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POSITIVE QUASI-ORDERS WITH THE COMMON MULTIPLE PROPERTY ON A SEMIGROUP*

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Abstract. Positive quasi-orders with the *cm*-property have a key role in semilattice and chain decompositions of semigroups. The purpose of this paper is to give some new properties of these quasi-orders.

The idea of studying of semilattice decompositions of semigroups through its quasi-orders satisfying certain conditions was born in the paper of T. Tamura [11]. There he proved that the lattice of semilattice congruences on a semigroup is isomorphic to the lattice of its lower-potent, positive half-congruences. Using the result given below as Lemma 1, the authors in [4] gave another formulation of this theorem, in which lower-potent, positive half congruences were replaced by positive quasi-orders with the *cm*-property. They developed this theory establishing connections between these quasi-orders and certain sublattices of the lattice of ideals of a semigroup. In terms of completely prime ideals they also characterized chain decompositions of semigroups.

In this paper we give an alternative approach to the study of semilattice and chain decompositions and we characterize positive quasi-orders with the *cm*-property in terms of filters of a semigroup. Some of the obtained results generalize some results of M. Petrich [6,7] and of the authors [3], concerning the greatest semilattice decomposition of a semigroup and some chain decompositions.

Throughout this paper $\mathbf{Z}^+$, will denote the set of all positive integers. For a binary relation ξ on a set X, ξ^∞ will denote the transitive closure of ξ, ξ^{-1} will denote the relation defined by: $a\,\xi^{-1}\,b \;\Leftrightarrow\; b\,\xi\,a$, for $a \in X$, $a\xi = \{x \in X \mid a\,\xi\,x\}$, $\xi a = \{x \in X \mid x\,\xi\,a\}$, and for $A \subseteq X$, sets $A\xi = \bigcup_{a\in A} a\xi$ and $\xi A = \bigcup_{a\in A} \xi a$ will be called the *left coset* and the *right coset* of ξ, of a subset A, respectively. For a quasi-order ξ on a set X, $\xi \cap \xi^{-1}$ is an equivalence relation called the *natural equivalence*. The poset of quasi-orders on a non-empty set X is a complete lattice and it will be denoted by $\mathcal{Q}(X)$.

A binary relation ξ on a semigroup S is: *positive*, if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$, *lower-potent*, if $a^2\,\xi\,a$, for any $a \in S$, *linear*, if for all $a, b \in S$, $a\,\xi\,b$ or $b\,\xi\,a$, *compatible*, if for $a, b, x \in S$, $a\,\xi\,b$ implies $ax\,\xi\,bx$ and $xa\,\xi\,xb$, a *half-congruence*, if it

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is a compatible quasi-order, and it satisfies the *cm-property*, if for $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$, [11].

Let L be a complete lattice with the zero 0 and the unity 1. A subset K of L is *closed for meets* (*joins*) if it contains the meet (join) of any its non-empty subset, and it is a *closed subset* of L if it is closed both for meets and joins. Clearly, any closed subset of L is its sublattice and it is complete. A sublattice K of L is a *1-sublattice* (*0-sublattice*) of L if $1 \in K$ ($0 \in K$), and it is a *0,1-sublattice* of L if $0, 1 \in K$.

For a set X, $\mathcal{P}(X)$ will denote the lattice of all subsets of X. If K is a subset of $\mathcal{P}(X)$, closed for meets and containing its unity, then for any $a \in X$, there exists a smallest element of K containing a, in notation $K(a)$, called the *principal element* of K generated by a. A subset A of a semigroup S is *consistent* if for any $x, y \in S$, $xy \in A$ implies $x, y \in A$. Consistent subsemigroups of S and the empty subset of S will be called *filters* of S. The set of all filters of S is a subset of $\mathcal{P}(S)$ closed for meets and it contains the unity of $\mathcal{P}(S)$, and the principal element of this set generated by $a \in S$, in notation $N(a)$, will be called the *principal filter* of S generated by a.

For undefined notions and notations we refer to [1], [2], [5], [7] and [8].

We start with the following:

Proposition 1. *The lattice of quasi-orders on a non-empty set X is dually isomorphic to the lattice of closed 0,1-sublattices of $\mathcal{P}(X)$.*

Note that an isomorphism between these lattices is given by $\xi \mapsto K_\xi$, where $\xi \in \mathcal{Q}(X)$ and related closed 0,1-sublattice K_ξ of $\mathcal{P}(X)$ is defined by

$$K_\xi = \{A \in \mathcal{P}(X) \mid \xi A = A\}. \tag{1}$$

Note also that the principal elements of K_ξ have a representation $K(a) = \xi a$, for any $a \in X$, and that the inverse of the given mapping is $K \mapsto \xi_K$, where K is a closed 0,1-sublattice K of $\mathcal{P}(X)$ and related quasi-order ξ_K on S is defined by:

$$a\,\xi_K\,b \quad \Leftrightarrow \quad K(a) \subseteq K(b), \qquad\qquad (a, b \in S). \tag{2}$$

Another isomorphism can be obtained if we assume left cosets of ξ instead of right cosets.

As we seen in [4], positive quasi-orders on a semigroup S can be characterized by some sublattices of the lattice of ideals of S, using the left cosets of these quasi-orders. Here, using the right cosets, we obtain the following:

Theorem 1. *The lattice of positive quasi-orders on a semigroup S is dually isomorphic to the lattice of closed 0,1-sublattices of the lattice of consistent subsets of S.*

Proof. Note first that consistent subsets of S form a closed 0,1-sublattice of $\mathcal{P}(S)$ and let L denote this lattice.

With respect to Proposition 1, it is enough to prove that for $\xi \in \mathcal{Q}(S)$, ξ is positive if and only if K_ξ is a sublattice of L. Really, by Lemma 1 [4], ξ is positive if and only if ξa is a consistent subset of S, for any $a \in S$. On the other hand, since L is a closed sublattice of $\mathcal{P}(S)$, then $K_\xi \subseteq L$ if and only if all principal elements of K_ξ are in L. This completes our proof. □

Further we consider lattices of quasi-orders with the *cm*-property.

Theorem 2. *The lattice of quasi-orders on a semigroup S satisfying the cm-property is dually isomorphic to the lattice of closed 0,1-sublattices of $\mathcal{P}(S)$ whose principal elements are subsemigroups of S.*

Proof. Immediately we check that the intersection of any family of quasi-orders satisfying the *cm*-property is a quasi-order that satisfies the *cm*-property. Since the universal relation on S satisfies also the *cm*-property, then really, the poset of quasi-orders satisfying the *cm*-property is a complete lattice.

Further, a quasi-order ξ on S satisfies the *cm*-property if and only if ξa is a subsemigroup of S, for any $a \in S$, i.e. if principal elements of K_ξ are subsemigroups of S, whence we obtain the assertion of the theorem. □

As we seen in the proof of the previous theorem, quasi-orders on a semigroup with the *cm*-property form a complete lattice. The smallest element of this lattice is characterized by the following:

Theorem 3. *The relation ζ on a semigroup S defined by:*

$$a \,\zeta\, b \;\Leftrightarrow\; (\exists n \in \mathbf{Z}^+)\; a = b^n, \qquad\qquad (a, b \in S),$$

is the smallest quasi-order on S satisfying the cm-property.

Proof. This follows immediately. □

By Theorem 3.1 [11], in its form given in [4] as Tamura's theorem, the lattice of semilattice congruences on a semigroup is isomorphic to the lattice of positive quasi-orders on this semigroup satisfying the *cm*-property. The authors in [4] characterized these lattices in terms of ideals of a semigroup. Here we give another characterization of these lattices, in terms of filters.

Theorem 4. *The lattice of positive quasi-orders on a semigroup S satisfying the cm-property is dually isomorphic to the lattice of closed 0,1-sublattices of $\mathcal{P}(S)$ whose principal elements are filters of S.*

Proof. By Lemma 2 [4], a quasi-order ξ on a semigroup S is positive and it satisfies the *cm*-property if and only if ξa is a filter of S, for any $a \in S$, and the last condition clearly hold if and only if the principal elements of K_ξ are filters of S. Now, by Proposition 1 we obtain the assertions of this theorem. □

Now, the lattice of semilattice decompositions of a semigroup can also be characterized by:

Corollary 1. *The lattice of semilattice decompositions of a semigroup S is isomorphic to the lattice of closed 0,1-sublattices of $\mathcal{P}(S)$ whose principal elements are filters of S.*

Also, as a consequence of Theorem 4 we obtain the following result of M. Petrich [6] (see also [7]), concerning the smallest semilattice congruence on a semigroup:

Corollary 2. *The relation ϱ on a semigroup S defined by*

$$a \,\varrho\, b \quad \Leftrightarrow \quad N(a) = N(b) \qquad\qquad (a, b \in S),$$

is the smallest semilattice congruence on S.

Proof. Let K denote the set of all subsets of S that are unions of principal filters of S. Then K is a closed 0,1-sublattice of $\mathcal{P}(S)$ and its principal elements are exactly the principal filters of S, so K is the greatest closed 0,1-sublattice of $\mathcal{P}(S)$ whose principal elements are filters of S. Thus, the relation ξ_K defined as in (2) is the smallest positive quasi-order on S satisfying the *cm*-property and $\varrho = \xi_K \cap \xi_K^{-1}$, so ϱ is the smallest semilattice congruence on S. □

By the following theorem we give an interesting feature of the smallest positive quasi-order with the *cm*-property:

Theorem 5. *In the lattice $\mathcal{Q}(S)$ of quasi-orders on a semigroup S, the smallest positive quasi-order on S satisfying the cm-property is the join of the smallest positive quasi-order on S and the smallest quasi-order on S satisfying the cm-property.*

Proof. T. Tamura [9] proved that the smallest positive quasi-order on a semigroup S satisfying the *cm*-property is the transitive closure of the relation $\longrightarrow$ on S defined by:

$$a \longrightarrow b \quad \Leftrightarrow \quad (\exists n \in \mathbf{Z}^+)(\exists x, y \in S)\ xay = b^n \qquad\qquad (a, b \in S).$$

On the other hand, $\longrightarrow = \xi \cdot \zeta$, where ξ denotes the division relation on S, which is the smallest positive quasi-order on S, and "$\cdot$" denotes the product of relations. Thus, $\longrightarrow^\infty = (\xi \cdot \zeta)^\infty$, and since ξ and ζ are quasi-orders, then $(\xi \cdot \zeta)^\infty$ is the join of ξ and ζ in $\mathcal{Q}(S)$. □

Let us emphasize that the lattice of semilattice congruences on a semigroup S is a principal dual ideal of the lattice of congruences on S. A similar result for positive quasi-orders with the *cm*-property will be proved in Theorem 6. First we give the following lemma:

Lemma 1. *A positive quasi-order ξ on a semigroup S satisfies the cm-property if and only if it is a lower-potent half-congruence.*

Proof. A proof of this lemma is contained in the proof of Theorem 4.9. [11]. □

Now we are ready to prove the following:

Theorem 6. *The lattice of positive quasi-orders on a semigroup S with the cm-property is a principal dual ideal of the lattice of half-congruences on S.*

Proof. It is well-known (see [9], [10] or [4]) that $\longrightarrow^\infty$ is both the smallest element of the lattice of positive quasi-orders on S with the *cm*-property and the smallest element of the lattice of positive lower-potent quasi-orders on S, which is the principal dual ideal of $\mathcal{Q}(S)$ generated by $\longrightarrow^\infty$. By this it follows that for any half-congruence ξ on S, $\longrightarrow^{\infty} \subseteq \xi$ if and only if ξ is positive and lower-potent, and also, by Lemma 1, ξ is positive and lower-potent if and only if it is positive and it satisfies the *cm*-property. Hence, the lattice of positive quasi-orders on S satisfying the *cm*-property is a principal dual ideal generated by $\longrightarrow^{\infty}$ of the lattice of half-congruences on S. □

Note that the lattice of congruence relations on a semigroup S is a closed sublattice of the lattice of equivalence relations on S (see [1, Chapter VI, §4] or [5,§10]). Similarly it can be proved that the lattice of half-congruences on S is a closed sublattice of $\mathcal{Q}(S)$, so by Theorem 6, the lattice of positive quasi-orders with the *cm*-property is also a closed sublattice of $\mathcal{Q}(S)$.

As we noted before, chain decompositions viewed by quasi-orders were considered by the authors in [4], where they were characterized in terms of completely prime ideals. Here we give its another characterization, through filters:

Theorem 7. *The poset of linear positive quasi-orders on a semigroup S satisfying the cm-property is dually isomorphic to the poset of closed 0,1-sublattices of $\mathcal{P}(S)$ consisting of filters of S.*

Proof. This follows immediately by Theorem 4. □

By the previous theorem we obtain the following corollary concerning chain decompositions of a semigroup:

Corollary 3. *The poset of chain decompositions of a semigroup S is isomorphic to the poset of closed 0,1-sublattices of $\mathcal{P}(S)$ consisting of filters of S.*

Example 1. As demonstrated in [4], the poset of linear positive quasi-orders on a semigroup, satisfying the *cm*-property, is isomorphic to the poset of its chain congruences. Generally, these posets are not lattices. For example, the poset of chain congruences on a semigroup that is a direct product of two two-element chains is not a lattice.

Finally we characterize all chain homomorphic images of a semigroup:

Theorem 8. *For any semigroup S, the poset of principal elements of any closed 0,1-sublattice of $\mathcal{P}(S)$ consisting of filters of S is a chain.*

Furthermore, any chain homomorphic image of S is isomorphic to the chain of principal elements of some closed 0,1-sublattice of $\mathcal{P}(S)$ consisting of filters of S.

Proof. Let K be any closed 0,1-sublattice of $\mathcal{P}(S)$ consisting of filters of S. By Theorem 7, the quasi-order ξ_K is linear, whence $K(a) = \xi_K a \subseteq \xi_K b = K(b)$ or $K(b) = \xi_K b \subseteq \xi_K a = K(a)$, for all $a, b \in S$, whence the poset of principal elements of K is a chain.

Further, let Y be any chain homomorphic image of S, let θ be a chain congruence on S that is a kernel of a homomorphism of S onto Y and let ξ be a linear, positive quasi-order on S with the *cm*-property whose natural equivalence is θ. As we proved before, the poset P of principal elements of K_ξ is a chain, which can be considered as a semigroup with respect to the set-theoretical union, by Lemma 3 [4]. Also, the mapping $a \mapsto \xi_K a$ is a homomorphism of S onto P whose kernel is θ, so P and Y are isomorphic chains. $\square$

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