A NEW APPROACH TO SOME GREATEST DECOMPOSITIONS OF SEMIGROUPS

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1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is to give a short presentation of new results concerning greatest decompositions of semigroups. The topic in question will be semilattice decompositions and decompositions of semigroups with zero: Orthogonal decompositions and decompositions into a right sum of semigroups.

The greatest semilattice decompositions were subject of interest of many mathematicians during last four decades. In this paper we present some new results of the authors [15] treating this topic. These results were obtained using a new approach to the problem and they join former results from this area.

To the difference from semilattice decompositions, studying of orthogonal decompositions and decompositions into a right sum is generally a new problem treated in the papers of the authors [11], [12].

In all the previous types of decompositions the authors use methods founded on the usage of various types of ideals (two-sided and one-sided) and the equivalence systems that generalize Green's equivalences.

This paper is divided into three chapters: In §1 we introduce basic notions and notations, §2 is devoted to semilattice decompositions and in §3 we present the basic results concerning decompositions of semigroups with zero.

Throughout this paper, Z^+ will denote the set of all positive integers, $J(a)$, $L(a)$ and $R(a)$ will denote the principal ideal, principal left ideal and the principal right ideal of a semigroup S and $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ will denote Green's relations of S.

For a binary relation ξ on a set A, ξ^n, $n \in Z^+$ will denote the n-th power of ξ in the semigroup of binary relations on A, and ξ^∞ will denote the transitive closure of ξ. Let $\mathfrak{S}$ denote the class of all semigroups. By a *type of relations* we mean any family $\vartheta = \{\vartheta_S \mid S \in \mathfrak{S}\}$ of relations such that ϑ_S is a relation of S, for each $S \in \mathfrak{S}$. If $S \in \mathfrak{S}$, then we say that ϑ_S is a *relation of the type* ϑ of S, and if we consider one fixed semigroup S, then we write simply ϑ instead ϑ_S. If ϑ is a type of relations, then a semigroup S is *ϑ-simple* if ϑ_S is the universal relation of S.

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Let A be a subset of a semigroup S. Then $\sqrt{A} = \{x \in S \mid (\exists n \in Z^+)\, x^n \in A\}$. If for $x, y \in S$, $xy \in A$ implies $x \in A$ or $y \in A$, then A is a *completely prime subset* of S. Clearly, the empty set is a completely prime subset of S. If for $x \in S$, $x^2 \in A$ implies $x \in A$, then A is a *completely semiprime subset* of S. If for $x, y \in S$, $xy \in A$ implies $x, y \in A$ ($xy \in A$ implies $x \in A$, $xy \in A$ implies $y \in A$), then A is a *consistent* (*left consistent*, *right consistent*) *subset* of S. Clearly, the empty set is a consistent subset of S.

A subset A of a semigroup $S = S^0$ is a *0-consistent* (*left 0-consistent*, *right 0-consistent*) *subset* of S if $A^\bullet$ is a consistent (left consistent, right consistent) subset of S.

A subsemigroup A of a semigroup S is a *filter* (*left filter*, *right filter*) of S if A is a consistent (right consistent, left consistent) subset of S.

Let a be an element of a semigroup S. By a *principal* (*left*, *right*) *radical* of S generated by a we mean the smallest completely semiprime (left, right) ideal of S containing a, i.e. the intersection of all completely semiprime (left, right) ideals of S containing a. By Σ_S we denote the *set of all principal radicals of* S. By a *principal* (*left*, *right*) *filter* of S generated by a we mean the smallest (left, right) filter of S containing a, i.e. the intersection of all (left, right) filters of S containing a.

Let A be an ideal of a semigroup S. If for $x, y \in S$, $xSy \subseteq A$ implies $x \in A$ or $y \in A$, then A is a *prime ideal* of S. It is well known that A is a prime ideal of S if and only if for ideals M, N of S, $MN \subseteq A$ implies $M \subseteq A$ or $N \subseteq A$. If for $x \in S$, $xSx \subseteq A$ implies $x \in A$, then A is a *semiprime ideal* of S.

A lattice L is *bounded* if it has a zero and a unity. A lattice L is *complete for joins* (*complete for meets*) if every nonempty subset of L has a join (meet) and it is *complete* if it is complete both for joins and for meets. An element a of a lattice L with the zero 0 is an *atom* of L if $a > 0$ and there exists no $x \in L$ such that $a > x > 0$. A complete Boolean algebra B is *atomic* if every element of B is the join of some set of atoms of B. If L is a distributive bounded lattice, then the set $\mathfrak{B}(L)$ of all elements of L having a complement in L is a Boolean algebra and it is called the *greatest Boolean subalgebra* of L. A lattice L complete for joins is *infinitely distributive for meets* if $a \wedge (\vee_{\alpha \in Y} x_\alpha) = \vee_{\alpha \in Y}(a \wedge x_\alpha)$, for every $a \in S$ and every nonempty subset $\{x_\alpha \mid \alpha \in Y\}$ of L. A nontrivial lattice L is *directly indecomposable* if it has the property: When L is a direct product of lattices L_i, $i \in I$, then there exists $i \in I$ such that L_i is isomorphic to L and $|L_j| = 1$, for every $j \in I$, $j \neq i$.

For a semigroup S, $\mathcal{I}d(S)$ will denote the lattice of all ideals of S. If $S = S^0$, then $\mathcal{I}d(S)$ is a complete lattice, infinitely distributive for meets, with the zero 0 and the unity S. Also, $\mathcal{LI}d(S)$ will denote the lattice of left ideals of a semigroup S defined on the following way: if $S = S^0$, then $\mathcal{LI}d(S)$ contains all of left ideals of S, and if S is without zero, then $\mathcal{LI}d(S)$ contains the empty set and all of left ideals of S. In both of this cases $\mathcal{LI}d(S)$ is a complete lattice, infinitely distributive for meets. Clearly, for a semigroup S without zero, the lattice $\mathcal{LI}d(S)$ is isomorphic to $\mathcal{LI}d(S^0)$. If $S = S^0$, $\mathcal{I}d^{\mathbf{c}}(S)$ will denote the set of all 0-consistent ideals of S and $\mathcal{LI}d^{\mathbf{c}}(S)$ will denote the set of all right 0-consistent left ideals of S.

Let 0 be a fixed element of a set S. Then S with a multiplication defined by: $xy = x$, if $x = y$, and $xy = 0$, otherwise, $x, y \in S$, is a semilattice called

Kronecker's semilattice.

For undefined notions and notations we refer to [2], [6], [13], [17], [18], [30], [31] and [40].

2. SEMILATTICE DECOMPOSITIONS

Semilattice decompositions of semigroups were first defined and studied by A.H.Clifford [16], 1941. After that, several authors worked on this very important topic. A significant contribution to the Theory of semilattice decompositions of semigroups was given by T.Tamura. A series of papers concerning this topic was opened by T.Tamura and N.Kimura. In the paper [55], 1954, they considered semilattice decompositions of commutative semigroups. The same authors in [56], 1955, and M.Yamada in [62], 1955, established the existence of the greatest semilattice decomposition of an arbitrary semigroup. T.Tamura [44], 1956, proved the fundamental result that components in the greatest semilattice decomposition of a semigroup are semilattice indecomposable. In [46], 1964, he described the smallest semilattice congruence on a semigroup, using the concept of contents. Various other characterizations of this congruence were given by the same and several other authors. M.Petrich [28], 1964, gave a characterization of this congruence using completely prime ideals and filters. Another connection among these concepts was given by R.Šulka [43], 1970. T.Tamura in [50], 1972, and [52], 1973, proved that $\longrightarrow^{\infty} \cap (\longrightarrow^{\infty})^{-1}$ is the smallest semilattice congruence of a semigroup and M.S.Putcha [36], 1974, proved this for the relation $(\longrightarrow \cap \longrightarrow^{-1})^{\infty}$. Finally, M.Ćirić and S.Bogdanović [15] gave a characterization of the greatest semilattice homomorphic image of a semigroup by completely semiprime ideals. Using completely semiprime subsets and ideals they defined an equivalence system that generalizes Green's equivalences and they developed a new method in the Theory of semilattice decompositions of semigroups.

For semilattice decompositions whose components are Archimedean we refer to the former survey article of the authors [9].

In Section 2.1 we present general results concerning the greatest semilattice decomposition of a semigroup and results concerning some special cases. Section 2.2 is devoted to semilattice decompositions by the relation λ.

2.1. The greatest semilattice decomposition.

Let $\mathfrak{C}$ be a class of semigroups. A congruence ξ of a semigroup S is a $\mathfrak{C}$-*congruence* of S if the factor S/ξ is in $\mathfrak{C}$. The partition and the factor determined by a $\mathfrak{C}$-congruence of a semigroup S are called a $\mathfrak{C}$-*decomposition* and a $\mathfrak{C}$-*homomorphic image* of S, respectively. If $\mathfrak{C}$ is a class of all bands, we have *band congruences*, *band decompositions* and *band homomorphic images*, if $\mathfrak{C}$ is a class of all semilattices, we have *semilattice congruences*, *semilattice decompositions* and *semilattice homomorphic images*, if $\mathfrak{C}$ is a class of all rectangular bands, we have *matrix congruences* and *matrix decompositions*, and if $\mathfrak{C}$ is a class of all left (right) zero bands, then we have *left* (*right*) *zero congruences* and *left* (*right*) *zero decompositions*. A $\mathfrak{C}$-congruence ξ of a semigroup S is a *smallest* $\mathfrak{C}$-*congruence* of S if ξ is contained in every $\mathfrak{C}$-congruence of S. The partition and the factor determined by the smallest $\mathfrak{C}$-congruence of a semigroup S are *greatest* $\mathfrak{C}$-*decomposition* and *greatest* $\mathfrak{C}$-*homomorphic image* of S, respectively. If $\mathfrak{C}$ is a variety of semigroups, then every semigroup have the smallest $\mathfrak{C}$-congruence, i.e. the greatest $\mathfrak{C}$-decomposition.

On a semigroup S we define a *relation of the type* $\longrightarrow$ by:

$$a \longrightarrow b \;\Leftrightarrow\; (\exists n \in Z^+)\, b^n \in J(a), \qquad (a, b \in S).$$

Let S be a semigroup. For $a \in S$ let

$$\Sigma_n(a) = \{x \in S \,|\, a \longrightarrow^n x\},\ n \in Z^+, \quad \Sigma(a) = \{x \in S \,|\, a \longrightarrow^\infty x\}.$$

An equivalent definition of these sets is the following:

$$\Sigma_1(a) = \sqrt{SaS},\ \Sigma_{n+1}(a) = \sqrt{S\Sigma_n(a)S},\ n \in Z^+,\ \Sigma(a) = \bigcup_{n \in Z^+} \Sigma_n(a).$$

Clearly, $\Sigma_n(a) \subseteq \Sigma_{n+1}(a)$, for each $n \in Z^+$, and $\Sigma(a)$ and $\Sigma_n(a)$, $n \in Z^+$, are completely semiprime subsets of S. On S we define *equivalences of the types* σ and σ_n, $n \in Z^+$, by

$$a\,\sigma\, b \;\Leftrightarrow\; \Sigma(a) = \Sigma(b), \qquad a\,\sigma_n\, b \;\Leftrightarrow\; \Sigma_n(a) = \Sigma_n(b),$$

$(a, b \in S)$.

Lemma 2.1 [15] *Let a be an element of a semigroup S. Then $\Sigma(a)$ is the principal radical of S generated by a.* □

Since $\longrightarrow^\infty$ is transitive, then $\sigma = \longrightarrow^\infty \cap (\longrightarrow^\infty)^{-1}$, so the result of T.Tamura [50] describing the smallest semilattice congruence of a semigroup can be formulated on the following way:

Theorem 2.1 *The relation σ on a semigroup S is the smallest semilattice congruence on S and every σ-class is semilattice indecomposable.* □

Let $— = \longrightarrow \cap (\longrightarrow)^{-1}$. M.S.Putcha [36] proved the following

Theorem 2.2 *The relation $—^\infty$ of a semigroup S is the smallest semilattice congruence on S, where $— = \longrightarrow \cap (\longrightarrow)^{-1}$.* □

A characterization of the greatest semilattice homomorphic image of a semigroup by principal radicals was given by M.Ćirić and S.Bogdanović [15]. This result is the following:

Theorem 2.3 *For elements a and b of a semigroup S,*

$$\Sigma(ab) = \Sigma(a) \cap \Sigma(b).$$

Furthermore, the set Σ_S of all principal radicals of S, ordered by inclusion, is the greatest semilattice homomorphic image of S. □

Theorem 2.3 gives many important consequences. As a first, a result of the authors [15] describing principal filters of semigroups. Corollary 2.2 is a result of M.Petrich [28], [30] that characterizes the smallest semilattice congruence of a semigroup with the help of principal filters. Corollaries 2.3 and 2.4 treat the well-known problem of representation of completely semiprime ideals by intersections of completely prime ideals. In Theory of semigroups this problem was considered by K.Iséki [23] and Ш.Шварц [42], and for its solution in the general case we refer to M.Petrich [30]. The same result was proved by the authors in [15] without use of Zorn's lemma arguments.

Let a be an element of a semigroup S. Then

$$N_n(a) = \{x \in S \mid x \longrightarrow^n a\},\ n \in Z^+, \quad N(a) = \{x \in S \mid x \ \longrightarrow^\infty \ a\}.$$

Clearly, $N(a)$ and $N_n(a)$, $n \in Z^+$, are consistent subsets of S.

Corollary 2.1 *Let a be an element of a semigroup S. Then $N(a)$ is the principal filter of S generated by a.* □

Corollary 2.2 *For elements a and b of a semigroup S, $a\,\sigma\,b$ if and only if $N(a) = N(b)$.* □

Corollary 2.3 *Let I be a completely semiprime ideal of a semigroup S and let $a \in S$ such that $a \notin I$. Then there exists a completely prime ideal P of S such that $I \subseteq P$ and $a \notin P$.* □

Corollary 2.4 *Every completely semiprime ideal of a semigroup S is an intersection of completely prime ideals of S.* □

Semilattice indecomposable semigroups were studied by several authors. In the next theorem T.Tamura [50] proved $(i) \ \Leftrightarrow \ (iii)$ and M.Petrich [28], [30] proved $(i) \ \Leftrightarrow \ (iv) \ \Leftrightarrow \ (v)$.

Theorem 2.4 *The following conditions on a semigroup S are equivalent:*

(i) *S is semilattice indecomposable;*

(ii) *S is σ-simple;*

(iii) $(\forall a, b \in S)\ a \ \longrightarrow^\infty \ b$*;*

(iv) *S has not proper completely semiprime ideals;*

(v) *S has not proper completely prime ideals.* □

Semigroups whose greatest semilattice homomorphic image is a chain will be treated by the next theorem. The conditions (ii), (iii) and (vii) are from M.Petrich [28], [30] and the rest is the result of the authors [15].

Theorem 2.5 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of σ-simple semigroups;*

(ii) *Σ_S is a chain;*

(iii) *the partially ordered set of all completely prime ideals of S is a chain;*

(iv) $\longrightarrow^\infty \ \cup(\ \longrightarrow^\infty \ )^{-1}$ *is equal to the universal relation of S;*

(v) *a union of an arbitrary nonempty family of filters of S is a filter of S;*

(vi) $(\forall a, b \in S)\ ab \ \longrightarrow^\infty \ a \vee ab \ \longrightarrow^\infty \ b$*;*

(vii) *every completely semiprime ideal of S is completely prime;*

(viii) *every principal radical of S is completely prime.*

Theorem 2.6 [15] *Let $n \in Z^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a band of σ_n-simple semigroups;*

(ii) *S is a semilattice of σ_n-simple semigroups;*

(iii) *every σ_n-class of S is a subsemigroup;*

(iv) $(\forall a \in S)\ a\ \sigma_n\ a^2$*;*

(v) $(\forall a, b \in S)\ a \longrightarrow^n b \ \Rightarrow \ a^2 \longrightarrow^n b$*;*

(vi) $(\forall a, b, c \in S)\ a \longrightarrow^n c \wedge b \longrightarrow^n c \ \Rightarrow \ ab \longrightarrow^n c$*;*

(vii) *for every $a \in S$, $\Sigma_n(a)$ is an ideal of S;*

(viii) $(\forall a, b \in S)\ \Sigma_n(ab) = \Sigma_n(a) \cap \Sigma_n(b)$*;*

(ix) *for every $a \in S$, $N_n(a)$ is a subsemigroup of S;*

(x) $\longrightarrow^n$ *is a quasi-order on S;*

(xi) $\sigma_n = \longrightarrow^n \cap (\longrightarrow^n)^{-1}$ *on S.*

Corollary 2.5 [15] *Let S be a finite semigroup. Then there exists $n \in Z^+$, $n \leq |S|$, such that S is a semilattice of σ_n-simple semigroups.* □

By Theorem 2.6 we obtain the results of R.Croisot [19], O.Anderson [1] and M.Petrich [28], [30] concerning semilattices of simple semigroups (se also A.H.Clifford and G.B.Preston [17]), and for $n = 1$ we obtain some results of M.S.Putcha [34], T.Tamura [49] and S.Bogdanović and M.Ćirić [7] for semilattices of Archimedean (i.e. σ_1-simple) semigroups. M.Ćirić and S.Bogdanović [14] gave and another result:

Theorem 2.7 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of Archimedean semigroups;*

(ii) $(\forall a, b \in S)(\forall k \in Z^+)\ a^k \longrightarrow ab$;

(iii) $(\forall a, b \in S)\ a^2 \longrightarrow ab$. □

Theorem 2.8 [15] *Let $n \in Z^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a chain of σ_n-simple semigroups;*

(ii) *for every $a \in S$, $\Sigma_n(a)$ is a completely prime ideal of S;*

(iii) *S is a semilattice of σ_n-simple semigroups and for every $a \in S$, $\Sigma_n(a)$ is a completely prime subset of S;*

(iv) *S is a semilattice of σ_n-simple semigroups and for all $a, b \in S$, $ab \longrightarrow^n a$ or $ab \longrightarrow^n b$;*

(v) *S is a semilattice of σ_n-simple semigroups and for all $a \in S$, $a \longrightarrow^n b$ or $b \longrightarrow^n a$.* □

A subset A of a semigroup S is *semiprimary* if

$$(\forall x, y \in S)(\exists n \in Z^+)\ xy \in A \Rightarrow x^n \in A \vee y^n \in A.$$

A semigroup S is *semiprimary* if all of its ideals are semiprimary [3], [4].

In [10] the authors showed that semiprimary semigroups are exactly chains of Archimedean semigroups. This is a part of the following

Theorem 2.9 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of Archimedean semigroups;*

(ii) $(\forall a, b \in S)\ ab \longrightarrow a \vee ab \longrightarrow b$;

(iii) *S is semiprimary;*

(iv) *for every ideal A of S, $\sqrt{A}$ is a completely prime ideal of S;*

(v) *for every ideal A of S, $\sqrt{A}$ is a completely prime subset of S.* □

2.2. Semilattices of λ-simple semigroups.

On a semigroup S we define a *relation of the type* $\overset{l}{\longrightarrow}$ by:

$$a \overset{l}{\longrightarrow} b \Leftrightarrow (\exists n \in Z^+)\ b^n \in L(a), \qquad (a, b \in S).$$

Let a be an element of a semigroup S. Then

$$\Lambda_n(a) = \{x \in S \mid a \longrightarrow^n x\},\ n \in Z^+, \quad \Lambda(a) = \{x \in S \mid a \longrightarrow^\infty x\}.$$

An equivalent definition of these sets is the following:

$$\Lambda_1(a) = \sqrt{SaS},\ \Lambda_{n+1}(a) = \sqrt{S\Lambda_n(a)S},\ n \in Z^+,\ \Lambda(a) = \bigcup_{n \in Z^+} \Lambda_n(a).$$

Clearly, $\Lambda_n(a) \subseteq \Lambda_{n+1}(a)$, for each $n \in Z^+$, and $\Lambda(a)$ and $\Lambda_n(a)$, $n \in Z^+$, are completely semiprime subsets of S. On S we define *equivalences of the types* λ and λ_n, $n \in Z^+$, by

$$a\,\lambda\, b \;\Leftrightarrow\; \Lambda(a) = \Lambda(b), \qquad a\,\lambda_n\, b \;\Leftrightarrow\; \Lambda_n(a) = \Lambda_n(b), \quad (a, b \in S).$$

Lemma 2.2 [15] *Let a be an element of a semigroup S. Then $\Lambda(a)$ is the principal left radical of S generated by a.* $\square$

If in Corollary 2.4. the expression "ideal" we replace with "left ideal", then the assertion do not holds. An example for this is the following semigroup:

$$\langle a, e \mid a^3 = a,\ e^2 = e,\ ae = ea^2 = e\rangle .$$

The conditions under which every completely semiprime left ideal is an intersection of completely prime left ideals are given by:

Theorem 2.10 [15] *The following conditions on a semigroup S are equivalent:*

(i) *every completely semiprime left ideal of S is an intersection of some family of completely prime left ideals of S;*

(ii) $(\forall a, b, c \in S)\ a \xrightarrow{l}{}^{\infty} c \wedge b \xrightarrow{l}{}^{\infty} c \;\Rightarrow\; ab \xrightarrow{l}{}^{\infty} c$;

(iii) *for every $a \in S$, $\{x \in S \mid x \xrightarrow{l}{}^{\infty} a\}$ is a subsemigroup of S.* $\square$

By the next theorem obtained by the authors in [15] various characterizations of semilattices of λ-simple semigroups are given. Afterwards several special cases will be treated.

Theorem 2.11 *The following conditions on a semigroup S are equivalent:*

(i) *S is semilattice of λ-simple semigroups;*

(ii) $(\forall a, b \in S)\ a \xrightarrow{l}{}^{\infty} ab$;

(iii) *for every $a \in S$, $\Lambda(a)$ is an ideal of S;*

(iv) *every completely semiprime left ideal of S is an ideal of S;*

(v) $(\forall a, b \in S)\ \Lambda(ab) = \Lambda(a) \cap \Lambda(b)$;

(vi) *for every $a \in S$, $\{x \in S \mid x \xrightarrow{l}{}^{\infty} a\}$ is a filter of S.* $\square$

Theorem 2.12 [15] *The following conditions on a semigroup S are equivalent:*

(i) *S is chain of λ-simple semigroups;*

(ii) *every left radical of S is a completely prime ideal of S;*

(iii) *S is a semilattice of λ-simple semigroups and for all $a, b \in S$, $ab \xrightarrow{l}{}^{\infty} a$ or $ab \xrightarrow{l}{}^{\infty} b$;*

(iv) *S is a semilattice of λ-simple semigroups and for all $a, b \in S$, $a \xrightarrow{l}{}^{\infty} b$ or $b \xrightarrow{l}{}^{\infty} a$.* $\square$

Theorem 2.13 [15] *Let $n \in Z^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $\xrightarrow{l}{}^{n}$ *is a quasi-order on S;*

(ii) $(\forall a \in S)\ a\,\lambda_n\, a^2$;

(iii) $(\forall a, b \in S)\ a \xrightarrow{l}{}^{n} b \;\Rightarrow\; a^2 \xrightarrow{l}{}^{n} b$;

(iv) *for every $a \in S$, $\Lambda_n(a)$ is a left ideal of S;*

(v) $\lambda_n = \xrightarrow{l}{}^{n} \cap \left(\xrightarrow{l}{}^{n}\right)^{-1}$. $\square$

Theorem 2.14 [7] *The following conditions on a semigroup S are equivalent:*

(i) $(\forall a,b\in S)\ a \xrightarrow{l} b \ \Rightarrow\ a^2 \xrightarrow{l} b$;

(ii) $(\forall a,b\in S)(\forall k\in Z^+)\ b^k \xrightarrow{l} ab$;

(iii) $(\forall a,b\in S)\ b^2 \xrightarrow{l} ab$. □

Theorem 2.15 [15] *Let $n\in Z^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of λ_n-simple semigroups;*

(ii) *$a\,\lambda_n\,a^2$, for every $a\in S$, and $a \xrightarrow{l}{}^n ab$, for all $a,b\in S$;*

(iii) *for every $a\in S$, $\Lambda_n(a)$ is an ideal of S;*

(iv) $(\forall a,b\in S)\ \Lambda_n(ab)=\Lambda_n(a)\cap\Lambda_n(b)$;

(v) *for every $a\in S$, $\{x\in S \mid x \xrightarrow{l}{}^n a\}$ is a filter of S.* □

Semilattices of left simple semigroups was studied by M.Petrich [28], [30]. Semilattices of left Archimedean (i.e. λ_1-simple) semigroups was described by M.S.Putcha [37]. S.Bogdanović [5] gave and another characterization of these semigroups:

Theorem 2.16 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left Archimedean semigroups;*

(ii) $(\forall a,b\in S)(\forall k\in Z^+)\ a^k \xrightarrow{l} ab$;

(iii) $(\forall a,b\in S)\ a \xrightarrow{l} ab$. □

By $\xrightarrow{r}$, ρ_n, $n\in Z^+$, and ρ we denote the dual relations for $\xrightarrow{l}$, λ_n, $n\in Z^+$, and λ, respectively, and $\tau_n=\lambda_n\cap\rho_n$, $n\in Z^+$, and $\tau=\lambda\cap\rho$. The following proposition shows that these relations generalize well-known Green's relations.

Proposition 2.1 [15] *If S is a semigroup, then*

$$\begin{array}{ccccccccccccc}
\mathcal{H} & \subseteq & \tau_1 & \subseteq & \tau_2 & \subseteq & \cdots & \subseteq & \tau_n & \subseteq & \cdots & \subseteq & \tau \\
\cap\!\!| & & \cap\!\!| & & \cap\!\!| & & & & \cap\!\!| & & & & \cap\!\!| \\
\mathcal{L} & \subseteq & \lambda_1 & \subseteq & \lambda_2 & \subseteq & \cdots & \subseteq & \lambda_n & \subseteq & \cdots & \subseteq & \lambda \\
\cap\!\!| & & \cap\!\!| & & & & & & & & & & \cap\!\!| \\
\mathcal{J} & \subseteq & \sigma_1 & \subseteq & \sigma_2 & \subseteq & \cdots & \subseteq & \sigma_n & \subseteq & \cdots & \subseteq & \sigma \\
\cup\!\!| & & \cup\!\!| & & & & & & & & & & \cup\!\!| \\
\mathcal{R} & \subseteq & \rho_1 & \subseteq & \rho_2 & \subseteq & \cdots & \subseteq & \rho_n & \subseteq & \cdots & \subseteq & \rho
\end{array}\ .$$

□

In contrast to semilattices and bands of σ_n-simple semigroups, semilattices and bands of λ- (λ_n-) simple semigroups do not determine the same class of semigroups. M.S.Putcha [35] described bands of left Archimedean and bands of t-Archimedean (i.e. τ_1-simple) semigroups.

Note that the congruence of a semigroup generated by τ is the smallest band congruence of this semigroup.

3. DECOMPOSITIONS OF SEMIGROUPS WITH ZERO

Some "classical" types of decompositions, like decompositions into a right and a left zero band of semigroups and matrix decompositions, degenerate in semigroups with zero. This requires some new decomposition methods specific to semigroups with zero. Here we present two such methods: Orthogonal decompositions and decompositions into a right sum of semigroups.

Orthogonal decompositions of semigroups were first defined and studied by E.C. Ляпин [26], [27], 1950, and by Ш.Шварц [41], 1951. After that, these decompositions were considered by many authors in several special cases (more informations about these the reader can find in the books [18] and [39] and in the survey article [9]). The existence of the greatest orthogonal decomposition of a semigroup with zero was established by S.Bogdanović and M.Ćirić [11]. In the same paper the authors proved also that summands of the greatest orthogonal decomposition of a semigroup with zero are orthogonal indecomposable. This result shouts to (but does not follow from) the existence of the smallest F-congruence of a semigroup, for a given finite set of identities F, discussed by T.Tamura, N.Kimura, M.Yamada, A.H.Clifford and G.B.Preston in 1955–65.

Decompositions of semigroups with zero into a right sum of semigroups are an analogue of decompositions of semigroups without zero into a right zero band of semigroups. For such decompositions the authors in [12] obtained results similar to the ones for orthogonal decompositions. A difference is that sometimes summands in the greatest decomposition into a right sum could be further decomposed into a right sum.

The author's approach to the question of decompositions of semigroups with zero, presented in this paper, is different from the one used by J.Dieudonné [20], 1942, in Theory of rings, and by Ш.Шварц [41], 1951, in Theory of semigroups. Their main tools were the cocle and 0-minimal ideals (two-sided and one-sided). Here the main role is captured by (right) 0-consistent (left) ideals. Note that the notion of (right, left) consistent subset was introduced by P.Dubreil [21], 1941. Also, the author's approach differs from the one of G.Lallement and M.Petrich [25], which studied decompositions of semigroups with zero by congruences whose corresponding factors are 0-rectangular bands. Instead of making of decompositions by congruences, the authors in [12] used (right) 0-consistent equivalences and other equivalence system that generalizes Green's relations.

In Sections 3.1 and 3.2 we treat the greatest orthogonal decomposition and the greatest decomposition into a right sum, respectively. In Section 3.3 we consider left 0-consistent, right 0-consistent and 0-consistent equivalences of a semigroup with zero, their mutual connections and connections with decompositions mentioned above. Results presented there give also mutual connections between these types of decompositions. Finally, in Section 3.4 we present a connection between these decompositions and decompositions of the lattice of ideals of a semigroup with zero into a direct product.

3.1. The greatest orthogonal decomposition.

A semigroup $S = S^0$ is an *orthogonal sum* of semigroups S_α, $\alpha \in Y$, in notation $S = \Sigma_{\alpha\in Y} S_\alpha$, if $S_\alpha \neq 0$, for all $\alpha \in Y$, $S = \cup_{\alpha\in Y} S_\alpha$ and $S_\alpha \cap S_\beta = S_\alpha S_\beta = 0$, for all $\alpha, \beta \in Y$, $\alpha \neq \beta$. In this case, the family $\mathcal{D} = \{S_\alpha | \ \alpha \in Y\}$ is an *orthogonal decomposition* of S and S_α are *orthogonal summands* of S or *summands* in $\mathcal{D}$. If $\mathcal{D}$ and $\mathcal{D}'$ are two orthogonal decompositions of a semigroup $S = S^0$, then we say that $\mathcal{D}$ is *greater than* $\mathcal{D}'$ if each member of $\mathcal{D}$ is a subset of some member of $\mathcal{D}'$. A semigroup $S = S^0$ is *orthogonal indecomposable* if $\mathcal{D} = \{S\}$ is the unique orthogonal decomposition of S.

Lemma 3.1 [11] *The following conditions for an ideal A of a semigroup*

$S = S^0$ *are equivalent:*

(i) A *is 0-consistent;*

(ii) A' *is an ideal of* S*;*

(iii) A *is an orthogonal summand of* S. $\square$

Let we introduce a *relation of the type* $\sim$ on a semigroup $S = S^0$ by

$$x \sim y \Leftrightarrow J(x) \cap J(y) \neq 0, \quad \text{for } x, y \in S^{\bullet}, \qquad 0 \sim 0.$$

Clearly, $\sim$ is reflexive and symmetric. For $a \in S$, $n \in Z^+$, let

$$\Delta_n(a) = \{x \in S \mid x \sim^n a\} \cup 0, \quad \Delta(a) = \{x \in S \mid x \sim^{\infty} a\} \cup 0 = \cup_{n \in Z^+} \Delta_n(a).$$

Clearly, $\Delta_n(0) = 0$, for each $n \in Z^+$, and $\Delta_n(a) \subseteq \Delta_{n+1}(a)$, for all $a \in S$, $n \in Z^+$. Also, let we introduce *equivalences of the types* δ and δ_n, $n \in Z^+$, on S by

$$a \,\delta\, b \Leftrightarrow \Delta(a) = \Delta(b), \qquad a \,\delta_n\, b \Leftrightarrow \Delta_n(a) = \Delta_n(b),$$

$(a, b \in S)$. For $a \in S$, Δ_a will denote the δ-class of a. Clearly, $\Delta_0 = \Delta(0) = 0$, and $\delta_n \subseteq \sim^n$, for each $n \in Z^+$.

By the following theorem principal 0-consistent ideals of a semigroup with zero are described.

Theorem 3.1 [11] *Let* $a \neq 0$ *be an element of a semigroup* $S = S^0$. *Then*

(a) $\Delta(a)$ *is the principal 0-consistent ideal of* S *generated by* a*;*

(b) $\Delta(a) = \Delta_a^0$*;*

(c) $\Delta(a)$ *is an orthogonal indecomposable semigroup.* $\square$

Because of Theorem 3.1, it follows that $\delta = \sim^{\infty}$ on every semigroup with zero.

Lemma 3.2 [12] *Let* A *be a 0-consistent ideal of a semigroup* $S = S^0$. *Then*

$$\mathcal{LId}(A) \subseteq \mathcal{LId}(S),\ \mathcal{LId}^{\mathbf{c}}(A) \subseteq \mathcal{LId}^{\mathbf{c}}(S),$$
$$\mathcal{Id}(A) \subseteq \mathcal{Id}(S),\ \mathcal{Id}^{\mathbf{c}}(A) \subseteq \mathcal{Id}^{\mathbf{c}}(S). \quad \square$$

Important features of the set of all 0-consistent ideals in the lattice of all ideals of a semigroup with zero are presented by the next theorem. This gives the main result of the theory of orthogonal decompositions of semigroups.

Theorem 3.2 [11] *For an arbitrary semigroup* $S = S^0$, $\mathcal{Id}^{\mathbf{c}}(S)$ *is a complete atomic Boolean algebra and* $\mathcal{Id}^{\mathbf{c}}(S) = \mathfrak{B}(\mathcal{Id}(S))$.

Furthermore, every complete atomic Boolean algebra is isomorphic to the Boolean algebra of 0-consistent ideals of some semigroup with zero. $\square$

Theorem 3.3 [11] *Every semigroup with zero have a greatest orthogonal decomposition and every summand of this decomposition is an orthogonal indecomposable semigroup.* $\square$

Note that summands in the greatest orthogonal decomposition of a semigroup $S = S^0$ are all the atoms of $\mathcal{Id}^{\mathbf{c}}(S)$.

Corollary 3.1 [11] *The following conditions on a semigroup* $S = S^0$ *are equivalent:*

(i) *S is orthogonal indecomposable;*
(ii) *S have not proper 0-consistent ideals;*
(iii) $(\forall a, b \in S^\bullet)\ a \sim^\infty b.$ □

3.2. The greatest decomposition into a right sum.

A semigroup $S = S^0$ is a *right sum* of semigroups $S_\alpha,\ \alpha \in Y$, in notation $S = \mathrm{R}\Sigma_{\alpha\in Y} S_\alpha$, if $S_\alpha \neq 0$, for all $\alpha \in Y$, $S = \cup_{\alpha\in Y} S_\alpha$, and $S_\alpha \cap S_\beta = 0$ and $S_\alpha S_\beta \subseteq S_\beta$, for all $\alpha, \beta \in Y,\ \alpha \neq \beta$. In this case, the family $\mathcal{D} = \{S_\alpha |\ \alpha \in Y\}$ is a *decomposition of S into a right sum* and S_α are *right summands* of S or *summands* in $\mathcal{D}$. If $\mathcal{D}$ and $\mathcal{D}'$ are two decompositions of a semigroup $S = S^0$ into a right sum, then we say that $\mathcal{D}$ is *greater than* $\mathcal{D}'$ if each member of $\mathcal{D}$ is a subset of some member of $\mathcal{D}'$. A semigroup $S = S^0$ is *indecomposable into a right sum* if $\mathcal{D} = \{S\}$ is the unique decomposition of S into a right sum.

Similarly we define *left sums* of semigroups and the related notions and notations.

Lemma 3.3 [12] *The following conditions for an ideal A of a semigroup $S = S^0$ are equivalent:*
(i) *A is right 0-consistent;*
(ii) *A' is a left ideal of S;*
(iii) *A is a right summand of S.* □

Let we introduce a *relation of the type* $\overset{\ell}{\sim}$ on a semigroup $S = S^0$ by

$$x \overset{\ell}{\sim} y \Leftrightarrow L(x) \cap L(y) \neq 0, \quad \text{for } x, y \in S^\bullet, \qquad 0 \overset{\ell}{\sim} 0.$$

Clearly, $\overset{\ell}{\sim}$ is reflexive and symmetric. For $a \in S,\ n \in Z^+$, let

$$\mathrm{K}_n(a) = \{x \in S \mid x \overset{\ell}{\sim}{}^n a\} \cup 0, \quad \mathrm{K}(a) = \{x \in S \mid x \overset{\ell}{\sim}{}^\infty a\} \cup 0 = \cup_{n\in Z^+} \mathrm{K}_n(a).$$

Clearly, $\mathrm{K}_n(0) = 0$, for each $n \in Z^+$, and $\mathrm{K}_n(a) \subseteq \mathrm{K}_{n+1}(a)$, for all $a \in S,\ n \in Z^+$. Also, let we introduce *equivalences of the types* κ and $\kappa_n,\ n \in Z^+$, on S by

$$a\,\kappa\, b \Leftrightarrow \mathrm{K}(a) = \mathrm{K}(b), \qquad a\,\kappa_n\, b \Leftrightarrow \mathrm{K}_n(a) = \mathrm{K}_n(b), \qquad (a, b \in S).$$

For $a \in S$, K_a will denote the κ-class of a. Clearly, $\mathrm{K}_0 = \mathrm{K}(0) = 0$.

Let $\overset{r}{\sim}$ be the relation obtained by replacement of principal left by principal right ideals in the definition for $\overset{\ell}{\sim}$, and let υ and $\upsilon_n,\ n \in Z^+$, be the relations obtained by replacement of $\overset{\ell}{\sim}$ by $\overset{r}{\sim}$ in definitions for κ and $\kappa_n,\ n \in Z^+$, respectively, and let $\mu = \kappa \cap \upsilon$, $\mu_n = \kappa_n \cap \upsilon_n,\ n \in Z^+$.

Principal right 0-consistent left ideals of a semigroup with zero characterize the following

Theorem 3.4 [12] *Let $a \neq 0$ be an element of a semigroup $S = S^0$. Then*
(a) $\mathrm{K}(a)$ *is the principal right 0-consistent left ideal of S generated by a;*
(b) $\mathrm{K}(a) = \mathrm{K}_a^0$*;*
(c) $\mathrm{K}(a)$ *contains not right 0-consistent left ideals of S different to* 0 *and* $\mathrm{K}(a)$. □

Because of Theorem 3.4, $\kappa = \overset{\ell}{\sim}{}^\infty$ on every semigroup with zero.

Some properties of orthogonal decompositions hold also for decompositions into a right sum:

Theorem 3.5 [12] *For an arbitrary semigroup* $S = S^0$, $\mathcal{LI}d^{\mathbf{c}}(S)$ *is a complete atomic Boolean algebra and* $\mathcal{LI}d^{\mathbf{c}}(S) = \mathfrak{B}(\mathcal{LI}d(S))$. $\square$

Theorem 3.6 [12] *Every semigroup with zero have a greatest decomposition into a right sum.* $\square$

In contrast to orthogonal decompositions, summands in the greatest decomposition of a semigroup with zero into a right sum, sometimes could be further decomposed into a right sum. An example for this was given in [12].

An analogue of a decomposition of a semigroup with zero into a right sum of semigroups is a decomposition of a semigroup without zero into a right zero band of semigroups, considered by M.Petrich in [29] and [31].

3.3. On (left, right) 0-consistent equivalences.

An equivalence ξ of a semigroup $S = S^0$ is *left* (*right*) *0-consistent* if for $x, y \in S$, $xy \neq 0$ implies $xy\,\xi\,x$ ($xy \neq 0$ implies $xy\,\xi\,y$), and ξ is *0-consistent* if it is both left and right 0-consistent. For a set X, $\mathcal{E}(X)$ will denote the *lattice of equivalences* (equivalence relations) of X, and for a semigroup $S = S^0$, $\mathcal{E}^{\mathbf{c}}(S)$ ($\mathcal{E}^{\mathbf{rc}}(S)$, $\mathcal{E}^{\mathbf{lc}}(S)$) will denote the set of all 0-consistent (right 0-consistent, left 0-consistent) equivalences of S.

Lemma 3.4 [12] *An equivalence* ξ *of a semigroup* $S = S^0$ *is 0-consistent* (*right 0-consistent, left 0-consistent*) *if and only if* $\sim\subseteq\xi$ ($\overset{\ell}{\sim}\subseteq\xi$, $\overset{r}{\sim}\subseteq\xi$). $\square$

Lemma 3.5 [12] *The following conditions for an equivalence* ξ *of a semigroup* $S = S^0$ *are equivalent:*

(i) ξ *is 0-consistent* (*right 0-consistent, left 0-consistent*)*;*

(ii) $(a\xi)^0$ *is a 0-consistent* (*right 0-consistent, left 0-consistent*) *subset of* S, *for every* $a \in S^\bullet$*;*

(iii) $(a\xi)^0$ *is an ideal* (*left ideal, right ideal*) *of* S, *for every* $a \in S^\bullet$. $\square$

Main features of sets $\mathcal{E}^{\mathbf{c}}(S)$, $\mathcal{E}^{\mathbf{rc}}(S)$ and $\mathcal{E}^{\mathbf{lc}}(S)$ in the lattice of all equivalences of a semigroup with zero and their mutual connections are given by the following

Theorem 3.7 [12] *For a semigroup* $S = S^0$, $\mathcal{E}^{\mathbf{c}}(S)$, $\mathcal{E}^{\mathbf{rc}}(S)$ *and* $\mathcal{E}^{\mathbf{lc}}(S)$ *are complete sublattices of* $\mathcal{E}(S)$. *The smallest elements of* $\mathcal{E}^{\mathbf{c}}(S)$, $\mathcal{E}^{\mathbf{rc}}(S)$ *and* $\mathcal{E}^{\mathbf{lc}}(S)$ *are* δ, κ *and* υ, *respectively.*

Furthermore, the join of an arbitrary subset of $\mathcal{E}(S)$ *containing at least one right 0-consistent equivalence and at last one left 0-consistent equivalence of* S *is a 0-consistent equivalence of* S. *Especially, the join of* κ *and* υ *is* δ. $\square$

Clearly, $\mathcal{E}^{\mathbf{c}}(S)$, $\mathcal{E}^{\mathbf{rc}}(S)$ and $\mathcal{E}^{\mathbf{lc}}(S)$ are principal dual ideals of $\mathcal{E}(S)$ generated by δ, κ and υ, respectively.

By the results of the previous two sections and by Lemma 3.5. we see a connection between the lattice $\mathcal{E}^{\mathbf{c}}(S)$ and orthogonal decompositions and between the lattice $\mathcal{E}^{\mathbf{rc}}(S)$ ($\mathcal{E}^{\mathbf{lc}}(S)$) and decompositions into a right (left) sum. Also, by Lemma 3.5. we see that decompositions into a right (left) sum are "finer" than orthogonal decompositions.

The following proposition give other generalization of Green's relations.

Proposition 3.1 [12] *If $S = S^0$, then*

$$\begin{array}{ccccccccccccc}
\mathcal{H} & \subseteq & \mu_1 & \subseteq & \mu_2 & \subseteq & \cdots & \subseteq & \mu_n & \subseteq & \cdots & \subseteq & \mu \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\mathcal{L} & \subseteq & \kappa_1 & \subseteq & \kappa_2 & \subseteq & \cdots & \subseteq & \kappa_n & \subseteq & \cdots & \subseteq & \kappa \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\mathcal{J} & \subseteq & \delta_1 & \subseteq & \delta_2 & \subseteq & \cdots & \subseteq & \delta_n & \subseteq & \cdots & \subseteq & \delta \\
\rotatebox[origin=c]{90}{$\subseteq$} & & & & & & & & & & & & \rotatebox[origin=c]{90}{$\subseteq$} \\
\mathcal{R} & \subseteq & \upsilon_1 & \subseteq & \upsilon_2 & \subseteq & \cdots & \subseteq & \upsilon_n & \subseteq & \cdots & \subseteq & \upsilon
\end{array} \quad . \quad \Box$$

Various types of orthogonal decompositions are described in the next theorems.

A semigroup $S = S^0$ is *0-δ_n-simple* if S has exactly two δ_n-classes, i.e. if $x \sim^n y$, for all $x, y \in S^\bullet$. The following theorem describe orthogonal sums of 0-δ_n-simple semigroups.

Theorem 3.8 [12] *Let $n \in Z^+$. Then the following conditions on a semigroup $S = S^0$ are equivalent:*

(i) *S is an orthogonal sum of 0-δ_n-simple semigroups;*

(ii) $(\forall x, y, a \in S)\ xy \neq 0 \ \Rightarrow\ [(x \sim^n a \ \vee y \sim^n a) \ \Rightarrow\ xy \sim^n a]$;

(iii) *for every $a \in S$, $\Delta_n(a)$ is an ideal of S;*

(iv) *$\sim^n$ is an equivalence relation on S;*

(v) *δ_n is a 0-consistent equivalence on S.* $\Box$

Corollary 3.2 [12] *Let S be a finite semigroup. Then there exists $n \in Z^+$, $n \leq |S|$, such that S is an orthogonal sum of 0-δ_n-simple semigroups.* $\Box$

A semigroup $S = S^0$ is *0-σ-simple* (*0-σ_n-simple*, $n \in Z^+$) if $a \longrightarrow^\infty b$ ($a \longrightarrow^n b$) for all $a, b \in S^\bullet$.

Theorem 3.9 [12] *The following conditions on a semigroup S are equivalent:*

(i) *S is an orthogonal sum of 0-σ-simple semigroups;*

(ii) $(\forall x, y \in S)\ xy \neq 0 \ \Rightarrow\ x\,\sigma\,y$;

(iii) $(\forall x, y \in S)\ xy \neq 0 \ \Rightarrow\ (xy \longrightarrow^\infty x \wedge xy \longrightarrow^\infty y)$;

(iv) *every principal radical of S is 0-consistent;*

(v) *every completely semiprime ideal of S is 0-consistent;*

(vi) *Σ_S is a Kronecker's semilattice and $\Sigma(0)$ is a 0-consistent ideal of S.*

Theorem 3.10 [12] *Let $n \in Z^+$. A semigroup $S = S^0$ is an orthogonal sum of 0-σ_n-simple semigroups if and only if*

$$(\forall x, y, a \in S)\ xy \neq 0 \ \Rightarrow\ [(x \longrightarrow^n a \vee\ y \longrightarrow^n a) \ \Rightarrow\ xy \longrightarrow^n a].$$

Theorem 3.11 [25] *A semigroup $S = S^0$ is an orthogonal sum of semigroups having 0 as a prime ideal if and only if the following conditions hold:*

(a) *0 is a semiprime ideal of S;* (b) $(\forall a, b, c \in S)\ aSb \neq 0 \wedge bSc \neq 0 \ \Rightarrow\ aSc \neq 0$. $\Box$

Information on connections between 0-primitivity of idempotents of (π-)regular semigroups and orthogonal decompositions the reader can find in the former survey article of the authors [9] and in the books of A.H.Clifford and G.B.Preston [18] and of O.Steinfeld [39].

3.4. Lattices of ideals of semigroups with zero.

In the following section we will present some theorems that give connections between orthogonal decompositions (decompositions into a right sum) of a semigroup with zero and of decompositions of the lattice of its ideals (left ideals) into a direct product.

Lemma 3.6 *Let L be a bounded lattice, infinitely distributive for meets. If $\{a_\alpha |\ \alpha \in Y\}$ is a subset of L for which*

$$\bigvee_{\alpha\in Y} a_\alpha = 1, \qquad a_\alpha \wedge a_\beta = 0,\ \textit{for } \alpha \neq \beta, \alpha, \beta \in Y,$$

then L is isomorphic to the direct product of its intervals $[0, a_\alpha]$, $\alpha \in Y$. □

Theorem 3.12 *Let L be a bounded lattice, infinitely distributive for meets. Then L has a decomposition into a direct product of directly indecomposable lattices if and only if $\mathfrak{B}(L)$ is a complete atomic Boolean algebra.* □

Corollary 3.3 *Let L be a bounded lattice, infinitely distributive for meets. Then L is directly indecomposable if and only if $\mathfrak{B}(L) = \{0, 1\}$.* □

Theorem 3.13 [12] *Let $\{S_\alpha |\ \alpha \in Y\}$ be the greatest orthogonal decomposition of a semigroup $S = S^0$. Then the lattice $\mathcal{I}d(S)$ is isomorphic to the direct product of lattices $\mathcal{I}d(S_\alpha)$, $\alpha \in Y$, and lattices $\mathcal{I}d(S_\alpha)$, $\alpha \in Y$, are directly indecomposable.* □

Theorem 3.14 [12] *The lattice of ideals of a semigroup $S = S^0$ is directly indecomposable if and only if S is orthogonal indecomposable.* □

Theorem 3.15 [12] *Let $\{S_\alpha |\ \alpha \in Y\}$ be the greatest decomposition of a semigroup $S = S^0$ into a right sum. Then the lattice $\mathcal{LI}d(S)$ is isomorphic to the direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$, which are directly indecomposable lattices.* □

Note that for $\alpha \in Y$, the interval $[0, S_\alpha]$ cannot be equal to $\mathcal{LI}d(S_\alpha)$. Because of that we give the following two results:

Corollary 3.4 [12] *Let $\{S_\alpha |\ \alpha \in Y\}$ be the greatest decomposition of a semigroup $S = S^0$ into a right sum. Then the lattice $\mathcal{LI}d(S)$ can be embedded into the direct product of lattices $\mathcal{LI}d(S_\alpha)$, $\alpha \in Y$.* □

Theorem 3.16 [12] *Let $\{S_\alpha |\ \alpha \in Y\}$ be the greatest orthogonal decomposition of a semigroup $S = S^0$. Then the lattice $\mathcal{LI}d(S)$ is isomorphic to the direct product of lattices $\mathcal{LI}d(S_\alpha)$, $\alpha \in Y$.* □

Results concerning decompositions of the lattice of ideals of a semigroup with zero into a direct product could be naturally extended to the lattice of ideals of a semigroup with kernel. Moreover, results concerning decompositions of the lattice of left ideals of a semigroup with zero into a direct product could be transferred to the lattice of left ideals of a semigroup without zero, and in this case these decompositions are connected with decompositions of this semigroup into a right zero band of semigroups.

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