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A note on radicals of Green's relations*

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Abstract. In this paper we characterize regularity of semigroups by radicals of Green's relations.

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A semigroup S is *left(right, intra-) regular* if $a \in Sa^2 (a \in a^2S,\ a \in Sa^2S)$ for all $a \in S$. Various characterizations of the regularity of semigroups by the Green's relations were investigated by A.H. Clifford and G.B. Preston [5], J.T. Sedlock [12], B. Pondělíček [11] and W.D. Miller [8]. For other characterizations of the semigroups considered, in these papers given in terms of two-sided, one-sided, bi- and quasi-ideals, we refer to the survey paper of S. Lajos [7]. In the present paper some new characterizations of the regularity, using the radicals of the Green's relations, will be given.

A semigroup S is *left* (*right*) *simple* if $a \in Sb$ ($a \in bS$) for all $a, b \in S$. A semigroup S is *right completely simple* if $a \in abS$ for all $a, b \in S$, [3]. Throughout this paper Z^+ will denote the set of all positive integers. A semigroup S is *left Archimedean* if for all $a, b \in S$ there exists $n \in Z^+$ such that $a^n \in Sb$, it is *left* (*right*) *weakly commutative* if for all $a, b \in S$ there exists $n \in Z^+$ such that $(ab)^n \in bS((ab)^n \in Sa)$, and it is *weakly commutative* if it is both left and right weakly commutative [11]. For a relation ρ on a

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semigroup S, the *radical* of ρ, in notation $\sqrt{\rho}$, is a relation introduced by L.N. Shevrin in [13] as follows:

$$(a,b) \in \sqrt{\rho} \iff (\exists m,n \in \mathbb{Z}^+)(a^m, b^n) \in \rho .$$

By $\mathcal{J}, \mathcal{R}, \mathcal{L}$ and $\mathcal{H}$ we denote the well known Green's relations.

For undefined notions and notations we refer to [5] and [9].

We start with the following:

THEOREM 1. *The following conditions on a semigroup S are equivalent:*

(i) *S is intra-regular;*

(ii) $\sqrt{\mathcal{J}} = \mathcal{J}$;

(iii) $\sqrt{\mathcal{L}} \subseteq \mathcal{J}$;

(iv) $\sqrt{\mathcal{H}} \subseteq \mathcal{J}$.

Proof. (i) $\Longrightarrow$ (ii) Let $a,b \in S$ and $(a,b) \in \sqrt{\mathcal{J}}$. Then there exist $m,n \in \mathbb{Z}^+$ such that $a^m \mathcal{J} b^n$. Since $a \mathcal{J} a^2$, for all $a \in S$, (see [5]), we then have that $a \mathcal{J} a^m \mathcal{J} b^n \mathcal{J} b$. So $a \mathcal{J} b$, whence $\sqrt{\mathcal{J}} \subseteq \mathcal{J}$ and since the opposite inclusion always holds, we have (ii).

(ii) $\Longrightarrow$ (i) Since $(a, a^2) \in \sqrt{\mathcal{L}}$ for all $a \in S$, we have by [5] that S is intra-regular. The rest of the proof is similar to the previous one and will be omitted. □

In a similar way we can prove the following theorem.

THEOREM 2. *A semigroup S is left (right) regular if and only if $\sqrt{\mathcal{L}} = \mathcal{L}$ ($\sqrt{\mathcal{R}} = \mathcal{R}$).*

Using the previous theorem and Theorem 4.3 [5] we obtain the following result.

THEOREM 3. *The following conditions on a semigroup S are equivalent:*

(i) *S is a union of groups;*

(ii) $\sqrt{\mathcal{L}} = \mathcal{L}$ & $\sqrt{\mathcal{R}} = \mathcal{R}$;

(iii) $\sqrt{\mathcal{H}} = \mathcal{H}$.

Next we prove

THEOREM 4. *A semigroup S is a semilattice of left simple semigroups if and only if $\sqrt{\mathcal{L}} = \mathcal{J}$.*

Proof. Assume that S is a semilattice Y of left simple semigroups S_α, $\alpha \in Y$. Let $a, b \in S$ and $a\mathcal{J}b$. Then there exists $\alpha \in Y$ such that $a, b \in S_\alpha$, whence $a\mathcal{L}b$. Thus $\mathcal{J} \subseteq \mathcal{L} \subseteq \sqrt{\mathcal{L}}$. By Theorem 4.2 [5] (see also Theorem 3 [3]) we have that S is left regular, and therefore S is intra-regular. By Theorem 1 we have that $\sqrt{\mathcal{L}} \subseteq \mathcal{J}$. Hence, $\sqrt{\mathcal{L}} = \mathcal{J}$.

Conversely, by Theorem 1 we have that S is intra-regular, whence by Theorem 4.4 [5], $ab\mathcal{J}ba$ for all $a, b \in S$. So $(ab, ba) \in \sqrt{\mathcal{L}}$ for all $a, b \in S$, i.e. there exists $n \in \mathbb{Z}^+$ such that $(ab)^n \in Sa$. Thus, S is right weakly commutative and by Proposition 1.1 [2], S is a semilattice Y of left Archimedean semigroups S_α, $\alpha \in Y$. For $a \in S_\alpha$, $\alpha \in Y$, there exist $x, y \in S_\alpha$ such that

$$a = xa^2y = x^n a(ay)^n \in x^n a S_\alpha a^2 \subseteq S_\alpha a^2 ,$$

for every $n \in \mathbb{Z}^+$. From this it follows that $a \in S_\alpha a^k$, for every $k \in \mathbb{Z}^+$, and since S_α is left Archimedean, we have that for every $a, b \in S_\alpha$ there exists $k \in \mathbb{Z}^+$ such that $a^k = zb$ for some $z \in S_\alpha$. Now, $a \in S_\alpha a^k \subseteq S_\alpha zb \subseteq S_\alpha b$. Thus, any S_α, $\alpha \in Y$, is left simple. Hence S is a semilattice of left simple semigroups. □

By Theorem 1.1 [11], using Theorems 2, 3 and 4, we have the following:

THEOREM 5. *A semigroup S is a semilattice of left groups if and only if $\sqrt{\mathcal{L}} = \mathcal{L} = \mathcal{J}$ & $\sqrt{\mathcal{R}} = \mathcal{R}$.*

Now we go to the main theorem of this paper.

THEOREM 6. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of groups ;*

(ii) $\sqrt{\mathcal{L}} = \mathcal{R}$;

(iii) $\sqrt{\mathcal{R}} = \mathcal{L}$;

(iv) $\sqrt{\mathcal{L}} = \sqrt{\mathcal{R}} = \mathcal{J}$;

(v) $\sqrt{\mathcal{H}} = \mathcal{J}$.

Proof. (i) $\Longrightarrow$ (ii) By Theorem 3 we have that $\sqrt{\mathcal{L}} = \mathcal{L}$ and since $\mathcal{L} = \mathcal{R}$ (see Theorem 12 [11]), we have the assertion (ii).

(ii) $\Longrightarrow$ (i) Since $(a, a^2) \in \sqrt{\mathcal{L}}$, for all $a \in S$, we then have that $a\mathcal{R}a^2$, for every $a \in S$. So, S is right regular ([5]). By Theorem 3 [3], S is a semilattice Y of right completely simple semigroups S_α, $\alpha \in Y$. Assume $a,b \in S_\alpha$, $\alpha \in Y$. Then by Theorem 2 [3], $a \in abS_\alpha$ and $ab \in aS_\alpha$. Thus $a\mathcal{R}ab$, whence $(a,ab) \in \sqrt{\mathcal{L}}$. So there exists $m \in \mathbf{Z}^+$ such that $(ab)^m \in Sa$, i.e. $(ab)^m = xa$ for some $x \in S_\beta$. From this it follows that $\alpha \leq \beta$, whence $(a)^{m+1} = abxa \in S_\alpha a$. By Proposition 1.1 [2] we have that S_α is a semilattice of left Archimedean semigroups, and since any S_α is semilattice-indecomposable, we have that S_α, $\alpha \in Y$ is left Archimedean. Since S_α is right completely simple, we have that for every $a,b \in S_\alpha$, $a = abx$ for some $x \in S_\alpha$, whence $a = a(bx)^n$, for every $n \in \mathbf{Z}^+$. But, S_α is left Archimedean, so $(bx)^k \in S_\alpha b$. Now

$$a = a(bx)^n \in aS_\alpha b \subseteq S_\alpha b\,.$$

Hence, any S_α, $\alpha \in Y$, is a left simple subsemigroup of S. Therefore, by Theorem 4.2 [5], S is left regular, and by Theorem 2, $\mathcal{L} = \sqrt{\mathcal{L}} = \mathcal{R}$. Since S is left and right regular, by Theorem 4.3 [5], we have that S is a union of groups, so by $\mathcal{L} = \mathcal{R}$ and Theorem 12 [11] we obtain that S is a semilattice of groups.

(v) $\Longrightarrow$ (i) By Theorem 1, S is intra-regular and by Theorem 4.4 [5], $ab\mathcal{J}ba$ for all $a,b \in S$. Thus $(ab,ba) \in \sqrt{\mathcal{H}}$ i.e. there exist $m,n \in \mathbf{Z}^+$ such that $(ab)^m\mathcal{H}(ba)^n$. Therefore, S is weakly commutative, and since S is intraregular, we then have by Theorem 1.1 [1], that S is a semilattice of groups.

(i) $\Longrightarrow$ (iii) The proof of this implication is similar to the proof of (i) $\Longrightarrow$ (ii).

(i) $\Longrightarrow$ (iv) Since $\mathcal{L} = \mathcal{R}$, then by Theorem 5 we have the assertion (iv).

(i) $\Longrightarrow$ (v) By the previous equivalences we have that

$$\sqrt{\mathcal{H}} = \sqrt{\mathcal{L}\cap\mathcal{R}} = \sqrt{\mathcal{L}} \cap \sqrt{\mathcal{R}} = \mathcal{J} \cap \mathcal{J} = \mathcal{J}\,. \qquad \square$$

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