n = S \times S$. It is clear that $S$ is a $\sigma_1$-simple semigroup if and only if it is Archimedean.

An element $a$ of a semigroup (ring) $S$ with zero 0 is called *nilpotent* if there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. A semigroup (ring) $S$ is a *nil-semigroup* (*nil-ring*) if all of its elements are nilpotents. By $E(S)$ ($Reg(R)$, $Nil(R)$) we denote the set of all idempotent (regular, nilpotent) elements of a semigroup (ring) $S$.

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If $R$ is a ring, $\mathcal{M}R$ will denote the multiplicative semigroup of $R$. A semigroup $S$ is called *Clifford's semigroup* if it is regular and idempotents of $S$ are central (or, equivalently, if $S$ is a semilattice of groups). A ring $R$ is said to be a *Clifford's ring* if $\mathcal{M}R$ is a Clifford's semigroup. For some characterizations of these rings see S. Lajos [8]. An element $a$ of a ring $R$ is *r.q.r.* (*right quasi regular*) if there exists $b \in R$ such that $a + b - ab = 0$. A right ideal $I$ of a ring $R$ is r.q.r. if all of its elements are r.q.r.

For undefined notions and notations we refer to [1] and [9].

M.S. Putcha [10] considered rings $R$ whose power of each element of $R$ lies in a subgroup of $R$ (strongly $\pi$-regular rings) and showed that in such a ring $(R, \cdot)$ is a semilattice of Archimedean semigroups if and only if the nilpotents of $R$ form an ideal of $R$. It can be posed the following more extensive problem.

Problem. Let $R$ be a ring such that $(R, \cdot)$ is a semilattice of $\sigma_n$-simple semigroups. Do $\Sigma_n(0)$ necessarily form a (ring) ideal of $R$? In the case that $\Sigma_n(0)$ is a ring ideal of $R$, is then $(R, \cdot)$ a semilattice of $\sigma_n$-simple semigroups?

A partial answer is given by the theorem below. In fact, the purpose of the present note is to extend the result of M.S. Putcha, [10], to $\pi$-regular rings.

For some related results we refer to [3], [5] and [6].

Theorem 1. *The following conditions on a ring $R$ are equivalent:*

(*i*) *$R$ is $\pi$-regular and $Nil(R)$ is an ideal of $\mathcal{M}R$;*

(*ii*) *$R$ is $\pi$-regular and $Nil(R)$ is an ideal of $R$;*

(*iii*) *$R$ is $\pi$-regular and an ideal extension of a nil-ring by a Clifford's ring;*

(*iv*) *$\mathcal{M}R$ is a semilattice of completely Archimedean semigroups;*

(*v*) *$R$ is $\pi$-regular and $\mathcal{M}R$ is a semilattice of Archimedean semigroups.*

*Proof.* (i)$\Longrightarrow$(ii). Let $J$ be the Jacobson's radical of $R$, let $a \in Nil(R)$ and let $x \in R$. Then $ax \in Nil(R)$, so $ax$ is r.q.r. Thus, $aR$ is r.q.r. so $a \in J$. Hence, $Nil(R) \subseteq J$.

Conversely, let $a \in J$. Then there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$, whence $a^n x R \subseteq aR$. Since $aR$ is r.q.r., then $a^n x R$ is r.q.r., so $a^n x \in J$, i.e. $a^n x \in J \cap E(R) = \{0\}$. Therefore, $a^n = 0$ so $a \in Nil(R)$.

Hence, $J = Nil(R)$, whence $Nil(R)$ is a ring ideal of $R$.

(ii)$\Longrightarrow$(iii). Let $N = Nil(R)$ and let $\varphi : R \to R/N$ be the natural homomorphism. Assume $u \in R/N$. Then there exists $a \in R$ such that

$u = \varphi(a)$ and there exists $n \in \mathbf{Z}^+$ and $x \in R$ such that $a^n = a^n x a^n$, whence

$$(a - axa^n)^n = a^n - a^n x a^n = 0,$$

Thus $a \equiv axa^n(\text{mod}N)$. Since $axa^n = (axa^n)(xa^{n-1})(axa^n)$, we then have that $axa^n \in Reg(S)$, whence $u = \varphi(a) = \varphi(axa^n) \in Reg(R/N)$. Therefore, $R/N$ is a regular ring.

Assume $a \in E(R/N),\ b \in R/N$. Then by Corollary 2. [2] it follows that $a = \varphi(e)$ and $b = \varphi(x)$ for some $e \in E(R)$ and $x \in R$. Since

$$(ex - exe)^2 = (xe - exe)^2 = 0,$$

then $ex \equiv exe(\text{mod}N) \equiv xe(\text{mod}N)$, whence

$$ab = \varphi(ex) = \varphi(xe) = ba.$$

Thus, idempotents of $R/N$ are central, so $R/N$ is a Clifford's ring.

(iii)$\Longrightarrow$(iv). Let $R$ be $\pi$-regular and let $R$ be an ideal extension of a nil-ring $N$ by a Clifford's ring $Q$. Let $\varphi : R \to Q$ be the natural homomorphism. Since $\mathcal{M}Q$ is a Clifford's semigroup, then $\mathcal{M}Q$ is a semilattice $Y$ of groups $G_\alpha,\ \alpha \in Y$. Let

$$R_\alpha = \varphi^{-1}(G_\alpha)\ , \quad \alpha \in Y\ .$$

Then it is easy to show that $\mathcal{M}R$ is a semilattice $Y$ of semigroups $R_\alpha,\ \alpha \in Y$. Also, it is clear that $R_\alpha$ are $\pi$-regular semigroups for all $\alpha \in Y$. Let $\alpha \in Y$ and let $e, f \in E(R_\alpha)$ such that $ef = fe = f$. Then

$$(e - f)^2 = (e - f)(e - f) = e - ef - fe + f = e - f.$$

On the other hand, since $G_\alpha$ is a group and $\varphi(e), \varphi(f) \in E(G_\alpha)$, then $\varphi(e) = \varphi(f)$, whence $e - f \in N$. Thus $e - f \in E(R) \cap N = \{0\}$, so $e = f$. Hence, $R_\alpha$ is a $\pi$-regular semigroup whose idempotents are primitive so by Theorem 1. [4] we obtain that $R_\alpha$ is an ideal extension of a completely simple semigroup by a nil-semigroup, i.e. that $R_\alpha$ is a completely Archimedean semigroup. Therefore, $(iv)$ holds.

(iv)$\Longrightarrow$(v) and (v)$\Longrightarrow$(i). This follows immediately. $\square$

*Remark.* By Theorem 1, for a semigroup $S$ which is a multiplicative semigroup of some ring, we obtain the following assertion: $S$ is a semilattice of completely Archimedean semigroups if and only if it is $\pi$-regular and a semilattice of Archimedean semigroups. For other semigroups this must not hold.

Namely, bicyclic semigroups are regular and simple and are not completely simple, i.e. these semigroups are $\pi$-regular and semilattices of Archimedean semigroups and are not semilattices of completely Archimedean semigroups.

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