Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6 (1992), 137–140

FILOMAT-20, Niš, Septembar 26–28, 1991

RIGHT π-INVERSE SEMIGROUPS AND RINGS

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Abstract. *In this paper we consider semigroups and rings whose multiplicative semigroups are completely π-regular and right π-inverse (for the definitions see below).*

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. A semigroup (ring) S is *π-regular* (*left π-regular, right π-regular*) if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^nSa^n$ ($a^n \in Sa^{n+1}$, $a^n \in a^{n+1}S$. A semigroup (ring) S is *completely π-regular* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^nxa^n$ and $a^nx = xa^n$. A semigroup S is *π-inverse* (*completely π-inverse*) if it is π-regular (completely π-regular) and every regular element has a unique inverse. A semigroup S is *right* (*left*) *π-inverse* if it is π-regular and $a = axa = aya$ implies $xa = ya$ ($ax = ay$). By $Reg(S)$ ($Gr(S)$, $E(S)$) we denote the set of all regular (completely regular, idempotent) elements of a semigroup S. If e is an idempotent of a semigroup (ring) S, then by G_e we denote the maximal subgroup of S with e as its identity. By $\mathcal{M}R$ we denote the multiplicative semigroup of a ring R. A semigroup S is a *nil-semigroup* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. An ideal extension S of a semigroup S is a *nil-extension* of T if S/T is a nil-semigroup.

M.S.Putcha [5] studied completely π-regular rings which are semilattices of Archimedean semigroups. Right π-inverse semigroups are studied in [2]. In this paper we consider semigroups and rings whose multiplicative semigroups are completely π-regular and right π-inverse, called right completely π-inverse. As a consequence we obtain a result of J.L.Galbiati and M.L.Veronesi [3] for completely π-inverse semigroups. Furthermore, we prove that in rings the notions of right completely π-inverse and right π-inverse coincide and in this case we have a ring in which multiplicative semigroup is a semilattice of nil-extensions of left groups.

Supported by Grant 0401A *of Science Fund of Serbia through Math. Inst. SANU*
1991 *Mathematics subject classification:* 20M10, 20M25, 16A30

THEOREM 1. *A semigroup S is completely π-regular if and only if S is π-regular and left (or right) π-regular.*

PROOF. Let S be a π-regular and a left π-regular semigroup. Then for every $a \in S$ there exists $x \in S$ and $r \in \mathbf{Z}^+$ such that $a^r = xa^{r+1}$, whence

$$a^{kr} = xa^{kr+1} \tag{1}$$

for all $k \in \mathbf{Z}^+$. Since S is π-regular we then have that for a^r there exists $y \in S$ and $m \in \mathbf{Z}^+$ such that

$$(a^r)^m = (a^r)^m y (a^r)^m \ ,$$

so by (1) we obtain that

$$a^{rm} = a^{rm} y (xa^{rm+1}) = a^{rm} yxa^{rm+1} \ .$$

Therefore, $a^{rm} \in a^{rm} S a^{rm+1}$, and by Theorem IV 2. [1] we have that S is completely π-regular.

The converse follows immediately. □

DEFINITION. *A semigroup S is right (left) completely π-inverse if S is completely π-regular and for all $a, x, y \in S$ by $a = axa = aya$ it implies that $xa = ya$ ($ax = ay$), i.e. if it is completely π-regular and right π-inverse.*

Right completely π-inverse semigroups we describe by the following theorem:

THEOREM 2. *The following conditions on a semigroup S are equivalent:*

(i) S is right completely π-inverse;

(ii) S is π-regular and

$$(\forall a \in S)(\forall f \in E(S))(\exists n \in \mathbf{Z}^+) \quad (af)^n = (faf)^n \ ; \tag{2}$$

(iii) S is π-regular and

$$(\forall a \in Reg(S))(\forall f \in E(S))(\exists n \in \mathbf{Z}^+) \quad (af)^n = (faf)^n \ . \tag{3}$$

PROOF. $(iii) \Rightarrow (i)$. Let we prove that S is completely π-regular. Let $a = axa$ for some $x \in S$. Then by (3) it follows that there exists $r \in \mathbf{Z}^+$ such that

$$a^r = (a(xa))^r = ((xa)a)^r = (xa^2)^r = xa^{r+1} \ .$$

Therefore, every regular element of S is right π-regular. Since S is π-regular, then for every $a \in S$ there exists $m \in \mathbf{Z}^+$ such that $a^m \in Reg(S)$, whence it follows that there exists $r \in \mathbf{Z}^+$ and $x \in S$ such that $(a^m)^r = x(a^m)^{r+1}$, so $a^{mr} \in Sa^{mr+1}$. Hence, S is π-regular and right π-regular so by Theorem 1. we obtain that S is completely π-regular semigroup. That S is right π-inverse follows by Theorem 1. [2].

$(i) \Rightarrow (ii)$. Let S be a right completely π-inverse semigroup. Assume $a \in S$ and $f \in E(S)$. Then there exists $k, m \in \mathbf{Z}^+$ such that $(af)^k \in G_g$ and $(faf)^m \in G_h$ for some $g, h \in E(S)$. By Lemma 1. [4] it follows that there exists $n \in \mathbf{Z}^+$ such that $(af)^n \in G_g$ and $(faf)^n \in G_h$. Now

$$g = ((af)^n)^{-1}(af)^n = ((af)^n)^{-1}(af)^n f = gf \ .$$

Similarly we obtain that $h = hf = fh$. Since

$$f(af)^r=(fa)^rf=(faf)^r$$

for all $r\in\mathbf{Z}^+$, we then have that

$$f(af)^n=(faf)^n=h(faf)^n=hf(af)^n=h(af)^n\ .$$

Thus

$$f(af)^n((af)^n)^{-1}=h(af)^n((af)^n)^{-1}\ ,$$

i.e. $fg=hg$, so $g(fg)=g(hg)$. Hence, $g=ghg=g^2$ (since $gf=g$). Since S is right π-inverse, then we obtain that $hg=g$. Therefore

$$fg=hg=g\ . \tag{4}$$

Moreover,

$$\begin{aligned}h=hf&=((faf)^n)^{-1}(faf)^nf=((faf)^n)^{-1}f(af)^nf\\&=((faf)^n)^{-1}f(af)^ngf=((faf)^n)^{-1}(faf)^ngf\\&=hgf=hg\ .\end{aligned}$$

By this and by (4) we obtain that $g=h$. Thus $(af)^n$ and $(faf)^n$ lies in the same subgroup G_g of S, so

$$(faf)^n=g(faf)^n=gf(af)^n=g(af)^n=(af)^n\ ,$$

since $gf=g$.

$(ii)\ \Rightarrow\ (iii)$. This follows immediately. □

Corollary 1. [3] *The following conditions on a semigroup S are equivalent:*

(i) *S is completely π-inverse;*

(ii) *S is π-regular and*

$$(\forall a\in S)(\forall f\in E(S))(\exists n\in\mathbf{Z}^+)\quad (af)^n=(fa)^n\ ;$$

(iii) *S is π-regular and*

$$(\forall a\in Reg(S))(\forall f\in E(S))(\exists n\in\mathbf{Z}^+)\quad (af)^n=(fa)^n\ .$$

By the following theorem we describe rings whose multiplicative semigroups are right π-inverse.

Theorem 3. *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a right π-inverse semigroup;*

(ii) *R is π-regular and $ae=eae$ for every $a\in R$, $e\in E(R)$;*

(iii) *R is π-regular and $(E(R),\cdot)$ is a right regular band;*

(iv) *$\mathcal{M}R$ is a semilattice of nil-extensions of left groups;*

(v) *$\mathcal{M}R$ is a right completely π-inverse semigroup.*

Proof. $(i)\ \Rightarrow\ (ii)$. Let $a\in R$ and let $e\in E(R)$. Then $g=e+ae-eae\in E(R)$. It is clear that $ge=g$ and $eg=e$. Now $g=g(eg)=g(geg)$, so $eg=geg$. By this it follows that $e=g$, so $ae=eae$.

$(ii)\ \Rightarrow\ (iii)$. This follows immediately.

$(iii)\ \Rightarrow\ (i)$. This follows by Theorem 1. [2].

$(i)\ \Rightarrow\ (iv)$. Let $a\in Reg(R)$, i.e. let $a=axa$ for some $x\in R$. By (ii) it follows that

$$a=a(xa)=(xa)a(xa)=xa^2\ ,$$

whence

$$a = axa = ax^2a^2 \in Gr(R) .$$

Hence, $Reg(R) = Gr(R)$, so by Corollary 3. [2] we obtain that $\mathcal{M}R$ is a semilattice of nil-extensions of left groups.

$(iv) \Rightarrow (i)$. This follows by Theorem 1. [3] and by Corollary 3. [3].

$(iv) \Rightarrow (v)$. This follows by Theorem 1. [2], by Corollary 3. [2] and by Theorem 2.

$(v) \Rightarrow (i)$. This follows immediately. □

By Theorem 3. we obtain the following:

COROLLARY 2. *The following conditions on a ring R are equivalent:*

(i) $\mathcal{M}R$ *is a π-inverse semigroup;*

(ii) R *is π-regular and* $ae = ea$ *for every* $a \in R$, $e \in E(R)$*;*

(iii) R *is π-regular and* $(E(R), \cdot)$ *is a semilattice;*

(iv) $\mathcal{M}R$ *is a semilattice of nil-extensions of groups;*

(v) $\mathcal{M}R$ *is a completely π-inverse semigroup.* □

REMARK. If a ring R has an identity element, then by Theorem 12. [5] it follows that all of the conditions from Theorem 3. and Corollary 2. are equivalent.

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