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Generation of Positive Lower-Potent Half-Congruences*

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Abstract. As known, positive lower-potent half-congruences on a semigroup play a crucial role in semilattice decompositions of semigroups. In the present paper, we determine necessary and sufficient conditions for a relation on a semigroup to the smallest positive lower-potent quasi-order containing it be compatible, and hence a half-congruence. The obtained results will be compared with some well-known results of T. Tamura from [8] and [9].

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T. Tamura in [10], proved a very nice theorem which connects semilattice congruences on a semigroup with its positive lower-potent half-congruences. More precisely, he proved that the natural equivalence of any positive lower-potent half-congeruence on a semigroup is a semilattice congruence. Furthermore, it has been proved that the mapping which to any lower-potent positive half-congruence associates its natural equivalence is an isomorphism between the lattice of positive lower-potent half-congruences and the lattice of semilattice congruences on a semigroup (such a version of this Tamura's theorem has been stated by the authors in [4]).

In further studying of semilattice congruences through lower-potent positive half-congruences the main problem is to build a lower-potent positive half-congruence from another simpler relation. Problems of this type were first treated by T. Tamura in [8], where he built the smallest positive lower-potent half-congruence starting from the division relation on a semigroup. A more general result has been obtained in his paper [9], where he proved that the smallest semilattice congruence containing a compatible relation ξ on a semigroup equals the natural equivalence of the smallest positive lower-potent quasi-order containing ξ.

As we will see later, construction of the smallest positive lower-potent quasi-order containing a given relation is relatively simple. A more difficult task is to make this quasi-order compatible. The purpose of this paper is to determine necessary and sufficient conditions for a relation on a semigroup to the smallest

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positive lower-potent quasi-order containing it be compatible, and hence a half-congruence. This will be done by Theorem 1 for reflexive positive relations, and by Theorem 2 for positive quasi-orders on a semigroup. At the end of the paper, we will give some connections between the obtained results and Tamura's results mentioned earlier.

The results given in this paper have been presented in an invited lecture at the *International Conference in Semigroups and Their Related Topics*, August 22, 1995, held at the Institute of Mathematics of Yunnan University, Kunming, Yunnan, China.

Through this paper $\mathbf{Z}^+$, will denote the set of all positive integers. For a semigroup S, S^1 will denote the semigroup obtained from S by adjoining an identity.

By a *quasi-order* we mean a reflexive transitive relation on a set. A compatible quasi-order on a semigroup is called a *half-congruence*. An equality relation on a set A will be denoted by Δ_A. If there is no danger of confusion we will simply write Δ instead of Δ_A. A relation ξ on a semigroup S is called *positive* if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$, and it is called *lower-potent* if $a^n\,\xi\,a$, for any $a \in S$ and any $n \in \mathbf{Z}^+$. *Clearly, if ξ is a quasi-order, then it is lower-potent if and only if $a^2\,\xi\,a$, for all $a \in S$.* By $\mid$ we denote the *division relation* on a semigroup S defined by

$$a \mid b \quad \Leftrightarrow \quad (\exists x, y \in S^1)\ b = xay.$$

As M. S. Putcha in [6] mentioned, and the authors in [4] stated in terms of the lattice of quasi-orders, the set of positive quasi-orders on a semigroup equals the principal dual ideal of the lattice of quasi-orders on S generated by the division relation on S.

For undefined notions and notations we refer to [1], [2], [3] and [5].

Starting from the division relation on a semigroup, Tamura in [8] constructed a new relation on S, denoted by Putcha in [7] by $\longrightarrow$, in the following way:

$$a \longrightarrow b \ \Leftrightarrow \ (\exists n \in \mathbf{Z}^+)\ a \mid b^n.$$

He also considered the transitive closure of this relation, for which he proved that its natural equivalence equals the smallest semilattice congruence on S. In another paper [10], Tamura turned this construction into a general method for construction of new relations from a given relation. Namely, to any relation π on a semigroup S he associated the relation $\xrightarrow{\pi}$ on S defined by

$$a \xrightarrow{\pi} b \quad \Leftrightarrow \quad (\exists n \in \mathbf{Z}^+)\ a\,\pi\,b^n,$$

and the transitive closure of $\xrightarrow{\pi}$, which will be denoted by $\overline{\pi}$. Except in the Tamura's paper, such constructed relations have been intensively used in a series of papers by M. S. Putcha. A sense of such a method is to build a lower-potent quasi-order from a given relation π. Namely, the following lemma can be easily proved:

Lemma 1. *The smallest lower-potent quasi-order on a semigroup S containing a relation ξ on S equals $\overline{\pi}$, where $\pi = \xi \cup \Delta$.*

On the other hand, a good feature of this construction is that it preserves positivity, so the following lemma can be easily verified.

Lemma 2. *The smallest positive lower-potent quasi-order on a semigroup S containing a relation ξ on S equals $\overline{\pi}$, where $\pi = \xi \cup |$.*

Therefore, if we start from a reflexive and positive relation π, then without serious problems we obtain that $\overline{\pi}$ is a positive lower-potent quasi-order. But, if we want to obtain that $\overline{\pi}$ is a half-congruence, i.e., that it is compatible, then more serious difficulties are arising. As we said before, the main task of this paper is to determine necessary and sufficient conditions for π to $\overline{\pi}$ be compatible. This will be done in the main theorems of this paper. To prove these theorems, we need one very useful feature of positive lower-potent quasi-orders which is given by the following lemma:

Lemma 3. *Let ξ be a positive lower-potent quasi-order on a semigroup S. Then*

$$a_1a_2\cdots a_n \;\xi\; a_{1\varphi}a_{2\varphi}\cdots a_{n\varphi},$$

for any $n \in \mathbf{Z}^+$, all $a_1, a_2, \dots, a_n \in S$ and any permutation φ of the set $\{1, 2, \dots, n\}$.

Proof. Since any permutation of a finite set is a product of transpositions of adjacent factors, then it is enough to prove that

$$xaby \;\xi\; xbay, \qquad \text{for all } a, b \in S,\ x, y \in S^1.$$

But a proof of this assertion can be easily derived from the proof of Theorem 2.3 of [9], completing the proof of the lemma. □

Note that the above lemma is closely related with Lemma II 3.5 of [5], since by Theorem 2 of [4], a quasi-order ξ on a semigroup S is lower-potent and positive if and only if $a\xi$ is a completely semiprime ideal of S, for each $a \in S$ (where $a\xi$ is defined by $a\xi = \{x \in S \mid a \;\xi\; x\}$).

Now we are ready to give the main theorem of this paper.

Theorem 1. *Let π be a reflexive positive relation on a semigroup S. Then $\overline{\pi}$ is compatible if and only if for all $a, b, c, d \in S$ the following condition holds*

$$a \;\pi\; c \;\&\; b \;\pi\; d \;\;\Rightarrow\;\; (\exists u \in S)\; a \mid u \;\&\; b \mid u \;\&\; u\overline{\pi}cd. \tag{1}$$

Proof. Let $\overline{\pi}$ be compatible. Assume $a, b, c, d \in S$ such that $a \;\pi\; c$ and $b \;\pi\; d$. Then by $a \;\overline{\pi}\; c$ and $b \;\overline{\pi}\; d$ it follows $ab \;\overline{\pi}\; cd$, so (1) holds for $u = ab$.

To prove the converse assume $a, b \in S$ such that $a \xrightarrow{\pi} b$, i.e., $a \;\pi\; b^n$, for some $n \in \mathbf{Z}^+$, and assume an arbitrary $c \in S$. By (1) it follows that

$$a \mid u \;\&\; c \mid u \;\&\; u \;\overline{\pi}\; b^n c, \tag{2}$$

for some $u \in S$. This means that $u = xay = pcq$, for some $x, y, p, q \in S^1$, so by positivity and lower-potency of $\overline{\pi}$, by (2) and Lemma 3 we obtain that

$$ac\,\overline{\pi}\,yxacqp\,\overline{\pi}\,xaypcq = u^2\,\overline{\pi}\,u\,\overline{\pi}\,b^nc\,\overline{\pi}\,b^nc^n\,\overline{\pi}\,(bc)^n\,\overline{\pi}\,bc.$$

Hence $a \xrightarrow{\pi} b$ implies $ac\,\overline{\pi}\,bc$, so by induction, we obtain that $\overline{\pi}$ is right compatible. Analogously we prove the left compatibility of $\overline{\pi}$. This completes the proof of the theorem. □

An immediate consequence of the previous theorem is the following.

Corollary 1. *The smallest positive lower-potent quasi-order on a semigroup S containing a relation ξ on S is compatible if and only if the relation $\pi = \xi \cup \mid$ satisfies the condition* (1)*.*

Especially, when π is a positive quasi-order, then condition (1) can be replaced by a more simple condition as follows.

Theorem 2. *Let π be a positive quasi-order on a semigroup S. Then $\overline{\pi}$ is compatible if and only if for all $a, b, c \in S$ the following condition holds*

$$a\,\pi\,c \;\&\; b\,\pi\,c \;\Rightarrow\; (\exists u \in S)\; a \mid u \;\&\; b \mid u \;\&\; u\overline{\pi}c. \tag{3}$$

Proof. The condition (3) follows immediately from (1), since $\overline{\pi}$ is a positive quasi-order.

The assumption that π is a positive quasi-order we use to prove that (3) implies (1). Indeed, if $a, b, c, d \in S$ such that $a\;\pi\;c$ and $b\;\pi\;d$, then $a\;\pi\;cd$ and $b\pi cd$, by positivity and transitivity of π, so by (3) we obtain that (2) holds. □

At the end of the paper we give some connections of the obtained results with some well-known results of Tamura from [8] and [9]. Recall first that one of the most important theorems in the theory of semilattice decompositions of semigroups is the following.

Theorem 3. [8] *The smallest semilattice congruence on a semigroup S equals the natural equivalence of $\overline{\pi}$, where $\pi = \mid$.*

Note that the central point of the proof of Tamura's theorem is the proof of compatibility of $\overline{\pi}$. Since the division relation $\mid$ satisfies the condition

$$a \mid c \;\&\; b \mid c \;\Rightarrow\; ab \mid c^2, \tag{4}$$

and hence the condition (3), then our results make this proof very simple.

In another paper Tamura proved a more general theorem.

Theorem 4. [9] *The smallest semilattice congruence on a semigroup S containing a compatible relation ξ on S equals the natural equivalence of the smallest positive lower-potent quasi-order on S containing ξ.*

This theorem has been proved without proving that the smallest positive lower-potent quasi-order containing ξ is compatible. This fact can be proved using our previous results, and it gives an additional explanation of Tamura's quoted result.

Theorem 5. *The smallest positive lower-potent quasi-order on a semigroup S containing a compatible relation ξ is also compatible.*

Proof. Let $\pi = \xi \cup |$. Assume $a, b, c, d \in S$ such that $a \; \pi \; c$ and $b \; \pi \; d$. We will conclude that (1) holds. We distinguish the following four cases:

(i) $a \; \xi \; c$ and $b \; \xi \; d$;
(ii) $a \mid c$ and $b \; \xi \; d$;
(iii) $a \; \xi \; c$ and $b \mid d$;
(iv) $a \mid c$ and $b \mid d$.

In case (iv) the desired conclusion follows by (4). Since case (iii) is symmetrical to case (ii), then it remains to draw the desired conclusion from cases (i) and (ii).

(i) In this case we obtain that $ab \; \xi \; cd$, since ξ is compatible, so (1) holds for $u = ab$.

(ii) In this case we have that $c = xay$, for some $x, y \in S^1$, and using compatibility of ξ we obtain that $xayb = cb \; \xi \; cd$, so (1) holds for $u = xayb$. □

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