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A NOTE ON CONGRUENCES ON ALGEBRAS

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Abstract. The purpose of this paper is to characterize algebraic classes closed under subdirect products and under homomorphisms by means of the properties of related posets of congruences. By the obtained results we deduce a recent result of the authors which gives a consequent characterization of varieties of algebras.

Throughout this paper all algebras and classes of algebras will be of a fixed type. By an *algebraic class* we mean a class of algebras closed under isomorphisms. The lattice of congruences on an algebra A is denoted by Con$A$. For a set A, Δ_A will denote the equality relation on A.

We say that a poset P is *dually directed* if any two-element subset (and any hence finite subset) of P has a lower bound in P. A subset K of a lattice L will be called a *meet* (*join*)*-subsemilattice* of L if $a \wedge b \in K$ ($a \vee b \in K$), for all $a, b \in K$, and it will be called a *dual order-ideal* of L if $[a) \subseteq K$, for any $a \in K$, where $[a) = \{x \in L \,|\, a \leq x\}$ is the *principal dual ideal* of L generated by a. A subset K of a complete lattice L will be called a *complete meet* (*join*)*-subsemilattice* of L if $\bigwedge X \in K$ ($\bigvee X \in K$), for any nonempty subset X of K, and it will be called a *complete sublattice* of L if it is both complete meet- and join-subsemilattice of L.

For undefined notions and notations we refer to [2], [3], [8] and [11].

In investigations in various algebraic theories we often work with a set of congruences of a given type on an algebra. Formally, a type $\mathfrak{T}$ congruences can be defined as a mapping which to any algebra A associates some subset (possibly empty) Con$_{\mathfrak{T}}A$ of Con$A$, and the elements of Con$_{\mathfrak{T}}A$ are called the *congruences of type* $\mathfrak{T}$, or shortly, $\mathfrak{T}$*-congruences* on A. The most frequently,

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types of congruences are determined by the membership of the related factors to a given class of algebras. More precisely, for a non-empty algebraic class $\mathfrak{C}$ of algebras, a congruence θ on an algebra A we call a $\mathfrak{C}$-*congruence* if A/θ belongs to $\mathfrak{C}$, and the set of all $\mathfrak{C}$-congruences on A we denote by $\mathrm{Con}_{\mathfrak{C}}A$.

The set $\mathrm{Con}_{\mathfrak{T}}A$ is a poset with respect to the usual ordering of relations, so we meet many questions concerning the order-theoretic properties of $\mathrm{Con}_{\mathfrak{T}}A$. For example, we meet questions such as: Does $\mathrm{Con}_{\mathfrak{T}}A$ has a smallest or greatest element? Is $\mathrm{Con}_{\mathfrak{T}}A$ a (complete) lattice? Is $\mathrm{Con}_{\mathfrak{T}}A$ a (complete) sublattice of $\mathrm{Con}A$?

The problems of this type were considered by T. Tamura and N. Kimura in [19], 1954, where they proved that the poset of semilattice congruences on an arbitrary semigroup S is a complete lattice. A more general result was obtained in another their paper [20], 1955, where they proved that for an arbitrary variety $\mathfrak{C}$ of semigroups, the poset $\mathrm{Con}_{\mathfrak{C}}S$ is a complete lattice, for any semigroup S. Various similar problems, concerning smallest or minimal congruences of certain types, were also discussed by T. Tamura in [13]–[18], and in another papers of this author.

The conditions on an algebraic class $\mathfrak{C}$ of algebras under which there exists the smallest $\mathfrak{C}$-congruence were noted (without proof) by N. Kimura in [9], 1958, and this result is given by the equivalence of the conditions (iii) and (iv) of the next theorem. As was also noted by N. Kimura, the same result was also given by E. J. Tully. Except the mentioned result, in the following theorem we also give two another equivalents of this result:

Theorem 1. *The following conditions for an algebraic class* $\mathfrak{C}$ *of algebras are equivalent:*

(i) $\mathrm{Con}_{\mathfrak{C}}A$ *is a complete meet-subsemilattice of* $\mathrm{Con}A$*, for any algebra* A *for which* $\mathrm{Con}_{\mathfrak{C}}A$ *is non-empty;*
(ii) $\mathrm{Con}_{\mathfrak{C}}A$ *is a complete meet-semilattice, for any algebra* A *for which* $\mathrm{Con}_{\mathfrak{C}}A$ *is non-empty;*
(iii) $\mathrm{Con}_{\mathfrak{C}}A$ *has a smallest element, for any algebra* A *for which* $\mathrm{Con}_{\mathfrak{C}}A$ *is non-empty;*
(iv) $\mathfrak{C}$ *is closed under subdirect products.*

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (iv). Assume a family $\{A_i \,|\, i \in I\}$ of algebras from $\mathfrak{C}$. Let A be an arbitrary subdirect product of this family and let $\{\theta_i \,|\, i \in\}$ be the corresponding family of factor congruences on A. Then $\theta_i \in \mathrm{Con}_{\mathfrak{C}}A$, for any $i \in I$, and $\cap_{i\in I}\theta_i = \Delta_A$. On the other hand, by the hypothesis we have that there exists a smallest element μ of $\mathrm{Con}_{\mathfrak{C}}A$, so by $\cap_{i\in I}\theta_i = \Delta_A$ it follows $\mu = \Delta_A$. Therefore, $\Delta_A \in \mathrm{Con}_{\mathfrak{C}}A$, and this means that $A \in \mathfrak{C}$, which was to be proved. Hence, $\mathfrak{C}$ is closed under subdirect products.

(iv) $\Rightarrow$ (i). Assume an algebra A for which $\mathrm{Con}_{\mathfrak{C}}A$ is non-empty, and assume an arbitrary family $\{\theta_i \mid i \in I\}$ of elements of $\mathrm{Con}_{\mathfrak{C}}A$. Let $\theta = \cap_{i\in I}\theta_i$. For $\vartheta_i = \theta_i/\theta$, $i \in I$, by the Second Isomorphism Theorem we have $\vartheta_i \in \mathrm{Con}A/\theta$ and $(A/\theta)/\vartheta_i \cong A/\theta_i$, for any $i \in I$. Now, by the Correspondence Theorem we have $\cap_{i\in I}\vartheta_i = \Delta_{A/\theta}$, whence it follows that A/θ is a subdirect product of algebras A/θ_i, $i \in I$. But, $A/\theta_i \in \mathfrak{C}$, for any $i \in I$, so by the hypothesis we obtain that $A/\theta \in \mathfrak{C}$, i.e. $\theta \in \mathrm{Con}_{\mathfrak{C}}A$, which was to be proved. Therefore, (i) holds. □

By a similar methodology the following can be proved:

Corollary 1. *The following conditions for an algebraic class $\mathfrak{C}$ of algebras are equivalent:*

(i) $\mathrm{Con}_{\mathfrak{C}}A$ *is a meet-subsemilattice of* $\mathrm{Con}A$*, for any algebra A for which* $\mathrm{Con}_{\mathfrak{C}}A$ *is non-empty;*
(ii) $\mathrm{Con}_{\mathfrak{C}}A$ *is a meet-semilattice, for any algebra A for which* $\mathrm{Con}_{\mathfrak{C}}A$ *is non-empty;*
(iii) $\mathrm{Con}_{\mathfrak{C}}A$ *is dually directed, for any algebra A for which* $\mathrm{Con}_{\mathfrak{C}}A$ *is non-empty;*
(iv) $\mathfrak{C}$ *is closed under finite subdirect products.*

By the next theorem we characterize algebraic classes closed under homomorphisms.

Theorem 2. *The following conditions for an algebraic class $\mathfrak{C}$ of algebras are equivalent:*

(i) $\mathrm{Con}_{\mathfrak{C}}A$ *is a dual order ideal of* $\mathrm{Con}A$*, for any algebra A;*
(ii) $\mathrm{Con}_{\mathfrak{C}}A$ *is a complete join-subsemilattice of* $\mathrm{Con}A$*, for any algebra A;*
(iii) $\mathrm{Con}_{\mathfrak{C}}A$ *is a join-subsemilattice of* $\mathrm{Con}A$*, for any algebra A;*
(iv) $\mathfrak{C}$ *is closed under homomorphisms.*

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (iv). Let $\mathrm{Con}_{\mathfrak{C}}A$ be a join-subsemilattice of $\mathrm{Con}A$ for any algebra A. To prove that $\mathfrak{C}$ is closed under homomorphisms, assume $A \in \mathfrak{C}$ and a homomorphism φ of A onto an algebra H. For $i = 1, 2$, let $A_i = A$, $\varphi_i = \varphi$, let π_i be the projection homomorphism of $A_1 \times A_2$ onto A_i, let $P = \{p \in A_1 \times A_2 \mid p\pi_1\varphi_1 = p\pi_2\varphi_2\}$, let π_i' be the restriction of π_i to P, let $\theta_i = \ker\pi_i$ and let $\theta = \theta_1 \vee \theta_2$ in $\mathrm{Con}P$. Since $P/\theta_i \cong A_i$ then $\theta_i \in \mathrm{Con}_{\mathfrak{C}}P$, for $i = 1, 2$, and $\theta \in \mathrm{Con}_{\mathfrak{C}}P$, by the hypothesis.

On the other hand, there exists a homomorphism ψ of P onto H such that $\psi = \pi_1'\varphi_1 = \pi_2'\varphi_2$. Clearly $\theta_i \subseteq \ker\psi$, for $i = 1, 2$, so $\theta \subseteq \ker\psi$. Conversely, if $(p, q) \in \ker\psi$ and $r = (p\pi_1', q\pi_2')$, then $r \in P$, $(p, r) \in \theta_1$ and $(r, q) \in \theta_2$, whence $(p,q) \in \theta_1\theta_2 \subseteq \theta$. Thus, $\ker\psi = \theta$, so $P/\theta \cong H$, and by $\theta \in \mathrm{Con}_{\mathfrak{C}}P$ we obtain $H \in \mathfrak{C}$, which was to be proved.

(iv) $\Rightarrow$ (i). Assume an arbitrary algebra A, $\theta \in \mathrm{Con}_{\mathfrak{C}}A$ and $\vartheta \in \mathrm{Con}A$ such that $\theta \subseteq \vartheta$. By the Second Isomorphism Theorem, A/ϑ is a homomorphic image of A/θ, and $A/\theta \in \mathfrak{C}$, whence $A/\vartheta \in \mathfrak{C}$, i.e. $\vartheta \in \mathrm{Con}_{\mathfrak{C}}A$, which verifies that $\mathrm{Con}_{\mathfrak{C}}A$ is a dual order-ideal of $\mathrm{Con}A$. □

Let us observe that the algebra P constructed in the proof of the previous theorem is obtained starting from algebras A_1 and A_2, and using their common homomorphic image H and homomorphisms φ_1 and φ_2. Such a construction is known as the *pullback product* of algebras A_1 and A_2 with respect to H and homomorphisms φ_1 and φ_2. This concept was introduced by L. Fuchs in [6], 1952, and after that it was studied by I. Fleischer in [7], 1955, and G. H. Wenzel in [21], 1968. As was proved by G. H. Wenzel, pullback products are exactly the subdirect products whose related systems of factor congruences satisfy conditions which can be viewed as a generalization of the conditions of the famous Chinese Remainder Theorem. In Theory of semigroups, pullback products are known as *spined products*.

As an immediate consequence of Theorems 1 and 2 we obtain the following theorem concerning varieties of algebras, proved by the authors in [5].

Theorem 3. *The following conditions on an algebraic class $\mathfrak{C}$ of algebras are equivalent:*

(i) $\mathrm{Con}_{\mathfrak{C}}A$ *is a complete sublattice of* $\mathrm{Con}A$*, for any algebra A;*
(ii) $\mathrm{Con}_{\mathfrak{C}}A$ *is a principal dual ideal of* $\mathrm{Con}A$*, for any algebra A;*
(iii) $\mathfrak{C}$ *is a variety.*

Recall that the various other characterizations of varieties of algebras were given by G. Birkhoff in [1], 1935, S. R. Kogalovskiĭ in [10], 1965, and B. M. Schein in [12], 1965.

The following theorem is a consequence of the previous one:

Theorem 4. *A variety $\mathfrak{C}$ of algebras is congruence-distributive* (*resp. congruence-modular*) *if and only if the lattice* $\mathrm{Con}_{\mathfrak{C}}A$ *is distributive* (*resp. modular*)*, for any algebra A.*

Proof. For an arbitrary algebra A, by the Correspondence Theorem and Theorem 3 we have that $\mathrm{Con}_{\mathfrak{C}}A \cong \mathrm{Con}A/\mu$, where μ denotes the smallest $\mathfrak{C}$-congruence on A. Since $A/\mu \in \mathfrak{C}$, then the congruence-distributivity (resp. congruence-modularity) of $\mathfrak{C}$ implies the distributivity (resp. modularity) of $\mathrm{Con}A/\mu$, and hence of $\mathrm{Con}_{\mathfrak{C}}A$.

The converse is obvious. □

A very interesting result concerning the lattice $\mathrm{Con}_{\mathfrak{C}}A$, for a variety $\mathfrak{C}$ of algebras and an algebra A, was given by M. Petrich in [11], 1973. He proved that if $\mathfrak{D}$ is the class of all subdirectly irreducible algebras from $\mathfrak{C}$, then the poset $\mathrm{Con}_{\mathfrak{D}}A$ is meet-dense in $\mathrm{Con}_{\mathfrak{C}}A$, i.e. any $\mathfrak{C}$-congruence on A can be represented as the intersection of some family of $\mathfrak{D}$-congruences on A. He also gave some significant applications of this result in Theory of semigroups, for constructions of the smallest congruences corresponding to certain varieties of semigroups.

The results obtained above have also very important applications in Theory of semigroups, in investigations of various decompositions of semigroups. For example, if the class $\mathfrak{C}$ is assumed to be the class of all chains (considered as semigroups), then $\mathfrak{C}$ is not closed under subdirect products, so there are semigroups which have not a greatest chain decomposition.

Let $\mathfrak{C}$ be the class of all semigroups with zero. Then $\mathfrak{C}$ is closed under homomorphisms and finite subdirect products, but not under infinite subdirect products (it is closed only under infinite pullback products). Therefore, for any semigroup S, $\mathrm{Con}_{\mathfrak{C}}S$ is a dual ideal of $\mathrm{Con}S$, but there are semigroups having no a smallest $\mathfrak{C}$-congruence. Minimal elements in $\mathrm{Con}_{\mathfrak{C}}S$ were investigated by T. Tamura in [15], 1965.

More information about decompositions of semigroups having the greatest one can be found in M. Ćirić and S. Bogdanović [4], 1995.

References

[1] G. Birkhoff, *On the structure of abstract algebras*, Proc. Camb. Phil. Soc. **31** (1935), 433–454.

[2] G. Birkhoff, *Lattice theory*, Amer. Math. Soc, Coll. Publ. Vol. 25, (3rd. edition, 3rd. printing), Providence, 1979.

[3] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, 1983.

[4] M. Ćirić and S. Bogdanović, *Theory of greatest decompositions of semigroups* (*A survey*), FILOMAT (Niš) (S. Bogdanović, M. Ćirić and Ž. Perović, eds.) **9:3** (1995), 385–426.

[5] M. Ćirić and S. Bogdanović, *Posets of* $\mathfrak{C}$*-congruences*, Algebra Universalis **36** (1996), 423–424..

[6] L. Fuchs, *On subdirect unions* I, Acta. Math. Acad. Sci. Hungar. **3** (1952), 103–120.

[7] I. Fleischer, *A note on subdirect products*, Acta. Math. Acad. Sci. Hungar. **6** (1955), 463–465.

[8] G. Grätzer, *General Lattice Theory*, Akademie-Verlag, Berlin, 1978.

[9] N. Kimura, *On Some Existence Theorems on Multiplicative Systems.* I. *Greatest Quotient*, Proc. Japan Acad. **34** (1958), 305–309.

[10] S. R. Kogalovskii, *On the theorem of Birkhoff*, Uspehi Mat. Nauk. **20 (5)** (1965), 206–207. (in Russian)

[11] M. Petrich, *Introduction to semigroups*, Merill, Ohio, 1973.

[12] B. M. Schein, *On the theorem of Birkhoff-Kogalovskii*, Uspehi Mat. Nauk. **20 (6)** (1965), 173–174. (in Russian)
[13] T. Tamura, *The theory of construction of finite semigroups* I, Osaka Math. J. **8** (1956), 243–261.
[14] T. Tamura, *The theory of operations on binary relations*, Trans. Amer. Math. Soc. **120** (1965), 343–358.
[15] T. Tamura, *Semigroups with a maximal homomorphic image having zero*, Proc. Japan Acad. **41** (1965), 681–685.
[16] T. Tamura, *Minimal or smallest relation of a given type*, Proc. Japan Acad. **42** (1966), 111–114.
[17] T. Tamura, *Decomposability of extension and its application to finite semigroups*, Proc. Japan Acad. **43** (1967), 93–97.
[18] T. Tamura, *Maximal or greatest homomorphic images of a given type*, Canad. J. Math. **20** (1968), 264–271.
[19] T. Tamura and N. Kimura, *On decompositions of a commutative semigroup*, Kodai Math. Sem. Rep. **4** (1954), 109–112.
[20] T. Tamura and N. Kimura, *Existence of a greatest decomposition of a semigroup*, Kodai Math. Sem. Rep. **7** (1955), 83–84.
[21] G. H. Wenzel, *Note on a subdirect representation of universal algebras*, Acta. Math. Acad. Sci. Hungar. **18** (1967), 329–333.

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