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SEMIGROUPS IN WHICH ANY PROPER IDEAL IS SEMILATTICE INDECOMPOSABLE*

Stojan Bogdanović, Miroslav Ćirić and Nebojša Stevanović

Dedicated to Professor S. Prešić on the occasion of his 65th birthday

Abstract

The main purpose of this note is to study semigroups all of whose proper ideals from an arbitrary nontrivial complete 1-sublattice of the lattice of ideals are semilattice indecomposable or archimedean semigroups. We also determine some conditions under which there exists the largest proper ideal in this lattice.

A significant problem of Semigroup theory is to study semigroups all of whose proper subsemigroups or ideals have certain properties. Semigroups whose proper ideals are groups, commutative or archimedean semigroups were investigated by Schwarz in [11], Tamura in [12], Bogdanović in [2, 3, 4, 5], Bogdanović and Ćirić in [6] and others. In the present paper we consider any complete 1-sublattice $Id^{\pi}(S)$ of the lattice $Id(S)$ of ideals of a semigroup S. As was shown in [8], it is uniquely determined by some positive quasi-order π on S. We study the set $M_{\pi}(S)$ of all elements of S that generate a proper ideal from $Id^{\pi}(S)$, for which we show that it is an ideal and the union of all proper ideals from $Id^{\pi}(S)$, and we find some conditions under which $M_{\pi}(S)$ is also a proper ideal of S. Finally, we determine the conditions under which any proper ideal from $Id^{\pi}(S)$ is a semilattice indecomposable or archimedean semigroup.

By a *complete 1-sublattice* of a complete lattice L we mean any complete sublattice of L containing the unity of L. For a semigroup S, S^1 denotes the semigroup obtained from S by adjoining the unity. The *division relation* $\mid$ on S is defined by: $a \mid b$ if and only if $b = xay$, for some $x, y \in S^1$, the relation $\longrightarrow$ on S is defined by: $a \longrightarrow b$ if and only if $a \mid b^n$, for some natural number n, and $\longrightarrow^{\infty}$ denotes the transitive closure of $\longrightarrow$. The lattice of all ideals of S is denoted by $Id(S)$. Any ideal of S different than S is called a *proper ideal* of S. An ideal I of S is called *completely semiprime* if for any $a \in S$, $a^2 \in I$ implies $a \in I$, and it is called *completely prime* if for any $a, b \in S$, $ab \in I$ implies that $a \in I$ or $b \in I$. The set of all completely semiprime ideals of S, which is a complete 1-sublattice of $Id(S)$, is denoted by $Id^{cs}(S)$.

By a *quasi-order* on a set A we mean any reflexive and transitive binary relation π on A, and the pair (A, π) is called a *quasi-ordered set*. For $a \in A$ we

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set $a\pi = \{x \in A \,|\, a\,\pi\, x\}$ and for $X \subseteq A$ we set $X\pi = \bigcup_{x\in X} x\pi$. In other words, $a\pi$ and $X\pi$ are the filters (dual ideals) of a quasi-ordered set (A, π) generated by $\{a\}$ and X, respectively. By π^{-1} we denote a relation on A defined by: $a\,\pi^{-1}\,b$ if and only if $b\,\pi\,a$. The relation $\widetilde{\pi} = \pi \cap \pi^{-1}$ is the greatest equivalence relation contained in π and it is called the *natural equivalence* of π. As known, $a\,\widetilde{\pi}\,b$ if and only if $a\pi = b\pi$, and the set of all $\widetilde{\pi}$-classes is partially ordered, where the partial order $\leq$ is defined by: $a\widetilde{\pi} \leq b\widetilde{\pi}$ if and only if $a\,\pi\,b$ (cf. [1] and [8]). The partially ordered set of all quasi-orders on A is a complete lattice and it is denoted by $\mathcal{Q}(S)$.

Let π be a quasi-order on a semigroup S. We say that π is *positive* if $a\,\pi\,ab$ and $b\,\pi\,ab$, for all $a, b \in S$, that it is *lower-potent* if $a^2\,\pi\,a$, for any $a \in S$, and that it satisfies the *cm-property* (*common multiple property*, [13, 14]) if for any $a, b, c \in S$, $a\,\pi\,c$ and $b\,\pi\,c$ implies $ab\,\pi\,c$. The set of all positive quasi-orders on S is the principal filter of the lattice $\mathcal{Q}(S)$ generated by the division relation on S, whereas the set of all positive lower-potent quasi-orders on S is the principal filter of $\mathcal{Q}(S)$ generated by the quasi-order $\longrightarrow^{\infty}$. If π is the division relation on S, then $\widetilde{\pi}$ is the well known Green's $\mathcal{J}$-relation, and if $\pi = \longrightarrow^{\infty}$, then $\widetilde{\pi}$ is the smallest semilattice congruence on S, and the partially ordered set of all $\widetilde{\pi}$-classes is the greatest semilattice homomorphic image of S (cf. [13], [14], [7] and [8]). As known, a semigroup S is *semilattice indecomposable*, i.e. the universal relation on S is the only semilattice congruence on S, if and only if $a \longrightarrow^{\infty} b$, for all $a, b \in S$ (cf. [13]). If $a \longrightarrow b$ for all $a, b \in S$, then S is called an *archimedean semigroup*.

Let π be a positive quasi-order on a semigroup S. An ideal I of S is called a π-*ideal* if $I\pi = I$, i.e. if $a \in I$ implies $a\pi \subseteq I$, for any $a \in S$. In other words, I is a π-ideal of S if and only if it is an ideal of S and a filter of the quasi-ordered set (S, π). The set of all π-ideals of S is denoted by $Id^{\pi}(S)$. As was proved in [8], $Id^{\pi}(S)$ is a complete 1-sublattice of $Id(S)$ and the mapping $\pi \mapsto Id^{\pi}(S)$ is a dual isomorphism of the lattice of all positive quasi-orders on S onto the lattice of all complete 1-sublattices of $Id(S)$. The same mapping also determines a dual isomorphism of the lattice of positive lower-potent quasi-orders on S onto the lattice of all complete 1-sublattices of $Id^{\mathrm{cs}}(S)$. Hence, if π is the division relation on S, then $Id^{\pi}(S) = Id(S)$, and if $\pi = \longrightarrow^{\infty}$, then $Id^{\pi}(S) = Id^{\mathrm{cs}}(S)$.

For undefined notions and notations we refer to [1], [5], [6] and [10].

Let π be a positive quasi-order on a semigroup S. Then we set

$$M_{\pi}(S) = \{a \in S \,|\, a\pi \subset S\}.$$

The proof of the first lemma is immediate and it will be omitted.

Lemma 1. *Let π be a positive quasi-order on a semigroup S. Then $M_{\pi}(S) = \varnothing$ if and only if S does not have proper π-ideals.*

Next we consider the case when $M_{\pi}(S)$ is nonempty.

Theorem 1. *Let π be any positive quasi-order π on a semigroup S such that S has a proper π-ideal. Then $M_\pi(S)$ is the union of all proper π-ideals of S.*

If, in addition, π satisfies the cm-property, then $M_\pi(S)$ is a completely prime ideal of S.

Proof. Let U denote the union of all proper π-ideals of S and let I be any proper π-ideal of S. For any $a \in I$ we have that $a\pi \subseteq I \subset S$, so $a \in M_\pi(S)$. Thus, $I \subseteq M_\pi(S)$ and we conclude that $U \subseteq M_\pi(S)$. On the other hand, for any $a \in M_\pi(S)$ we have that $a\pi$ is a proper ideal of S, that is $a\pi \subseteq U$, and hence $a \in U$. Therefore, $M_\pi(S) \subseteq U$, and we have proved that $M_\pi(S) = U$.

Suppose now that π satisfies the cm-property, and let $a, b \in S$ such that $ab \in M_\pi(S)$. If $a \notin M_\pi(S)$ and $b \notin M_\pi(S)$, i.e. if $a\pi = S$ and $b\pi = S$, then by Lemma 2 of [8] we have that $(ab)\pi = a\pi \cap b\pi = S$, which contradicts the hypothesis $ab \in M_\pi(S)$. Thus, we conclude that $a \in M_\pi(S)$ or $b \in M_\pi(S)$, so we have proved that $M_\pi(S)$ is a completely prime ideal of S. □

Now we determine some conditions under which $M_\pi(S)$ is a proper ideal.

Theorem 2. *Let π be any positive quasi-order on a semigroup S and suppose that S has at least one proper π-ideal. Then the following conditions are equivalent:*

(i) *$M_\pi(S)$ is a proper ideal of S;*
(ii) *S has a largest proper π-ideal;*
(iii) *The partially oredered set of $\widetilde{\pi}$-classes has a least element.*

Proof. (i)⇒(ii). Since $Id^\pi(S)$ is a complete 1-sublattice of $Id(S)$, then by Theorem 1 we have that $M_\pi(S) \in Id^\pi(S)$. Therefore, if $M_\pi(S)$ is a proper ideal of S, then it is the largest proper π-ideal of S, again by Theorem 1.

(ii)⇒(i). Let S has a largest π-ideal U. Then U is the union of all proper π-ideals of S, and by Theorem 1 it follows that $U = M_\pi(S)$. Therefore, $M_\pi(S)$ is a proper ideal of S.

(i)⇒(iii). Let $X = S \setminus M_\pi(S)$. If $a, b \in X$. then $a\pi = S = b\pi$, so by Proposition 1 of [8] it follows that $(a, b) \in \widetilde{\pi}$. Thus, X is contained in some $\widetilde{\pi}$-class C of S. On the other hand, for any $c \in C$ and $a \in X$ we have that $(c, a) \in \widetilde{\pi}$, whence $c\pi = a\pi = S$, so $c \in X$. Therefore $X = C$, i.e. X is a $\widetilde{\pi}$-class of S. Let Y be any $\widetilde{\pi}$-class of S and let $a \in X$ and $b \in Y$ be arbitrary elements. Then $a\pi = S$ and $b \in S$, whence $a \,\pi\, c$. This means that $X \leq Y$ in the partially ordered set of all $\widetilde{\pi}$-classes of S, so we have proved that X is the least element in this partially ordered set.

(iii)⇒(i). Let X be the least element in the partially ordered set of $\widetilde{\pi}$-classes of S. First we prove that $X = \{a \in S \mid a\pi = S\}$. Let $a \in X$ and $b \in S$. Then $a\widetilde{\pi} \leq b\widetilde{\pi}$ implies $a \,\pi\, b$, so $b \in a\pi$. This means that $a\pi = S$. Conversely, let $a \in S$ such that $a\pi = S$ and let $b \in X$ be an arbitrary element. Then $a\pi = S$ yields $a \,\pi\, b$, whereas by $b\widetilde{\pi} \leq a\widetilde{\pi}$ it follows $b \,\pi\, a$. Thus $(a, b) \in \pi \cap \pi^{-1} = \widetilde{\pi}$, so we have that $a \in X$. Now we have that $\varnothing \neq X \neq S$, since S has at least one proper π-ideal, whence it follows that $M_\pi(S) = S \setminus X$ is a proper ideal of S. □

Note that the condition (ii) of the above theorem means that the lattice $Id^{\pi}(S)$ has a unique dual atom.

Let $M_{\mathrm{cs}}(S)$ denote the union of all proper completely semiprime ideals of a semigroup S. By Theorem 2 we obtain the following consequence.

Corollary 1. *The following conditions on a semigroup S are equivalent:*

(i) *$M_{\mathrm{cs}}(S)$ is a proper ideal of S;*
(ii) *S has a largest proper completely semiprime ideal;*
(iii) *S has a largest proper completely prime ideal;*
(iii) *The greatest semilattice homomorphic image of S has a unity.*

By the previous corollary it follows that the largest proper completely semiprime ideal of a semigroup, if it exists, is completely prime.

The following theorem describes semigroups whose proper π-ideals are semilattice indecomposable semigroups.

Theorem 3. *Let π be a positive quasi-order on a semigroup S and suppose that S has at least one proper π-ideal. Then any proper π-ideal is a semilattice indecomposable semigroup if and only if $M_{\pi}(S)$ is a semilattice indecomposable semigroup.*

Proof. Let any proper π-ideal of S be a semilattice indecomposable semigroup and let $a, b \in M_{\pi}(S)$. Then $a\pi$ and $b\pi$ are proper π-ideals of S, and they are semilattice indecomposable. Moreover, $a, ab \in a\pi$ and $b, ab \in b\pi$, whence $a \longrightarrow^{\infty} ab$ and $ab \longrightarrow^{\infty} a$ in $a\pi$ and $b \longrightarrow^{\infty} ab$ and $ab \longrightarrow^{\infty} b$ in $b\pi$. Since the ideals $a\pi$ and $b\pi$ are contained in $M_{\pi}(S)$, by Theorem 1, then we have that $a \longrightarrow^{\infty} ab \longrightarrow^{\infty} b$ and $b \longrightarrow^{\infty} ab \longrightarrow^{\infty} a$ in $M_{\pi}(S)$. Therefore, $M_{\pi}(S)$ is a semilattice indecomposable semigroup.

Conversely, let $M_{\pi}(S)$ be a semilattice indecomposable semigroup. For any proper π-ideal I of S, by Theorem 1 we have that I is an ideal of $M_{\pi}(S)$, and by Theorem 3.4 and Corollary 3.9 of [9] we have that any ideal of a semilattice indecomposable semigroup is also semilattice indecomposable. □

In a similar way we prove the next corollary which generalizes some results from [3], [5] and [6].

Corollary 2. *Let π be a positive quasi-order on a semigroup S and suppose that S has at least one proper π-ideal. Then any proper π-ideal is an archimedean semigroup if and only if $M_{\pi}(S)$ is an archimedean semigroup.*

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Contact address:
Miroslav Ćirić
Faculty of Sciences and Mathematics,
Department of Mathematics
Ćirila i Metodija 2, P. O. Box 91,
18000 Niš, Yugoslavia
e-mail: `ciricm@bankerinter.net`
`mciric@archimed.filfak.ni.ac.yu`