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UNIFORMLY π-REGULAR RINGS AND SEMIGROUPS: A SURVEY

To the Memory of
Professor Hisao Tominaga

CONTENTS

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Introduction

The main aim of this paper is to give a survey of the most important structural properties of uniformly π-regular rings and semigroups. It is well-known that there are many similarities between certain types of semigroups and related rings. For example, we will see in Theorem 2.1 that the regularity of a semigroup can be characterized by means of the properties of its left and right ideals, and in the same way, the regularity of a ring can be characterized through its ring left and right ideals. On the other hand, there are many significant differences between the properties of certain types of semigroups and the properties of related rings. For example, many concepts such as the left, right and complete regularity and other, are different in Theory of semigroups, but they coincide in Theory of rings. One of the main goals of this paper is exactly to underline both the similarities and differences between related types of rings and semigroups. For that purpose many interesting results of Theory of rings or Theory of semigroups will be omitted here if they are not similar or essentially different than the corresponding result of another theory.

There are two central places in the paper. The first one is Theorem 5.11 which asserts that a π-regular ring is uniformly π-regular if and only if it is an ideal extension of a nil-ring by a Clifford ring. This theorem makes possible to represent such rings by the Everett's sums of nil-rings and Clifford rings. This has shown oneself to be very useful in many situations. For example, using Theorem 5.11, a lot of known results concerning uniformly π-regular semigroups can be very successfully applied in Theory of rings.

Another crucial result is Theorem 5.44. This theorem describes rings whose multiplicative semigroups are nil-extensions of unions of groups and it asserts that such rings are exactly the direct sums of nil-rings and Clifford rings. We present numerous known methods for decomposition of semigroups into a nil-extension of a union of groups and we show that these methods have very significant applications in Theory of rings, in decompositions of rings into the direct sum of a nil-ring and a Clifford ring.

The purpose of this paper is twofold. At first, we intend to present the known results concerning uniformly π-regular semigroups and applications of these results in Theory of rings. On the other hand, we want also to interest ring-theoretists and semigroup-theoretists for more intensive investigations in the considered area.

The paper is divided into six sections. In the first section we introduce the necessary notions and notations and we present the main results concerning ideal extensions of rings and their representation by the known Everett's sums of rings. In Sections 2 and 3 we introduce the notions of a regular, π-regular, completely

π-regular and periodic ring and semigroup, and of a completely Archimedean semigroup and we describe their basic properties. Structural characterizations of completely regular semigroups and rings are given in Section 4. The main tools that we use there, are certain decomposition methods: semilattice decompositions, in the case of semigroups, and subdirect sum decompositions, in the case of rings.

The main part of the whole paper is Section 5. In this section we first give structural descriptions of uniformly π-regular semigroups and rings. After that we present various characterizations of semigroups decomposable into a nil-extension of a union of groups, and using these results we characterize the rings decomposable into the direct sum of a nil-ring and Clifford ring.

Finally, in Section 6 we present certain applications of the results given in the previous section. Here we study various types of semigroup identities satisfied on the various classes of semigroups and rings. The classes of all identities satisfied on the classes of the semilattices of Archimedean semigroups, the nil-extensions of unions of groups, the bands of π-regular semigroups are described. The main result in the part about the rings satisfying certain semigroup identities is the characterization of all rings satisfying a semigroup identity of the form $x_1x_2\cdots x_n = w(x_1, x_2, \dots, x_n)$, where $|w| \geq n+1$, given in the Theorem 6.29.

1. Preliminaries

In this section we introduce necessary notions and notations.

1.1. Basic notions and notations. Throughout this paper $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}^0$ the set of all non-negative integers, and $\mathbb{Z}$ will denote the ring of integers. By $\mathbb{Z}\langle x, y\rangle$ we will denote the ring of all polynomials with the variables x and y and the coefficients in $\mathbb{Z}$.

For a semigroup (ring) S, $E(S)$ will denote the set of all idempotents of S, and for $A \subseteq S$, $\sqrt{A}$ will denote the subset of S defined by $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{N})\, x^n \in A\}$. For a ring R, $\mathcal{M}R$ will denote the multiplicative semigroup of R. A subset A of a semigroup (ring) S is called *completely semiprime* if for $x \in S$, $x^2 \in A$ implies $x \in A$, *completely prime* if for $x, y \in S$, $xy \in A$ implies that either $x \in A$ or $y \in A$, *left consistent* if for $x, y \in S$, $xy \in A$ implies $x \in A$, *right consistent* if for $x, y \in S$, $xy \in A$ implies $y \in A$, and it is *consistent* if it is both left and right consistent.

The expression $S = S^0$ means that S is a semigroup with the zero 0. Let S be a semigroup (ring) with the zero 0. An element $a \in S$ is called a *nilpotent element* (or a *nilpotent*) if there exists $n \in \mathbb{N}$ such that $a^n = 0$, and the smallest number $n \in \mathbb{N}$ having this property is called the *index of nilpotency* of a. The set of all nilpotents of S is denoted by $\mathrm{Nil}(S)$, and also $N_2(S) = \{a \in S \mid a^2 = 0\}$. A semigroup (ring) whose any element is nilpotent is called a *nil-semigroup* (*nil-ring*). For $n \in \mathbb{N}$, $n \geq 2$, a semigroup (ring) S is called *n-nilpotent* if $S^n = 0$, and is called *nilpotent* if it is n-nilpotent, for some $n \in \mathbb{N}$, $n \geq 2$. A 2-nilpotent semigroup (ring) is called a *null-semigroup* (*null-ring*).

For a semigroup S we say that is an *ideal extension* of a semigroup T by a semigroup Q if T is an ideal of S and the factor semigroup S/T is isomorphic to Q. An ideal extension of a semigroup S by a nil-semigroup (resp. n-nilpotent semigroup, nilpotent semigroup, null-semigroup) is called a *nil-extension* (resp. *n-nilpotent extension*, *nilpotent extension*, *null-extension*) T. A subsemigroup T of a semigroup S is called a *retract* of S if there exists a homomorphism φ of S onto T such that $a\varphi = a$, for any $a \in T$, and then φ is called a *retraction* of S onto T. An ideal extension S of a semigroup T is called a *retractive extension* of T if T is a retract of S.

By A^+ we denote the *free semigroup* over an alphabet A and by A^* we denote the *free monoid* over A. For $n \in \mathbb{N}$, $n \geq 4$, $A_n = \{x_1, x_2, \ldots, x_n\}$, $A_3 = \{x, y, z\}$ and $A_2 = \{x, y\}$. For a word $w \in A^+$, w^+ will denote the set $w^+ = \{w^n \mid n \in \mathbb{N}\}$. By $|w|$ we denote the *length* of a word $w \in A^+$ and by $|x|_w$ we denote the number of appearances of the letter $x \in A$ in the word $w \in A^+$. A word $v \in A^+$ is a *left* (*right*) *cut* of a word $w \in A^+$ if $w = vu$ ($w = uv$), for some $u \in A^*$, and v is a *subword* of w if $w = u'vu''$, for some $u', u'' \in A^*$. For $w \in A^+$ such that $|w| \geq 2$, by $h^{(2)}(w)$ ($t^{(2)}(w)$) we denote the left (right) cut of w of the length 2. By $h(w)$ ($t(w)$) we denote the first (last) letter of a word $w \in A^+$, called the *head* (*tail*) of w, and by $c(w)$ we denote the set of all letters which appear in w, called the *content* of w [**246**]. An expression $w(x_1, \ldots, x_n)$ will mean that w is a word with $c(w) = \{x_1, \ldots, x_n\}$. If $w \in A^+$ and $i \in \mathbb{N}$, $i \leq |w|$, then $l_i(w)$ ($r_i(w)$) will denote the left (right) cut of w of the length i, $c_i(w)$ will denote the i-th letter of w and for $i, j \in \mathbb{N}$, $i, j \leq |w|$, $i \leq j$, $m_i^j(w)$ will denote the subword w determined by: $w = l_{i-1}(w)m_i^j(w)r_{|w|-j}(w)$. For $n \in \mathbb{N}$, Π_n will denote the word $x_1x_2\ldots x_n \in A_n^+$. If $w \in A^+$ and $x \in A$, then $x \underset{l}{\parallel} w$ ($x \underset{r}{\parallel} w$) if $w = xv$ ($w = vx$), $v \in A^+$ and $x \notin c(v)$. Otherwise we write $x \underset{l}{\nparallel} w$ ($x \underset{r}{\nparallel} w$).

Let $n \in \mathbb{N}$, $w \in A_n^+$ and let S be a semigroup. By the *value* of the word w in S, in a *valuation* $a = (a_1, a_2, \ldots, a_n)$, $a_i \in S$, $i \in \{1, 2, \ldots, n\}$, in notation $w(a)$ or $w(a_1, a_2, \ldots, a_n)$, we mean the element $w\varphi \in S$, where $\varphi : A_n^+ \to S$ is the homomorphism determined by $x_i\varphi = a_i$, $i \in \{1, 2, \ldots, n\}$. Also, we then say that for $i \in \{1, 2, \ldots, n\}$, the letter x_i *assumes the value* a_i in S, in notation $x_i := a_i$. For two words $u, v \in A_n^+$, the formal expression $u = v$ we call an *identity* (or a *semigroup identity*) over the alphabet A_n, and for a semigroup S we say that it *satisfies the identity* $u = v$, in notation $S \models u = v$, if $u\varphi = v\varphi$, for any homomorphism φ from A_n^+ into S, i.e. if u and v have the same value for any valuation in S. The class of all semigroups satisfying the identity $u = v$ is denoted by $[u = v]$, and is called the *variety* determined by the identity $u = v$. Identities $u = v$ and $u' = v'$ over an alphabet A_n^+ are *p-equivalent* if $u' = v'$ can be obtained from $u = v$ by some permutation of letters. It is clear that p-equivalent identities determine the same variety.

Let φ be a homomorphism of a free semigroup A^+ into a semigroup S. For an identity over A, which is treated as a pair of words from A^+, we say that it is a *solution of the equation* $u\varphi = v\varphi$ if it is contained in the kernel of φ. Any trivial identity over A, i.e. an identity of the form $w = w$, is clearly a solution of the equation $u\varphi = v\varphi$, called the *trivial solution* of $u\varphi = v\varphi$. All other solutions of $u\varphi = v\varphi$, if they exist, are called *non-trivial solutions* of $u\varphi = v\varphi$.

Let Σ be a set of non-trivial identities over an alphabet A. i.e. a subset of $A^+ \times A^+$ having the empty intersection with the equality relation on A^+. For a semigroup S we say that it *satisfies variabily* the set Σ of identities, or that it *satisfies the variable identity* Σ, in notation $S \models_v \Sigma$, if for any homomorphism φ from A^+ to S, the equation $u\varphi = v\varphi$ has a solution in Σ (clearly, such solutions are non-trivial). The class of all semigroups which satisfy the variable identity Σ is denoted by $[\Sigma]_v$ and is called a *variable variety*.

A semigroup S is called a *band* (resp. *left zero band*, *right zero band*, *rectangular band*, *left regular band*, *right regular band*, *semilattice*) if it belongs to the variety $[x = x^2]$ (resp. $[xy = x]$, $[xy = y]$, $[x = x^2,\ xyx = x]$, $[x = x^2,\ xyz = xzy]$, $[x = x^2,\ xyz = yxz]$, $[x = x^2,\ xy = yx]$). If B is a band, we say that a semigroup S is a *band B of semigroups* if B is a homomorphic image of S. When B is semilattice (resp. left zero band, right zero band, rectangular band), then we say that S is a *semilattice* (resp. *left zero band*, *right zero band*, *matrix*) *of semigroups*.

In this paper we will use several semigroups given by the following presentations:

$$\mathbb{B}_2 = \langle a, b \,|\, a^2 = b^2 = 0,\ aba = a,\ bab = b \rangle$$
$$\mathbb{A}_2 = \langle a, e \,|\, a^2 = 0,\ e^2 = e,\ aea = a,\ eae = e \rangle$$
$$\mathbb{N}_m = \langle a \,|\, a^{m+1} = a^{m+2},\ a^m \neq a^{m+1} \rangle$$
$$\mathbb{L}_{3,1} = \langle a, f \,|\, a^2 = a^3,\ f^2 = f,\ a^2 f = a^2,\ fa = f \rangle$$
$$\mathbb{C}_{1,1} = \langle a, e \,|\, a^2 = a^3,\ e^2 = e,\ ae = a,\ ea = a \rangle$$
$$\mathbb{C}_{1,2} = \langle a, e \,|\, a^2 = a^3,\ e^2 = e,\ ae = a,\ ea = a^2 \rangle$$

where $m \in \mathbb{N}$, and $\mathbb{R}_{3,1}$ (resp. $\mathbb{C}_{2,1}$) will denote the dual semigroup of $\mathbb{L}_{3,1}$ (resp. $\mathbb{C}_{1,2}$). By $\mathbb{L}_2$ (resp. $\mathbb{R}_2$) we denote the two-element left zero (resp. right zero) semigroup. Let A_N^+ be the free semigroup over an alphabet $A_N = \{x_k \,|\, k \in \mathbb{N}\}$ and let $I = \{u \in A_N^+ \,|\, (\exists x_i \in A_N)\, |x_i|_u \geq 2\}$. Then I is an ideal of A_N^+. By $\mathbb{D}_N$ we will denote the factor semigroup $(A_N^+)/I$. It is clear that D_N is isomorphic to the semigroup

$$(\{u \in A_N^+ | \Pi(u) = c(u)\} \cup \{0\}, \cdot),$$

where the multiplication " $\cdot$ " is defined by

$$u \cdot v = \begin{cases} uv & \text{if } u, v \neq 0 \text{ and } c(u) \cap c(v) = \varnothing \\ 0 & \text{otherwise} \end{cases}.$$

$\mathbb{D}_N$ is a nil-semigroup and it is not nilpotent.

The *principal twosided* (resp. *left*, *right*) *ideal*of a semigroup (ring) S generated by an element $a \in S$ will be denoted by (a) (resp. $(a)_L$, $(a)_R$). The *Green's relations*

$\mathcal{J}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a semigroup S are defined by

$$a\,\mathcal{L}\,b \;\Leftrightarrow\; (a)_L = (b)_L; \qquad a\,\mathcal{R}\,b \;\Leftrightarrow\; (a)_R = (b)_R;$$
$$a\,\mathcal{J}\,b \;\Leftrightarrow\; (a) = (b); \quad \mathcal{H} = \mathcal{L}\cap\mathcal{R}, \quad \mathcal{D} = \mathcal{L}\mathcal{R},$$

where $a, b \in S$. The *division relations* $|$, $\underset{l}{|}$, $\underset{r}{|}$ and $\underset{t}{|}$ on a semigroup S are defined by

$$a \underset{l}{|} b \;\Leftrightarrow\; b \in (a)_L, \qquad a \underset{r}{|} b \;\Leftrightarrow\; b \in (a)_R,$$
$$a \mid b \;\Leftrightarrow\; b \in (a), \qquad \underset{t}{|} = \underset{l}{|} \cap \underset{r}{|},$$

and the relations $\xrightarrow{l}$, $\xrightarrow{r}$, $\xrightarrow{t}$ and $\longrightarrow$ on S are defined by

$$a \xrightarrow{l} b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \underset{l}{|} b, \qquad a \xrightarrow{r} b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \underset{r}{|} b,$$
$$a \xrightarrow{t} b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \underset{t}{|} b, \qquad a \longrightarrow b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \mid b,$$

for $a, b \in S$.

If a semigroup T is a homomorphic image of a subsemigroup T' of a semigroup S, then we say that T *divides* S *through* T'. If the intersection of all ideals of a semigroup S is non-empty, then it is an ideal of S called the *kernel* of S. With respect to set-theoretical union and intersection, the set of all left ideals of a semigroup S, with the empty set included, is a lattice and it is denoted by $\mathcal{LI}d(S)$. By a *discrete partially ordered set* we mean a partially ordered set in which any two elements are incomparable. An element of a semigroup (ring) S is called *central* if it commutes with any element of S, and the set of all central elements of S is called the *center* of S. A ring without non-zero nilpotent elements is called a *reduced ring*.

For undefined notions and notations we refer to the books [**36**], [**48**], [**105**], [**106**], [**128**], [**144**], [**147**], [**153**], [**195**], [**210**], [**241**], [**243**], [**245**], [**246**], [**247**], [**270**], [**291**], [**292**], [**301**] and [**313**].

1.2. Everett's sums of rings. In this section we talk about the general problem of ideal extensions of rings. This problem is formulated in the following way: Given rings A and B, construct all ideal extensions of a ring A by a ring B, i.e. construct all rings R having the property that A is an ideal of R and the factor ring R/A is isomorphic to B. A solution of this problem was given by Everett in [**113**], 1942, and is referred here as the Everett's theorem.

The original version of the Everett's theorem can be found in the book of Rédei [**270**], 1961. The version which will be given here due to Müller and Petrich [**217**], 1971. The Everett's construction, given in such a version, is a combination of the well-known Schreier's construction of all extensions of a group by another, and the construction of all ideal extensions of a semigroup by a semigroup with zero, due to Yoshida [**348**], 1965. Namely, as in the group case, one chooses a system of representatives of the cosets of A in R, and as in the semigroup case, one makes a bitranslation of A by any of these representatives. Moreover, because the representatives are chosen in different cosets, two "factor systems", one for addition and one for multiplication, have to be introduced. For more information concerning Schreier's extensions of groups we refer to Hall [**131**] and Rédei [**270**], 1961, and for more information about ideal extensions of semigroups we refer to the survey article written by Petrich [**240**], 1970, and the book of the same author [**241**], 1973.

To present the Everett's construction we need the notion of a translational hull of a ring. The translational hull occurs naturally when one is concerned with a construction of ideal extensions of semigroups, and seeing that ring extensions can be treated as their particular case, it appears also in ring theory, with the necessary modification that all the functions in the definition be additive.

Let R be a ring. An endomorphism λ (ϱ) of the additive group of R, written on the left (right), is a *left* (*right*) *translation* of R if $\lambda(xy) = (\lambda x)y$ ($(xy)\varrho = x(y\varrho)$), for all $x, y \in R$. A left translation λ and a right translation ϱ of R are *linked* if $x(\lambda y) = (x\varrho)y$, for all $x, y \in R$, and in such a case the pair (λ, ϱ) is called a *bitranslation* of R. It is sometimes convenient to consider a bitranslation (λ, ϱ) as a bioperator denoted by a single letter, say π, which acts as λ, if it is written on the left, and as ϱ, if it is written on the right, i.e. $\pi x = \lambda x$ and $x\pi = x\varrho$, for $x \in R$. For any $a \in R$, the *inner left* (*right*) *translation* induced by a is the mapping λ_a (ϱ_a) of R into itself defined by $\lambda_a x = ax$ ($x\varrho_a = xa$), for $x \in R$, and the pair $\pi_a = (\lambda_a, \varrho_a)$ is called the *inner bitranslation* of R induced by a.

A left translation λ and a right translation ϱ of a ring R are *permutable* if $(\lambda x)\varrho = \lambda(x\varrho)$, for all $x \in R$, and a set T of bitranslations of R is *permutable* if for all $(\lambda, \varrho), (\lambda', \varrho') \in T$, λ and ϱ' are permutable.

The set $\Lambda(R)$ ($\mathrm{P}(R)$) of all left (right) translations of a ring R is a ring under the addition and the multiplication defined by:

$$
\begin{aligned}
(\lambda + \lambda')x &= \lambda x + \lambda' x \qquad \big(\, x(\varrho + \varrho') = x\varrho + x\varrho' \,\big),\\
(\lambda\lambda')x &= \lambda(\lambda' x) \qquad \big(\, x(\varrho\varrho') = (x\varrho)\varrho' \,\big),
\end{aligned}
$$

for $\lambda, \lambda' \in \Lambda(R)$ ($\varrho, \varrho' \in \mathrm{P}(R)$) and $x \in R$. The subring $\Omega(R)$ of the direct sum of rings $\Lambda(R)$ and $\mathrm{P}(R)$, consisting of all bitranslations of R, is called the *translational hull* of R. More information about translational hulls of rings and semigroups can be found in [**240**] and [**241**].

Theorem 1.1. (Everett's theorem) *Let A and B be disjoint rings. Let θ be a function of B onto a set of permutable bitranslations of A, in notation $\theta : a \mapsto \theta^a \in \Omega(A)$, $a \in B$, and let $[,]$, $\langle , \rangle : B \times B \to A$ be functions such that for all $a, b, c \in B$ the following conditions hold:*

(E1) $\theta^a + \theta^b - \theta^{a+b} = \pi_{[a,b]}$;
(E2) $\theta^a \cdot \theta^b - \theta^{ab} = \pi_{\langle a,b\rangle}$;
(E3) $\langle ab, c\rangle + \langle a, b\rangle\theta^c = \langle a, bc\rangle + \theta^a\langle b, c\rangle$;

(E4) $[0,0]=0$;
(E5) $[a,b]=[b,a]$;
(E6) $[a,b]+[a+b,c]=[a,b+c]+[b,c]$;
(E7) $[a,b]\theta^c+\langle a+b,c\rangle=[ac,bc]+\langle a,c\rangle+\langle b,c\rangle$;
(E8) $\theta^a[b,c]+\langle a,b+c\rangle=[ab,ac]+\langle a,b\rangle+\langle a,c\rangle$.

Define an addition and a multiplication on $R=A\times B$ *by:*

(E9) $(\alpha,a)+(\beta,b)=(\alpha+\beta+[a,b],a+b)$;
(E10) $(\alpha,a)\cdot(\beta,b)=(\alpha\beta+\langle a,b\rangle+\theta^a\beta+\alpha\theta^b,ab)$,

$\alpha,\beta\in A,\ a,b\in B$. *Then* $(R,+,\cdot)$ *is a ring isomorphic to an ideal extension of* A *by* B.

Conversely, every ideal extension of A *by* B *can be so constructed.*

A ring constructed as in the Everett's theorem we call an *Everett's sum* of rings A and B *by a triplet* $(\theta;[,];\langle,\rangle)$ of functions and we denote it by $E(A,B;\theta;[,];\langle,\rangle)$. The representation of a ring R as an Everett's sum of some rings we call an *Everett's representation* of R.

More information about the Everett's theorem can be found in [**240**] and [**270**]. There we can see that an Everett's representation $E(A,B;\theta;[,];\langle,\rangle)$ of some ring R is determined by the choice of a set of representatives of the cosets of A in R. Namely, if for every coset $a\in B$ we choose a representative, in notation a', then the set $\{a'\mid a\in B\}$ determines the triplet $(\theta;[,];\langle,\rangle)$ in the following way:

(E11) $\alpha\theta^a=\alpha\cdot a',\theta^a\alpha=a'\cdot\alpha,\quad \alpha\in A,\ a\in B$;
(E12) $[a,b]=a'+b'-(a+b)',\quad a,b\in B$;
(E13) $\langle a,b\rangle=a'\cdot b'-(a\cdot b)',\quad a,b\in B$.

Although an Everett's representation of a ring is determined by the choice of representatives of the related cosets, for any such choice we obtain equivalent Everett's sums. The precise conditions under which two Everett's sums are equivalent were given by Müller and Petrich in [**217**], 1971, by the following theorem:

Theorem 1.2. *Two Everett's sums* $E(A,B;\theta;[,];\langle,\rangle)$ *and* $E(A,B;\theta';[,]';\langle,\rangle')$ *of rings* A *and* B *are equivalent if and only if there exists a mapping* $\xi:B\to A$ *such that* $0\xi=0$ *and for all* $a,b\in B$ *the following conditions hold:*

(a) $(\theta')^b=\theta^b+\pi_{b\xi}$;
(b) $[a,b]'=[a,b]+a\xi+b\xi-(a+b)\xi$;
(c) $\langle a,b\rangle'=\langle a,b\rangle+\theta^a(b\xi)+(a\xi)\theta^b+(a\xi)(b\xi)-(ab)\xi$.

Let $n\in\mathbb{N}$ and let $w\in A_n^+$. If $X_1,X_2,\dots,X_n$ are sets, then we will denote by $w(X_1,X_2,\dots,X_n)$ the set obtained by replacement of letters $x_1,x_2,\dots,x_n$ in w by sets $X_1,X_2,\dots,X_n$, respectively, considering the Cartesian multiplication of sets instead of the juxtapositions in w. Let R be a ring, let P be a set of permutable bitranslations of R and let μ be an element of the Cartesian n-th power of $R\cup P$. If at least one projection of μ is in R, then $w(\mu)$ will denote the element of R obtained by replacement of any letter x_i, $i\in\{1,2,\dots,n\}$, by the i-th projection of μ, considering the multiplications in $\mathcal{M}R$ and $\mathcal{M}\Omega(R)$ and acting of bitranslations from P on elements of R, instead of the juxtapositions in w. Otherwise, if all the projections of μ are in P, then $w(\mu)$ will denote the value of w in the semigroup $\mathcal{M}\Omega(R)$, for the valuation μ.

The following theorem, given by Ćirić, Bogdanović and Petković in [**94**], 1995, describes more complicated multiplications in Everett's sums of rings.

Theorem 1.3. *Let* $R = E(A, B; \theta; [\,,]; \langle\,,\rangle)$, *let* $n \in \mathbb{N}$, $n \geq 2$, *and assume that* $w = w(x_1, \dots, x_n) \in A_n^+$, $|w| = k$, $a = (a_1, \dots, a_n) \in B^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in A^n$, $\xi_i = (\alpha_i, a_i)$, $i \in \{1, \dots, n\}$, $\xi = (\xi_1, \dots, \xi_n)$, *and* $\theta^a = (\theta^{a_1}, \dots, \theta^{a_n})$. *Then for*

$$\beta = \sum_{j=1}^{k-2} \langle (l_j(w))(a), (c_{j+1}(w))(a)\rangle (r_{k-j-1}(w))(\theta^a) + \langle (l_{k-1}(w))(a), (t(w))(a)\rangle,$$

the following statements hold:

$$\text{(i)}\ \ w(\theta^a) = \theta^{w(a)} + \pi_\beta, \qquad \textit{and} \qquad \text{(ii)}\ \ w(\xi) = \Bigg(\sum_{\mu \in M_w} \Pi_k(\mu) + \beta, w(a)\Bigg),$$

where $M_w = w(X_1, \dots, X_n) - \{\theta^a\}$, $X_i = \{\alpha_i, a_i\}$, $i \in \{1, \dots, n\}$.

Furthermore, if $\theta^b A \theta^c = 0$, *for all* $b, c \in B$ *and if* $k \geq 3$, *then*

$$\beta = \langle (h(w))(a), (m_2^{k-1}(w))(a)\rangle \theta^{(t(w))(a)} + \langle (l_{k-1}(w))(a), (t(w))(a)\rangle.$$

There are many known constructions in Theory of rings which are special cases of Everett's sums. For example, the well known *split extension* of rings is in fact an Everett's sum of rings in which the functions $[\,,]$ and $\langle\,,\rangle$ are zero functions, i.e. $[a, b] = \langle a, b\rangle = 0$, for all a, b. In such a way we obtain also the well-known *Dorroh extension* of a ring by a ring of integers, which realizes an embedding of a ring into a ring with unity.

An interesting specialization of Everett's sums was given by Ćirić and Bogdanović in [**80**], 1990. An Everett's sum $E(A, B; \theta; [\,,]; \langle\,,\rangle)$ was called by them a *strong Everett's sum* if θ is a zero homomorphism of B into $\Omega(A)$, i.e. if $\theta^a = \pi_0$, for any $a \in B$. Such an Everett's sum is denoted by $E(A, B; [\,,]; \langle\,,\rangle)$, and a representation of a ring R by such an Everett's sum is called a *strong Everett's representation* of R. A ring R is called a *strong extension* of a ring A by a ring B if there exists a strong Everett's representation $R = E(A, B; [\,,]; \langle\,,\rangle)$.

Using the concept of strong extensions of rings, Ćirić and Bogdanović in [**80**], 1990, gave the following construction of nilpotent rings:

Theorem 1.4. *Let* $n \in \mathbb{N}$, $n \geq 2$. *A ring* R *is an* $(n+1)$*-nilpotent ring if and only if it is a strong extension of a null-ring by an* n*-nilpotent ring.*

Recall that by a null-ring we mean a 2-nilpotent ring.

The same authors investigated also some other strong extensions of rings, and some of the obtained results will be presented in the next sections. Here we will give only some general properties of strong extensions.

Theorem 1.5. *Any strong extension of a ring by a ring with identity is isomorphic to their direct sum.*

The previous result was obtained by Ćirić and Bogdanović in [**80**], 1990, who also stated the following problem: Is any strong extension of two rings isomorphic to their direct sum?

An example of an Everett's sum of two rings which is not equivalent to a strong Everett's sum of these rings is the following: Let $n \in \mathbb{N}$, $n \geq 2$, and let R be the ring of all $n \times n$ upper triangular matrices over a field F. The set N of nilpotents of R is the set of all matrices (a_{ij}) from R for which $a_{ij} = 0$, whenever $i \geq j$, and we have that N is an ideal of R, the factor ring R/N is isomorphic to the ring F^n, and by the previous theorem, R cannot be a strong extension of N by F^n.

Note that the previous theorem is similar to the following well-known result:

Theorem 1.6. *Let A be a ring with an identity. Then a ring R is an ideal extension of A if and only if A is a direct summand of R.*

This theorem is in fact an immediate consequence of the result given by Ćirić and Bogdanović in [**80**], 1990, concerning retractive extensions of rings. A subring A of a ring R is called a *retract* of R if there exists a homomorphism φ of R onto A such that $a\varphi = a$, for any $a \in A$. Such a homomorphism is called a *retraction* of R onto A. If R is an ideal extension of A and there exists a retraction of R onto A, we say that R is a *retractive extension* of A and that A is a *retractive ideal* of R.

Theorem 1.7. *A ring R is a retractive ideal of a ring R if and only if A is a direct summand of R.*

Note that any ideal A with an identity of a ring R is a retract of R. Namely, a retraction φ of R onto A is given by $x\varphi = xe$, where $x \in R$ and e is an identity of A.

More information concerning retractions of semigroups will be given in Section 5.

2. On π-regular semigroups and rings

In this section we present the main properties of regular and π-regular semigroups and rings.

2.1. The regularity in semigroups and rings. The regularity was first defined in Ring theory by von Neumann in [**224**], 1936, and after that this definition was naturally transmitted in Semigroup theory. By this definition, an element a of a ring (semigroup) R is a *regular element* if there exists $x \in R$ such that $a = axa$, and a ring (semigroup) is defined to be a *regular ring* (*regular semigroup*) if all its elements are regular. Thierrin, who first investigated some general properties of regular semigroups in [**322**], 1951, called them *inversive semigroups* (*demi-groupes* *inversifs*). The set of all regular elements of a semigroup (ring) S we call the *regular part* of S and we denote it by $\mathrm{Reg}(S)$.

Many very important kinds of rings are regular. For example, such a property have division rings, the full matrix ring over a division ring, the ring of linear transformations of a vector space over a division ring, and many other rings. This also holds for many significant concrete semigroups. For example, the full transformation semigroup of an arbitrary finite set is regular, and the statement that the full transformation semigroup of a set X is regular for any set X is equivalent to the famous Axiom of Choice. For more information about general properties of regular rings and semigroups we refer to the books: Goodearl [**128**], Steinfeld [**301**], Petrich [**245**] and others. Here we give only some their properties which we need in the further work.

Theorem 2.1. *The following conditions on a semigroup* (*ring*) S *are equivalent:*

(i) S *is regular;*
(ii) $A \cap B = BA$, *for any left ideal* A *and any right ideal* B *of* S*;*
(iii) *any one-sided ideal of* S *is globally idempotent and* BA *is a quasi-ideal of* S, *for any left ideal* A *and any right ideal* B *of* S*;*
(iv) *any principal left* (*right*) *ideal of* S *has an idempotent generator.*

The equivalence of conditions (i) and (ii) was established by Iséki in [**145**], 1956, for semigroups, and Kovács in [**160**], 1956, for rings. Similar characterizations of regular elements by principal one-sided ideals, and related characterizations of regular semigroups and rings, were given by Lajos in [**164**], 1961, for semigroups, and Szász in [**308**], 1961, for rings. For many information on other interesting properties of two-sided, one-sided, quasi- and bi-ideals of regular semigroups and rings we refer to the book of Steinfeld [**301**], 1978.

The equivalence of conditions (i) and (iii) was proved by Calais in [**67**], 1961, for semigroups, and by Steinfeld in [**301**], 1978, for rings. Finally, (i) $\Leftrightarrow$ (iv) was proved by von Neuman in [**224**], 1936 (see also Clifford and Preston [**105**], 1961).

If a is a regular element of a semigroup (ring) S, then the element x, whose existence was postulated by the definition of the regularity, can be chosen such that $a = axa$ and $x = xax$, and any element x satisfying this condition, which is not necessary unique, is called an *inverse* of a. This property of regular elements was first observed by Thierrin in [**323**], 1952. A regular semigroup (ring) whose any element has a unique inverse is called an *inverse semigroup* (*inverse ring*). Inverse semigroups were first defined and investigated by Vagner in [**335**], 1952, and [**337**], 1953, who called them *generalized groups*, and independently by Preston in [**252**], [**253**], [**254**], 1954. The most significant example of inverse semigroups is the semigroup of partial one-to-one mappings of a set X into itself, and is called the *symmetric inverse semigroup* on X. Just as any group can be embedded in a symmetric group, by the Cayley theorem, and any semigroup can be embedded in a full transformation semigroup, so every inverse semigroup can be embedded into a symmetric inverse semigroup. This result is due to Vagner [**332**], 1952, and Preston [**254**], 1954, and is known as the Vagner-Preston Representation Theorem.

For more information on inverse semigroups we refer to the books of Howie [**144, Chapter V**], 1976, and Petrich [**247**], 1984. Here we quote only some characterizations of these semigroups that we need in the further work.

Theorem 2.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is inverse;*
(ii) *S is regular and the idempotents of S commute;*
(iii) *any principal one-sided ideal of S has a unique idempotent generator.*

The implication (ii) $\Rightarrow$ (iii) was proved by Vagner in [**335**], 1952, and independently by Preston in [**252**], 1954, (i) $\Rightarrow$ (ii) was proved by Liber in [**197**], 1954, whereas the equivalence of all three conditions was proved by Munn and Penrose in [**219**], 1955.

A natural generalization of inverse semigroups was given by Venkatesan in [**338**], 1974, who defined a regular semigroup (ring) to be a *left inverse* (resp. *right inverse*) *semigroup* (*ring*) if for all $a, x, y \in S$, $a = axa = aya$ implies $ax = ay$ (resp. $a = axa = aya$ implies $xa = ya$). Left inverse semigroups are characterized by the following theorem:

Theorem 2.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is left inverse;*
(ii) *S is regular and $E(S)$ is a left regular band;*
(iii) *any principal left ideal of S has a unique idempotent generator.*

Another important kind of the regularity was introduced by Clifford in [**99**], 1941, who studied elements a of a semigroup S having the property that there exists $x \in S$ such that $a = axa$ and $ax = xa$, which we call now *completely regular elements*, and semigroups whose any element is completely regular, called *completely regular semigroups*. The complete regularity was also investigated by Croisot in [**107**], 1953, who also studied elements a of a semigroup S for which $a \in Sa^2S$ (resp. $a \in Sa^2$, $a \in a^2S$), called *intra-regular*(resp. *left regular*, *right regular*) *elements*, and semigroups whose every element is intra-regular (resp. left regular, right regular), called *intra-regular* (resp. *left regular*, *right regular*) *semigroups*. Analogously we define intra-, left, right and completely regular rings and elements of rings. As we will see in Section 4, the concepts of the left, right and completely regular rings coincide, and in Ring theory such rings are known under the names *strongly regular* and *Abelian regular rings*. The results of A. H. Clifford and R. Croisot from the above mentioned papers concerning intra-, left, right and completely regular semigroups will be also presented in Section 4. Here we give only some their results which characterizes completely regular elements of a semigroup:

Theorem 2.4. *The following conditions for an element a of a semigroup S are equivalent:*

(i) *a is completely regular;*

(ii) *a has an inverse which commutes with a;*
(iii) *a is contained in a subgroup of S;*
(iv) *a is left regular and right regular.*

In view of the previous theorem, completely regular elements are often called *group elements*, and the set of all completely regular elements of a semigroup (ring) S is denoted by $\mathrm{Gr}(S)$ and is called the *group part* of S. For any idempotent e of a semigroup S, $G_e = \{a \in S \mid a \in eS \cap Se,\ e \in aS \cap Sa\}$ is the *maximal subgroup* of S having e as its identity, and $\mathrm{Gr}(S)$ is a disjoint union of all maximal subgroups of S. The existence of maximal subgroups was established by Schwarz in [**278**], 1943, for periodic semigroups, and by Wallace in [**340**], 1953, and Kimura in [**155**], 1954, for an arbitrary semigroup. The sets of all left, right and intra-regular elements of a semigroup (ring) S are called the *left regular*, *right regular* and *intra-regular part* of S, and are denoted by $\mathrm{LReg}(S)$, $\mathrm{RReg}(S)$ and $\mathrm{Intra}(S)$, respectively.

For any pair $m, n \in \mathbb{N}^0$, $m + n > 1$, Croisot in [**107**], 1953, also defined an element a of a semigroup S to be *(m, n)-regular* if $a \in a^m S a^n$, where a^0 denotes the identity adjoined to S. He proved that for all $m, n \geq 2$, the $(m, 0)$-regularity is equivalent to the right regularity and the $(0, n)$-regularity is equivalent to the left regularity, and for all $m, n \in \mathbb{N}$ for which $m + n \geq 3$, the (m, n)-regularity of a semigroup is equivalent to the complete regularity. As we see, the intra-regularity is not included in this Croisot's concept. But, by Lajos and Szász in [**192**], 1975, for $p, q, r \in \mathbb{N}^0$, an element a of a semigroup S was defined to be *(p, q, r)-regular* if $a \in a^p S a^q S a^r$, and a semigroup S was defined to be a *(p, q, r)-regular semigroup* if any its element is (p, q, r)-regular. This definition obviously includes the intra-regularity and many other interesting concepts. For example, this definition includes the concept of quasi-regularity introduced by Calais in [**67**], 1961, as a generalization of the ordinary regularity, seeing that by Theorem 2.1, in a regular semigroup (ring) any its one-sided ideal is globally idempotent. Namely, J. Calais defined a semigroup (ring) to be *left quasi-regular* (resp. *right quasi-regular* if any its left ideal (resp. right ideal) is globally idempotent, and to be *quasi-regular* if it is both left and right quasi-regular. The corresponding definitions can be given for elements: an element a of a semigroup (ring) S is called *left quasi-regular* (resp. *right quasi-regular*) if the principal left ideal $(a)_L$ (resp. the principal right ideal $(a)_R$) generated by a is globally idempotent, and is called *quasi-regular* if it is both left and right quasi-regular. It is easy to see that a semigroup (ring) is (left, right) quasi-regular if and only if any its element is (left, right) quasi-regular. As Lajos and Szász proved in [**192**], 1975, the left quasi-regular and the right quasi-regular elements of a semigroup S are exactly the $(0, 1, 1)$-regular and the $(1, 1, 0)$-regular elements of S, respectively.

Note that this concept of quasi-regularity differs to the well-known concept of quasi-regularity of elements of rings which is used in the definition of the Jacobson radical of a ring.

2.2. The π-regularity in semigroups and rings. In order to give a generalization both of regular rings and of algebraic algebras and rings with minimum conditions on left or right ideals, Arens and Kaplansky in [**11**], 1948, and Kaplansky in [**150**], 1950, defined π-regular rings. Following their terminology, an element a of a semigroup (ring) S is called *π-regular* (resp. *left π-regular*, *right π-regular*, *completely π-regular*, *intra-π-regular*) if some its power is regular (resp. left regular, right regular, completely regular, intra-regular), and S is called a *π-regular* (resp. *left π-regular*, *right π-regular*, *completely π-regular*, *intra-π-regular*) if any its element is π-regular (resp. left π-regular, right π-regular, completely π-regular, intra-π-regular). In some origins several other names were used. For example, Putcha in [**255**], 1973, Galbiati and Veronesi in [**121**]–[**125**], Shum, Ren and Guo in [**289**], [**290**], [**272**] and [**273**], and others called π-regular semigroups *quasi regular*, whereas Edwards in [**112**], 1993, called them *eventually regular*. Completely π-regular semigroups were sometimes called *quasi-completely regular* or *group-bound*, and Shevrin in [**285**] and [**296**], 1994, called them *epigroups*. In theory of rings, completely π-regular rings are known as *strongly π-regular rings*, as they were called by Azumaya in [**14**], 1954. In order to unify the terminology used in this paper, we use the name completely π-regular both for semigroups and rings.

Some variations of the π-regularity were also investigated by Fuchs and Rangaswamy in [**119**], 1968. For a positive integer m, they called an element a of a semigroup (ring) S *m-regular* if the power a^m is regular, and *$\overline{m}$-regular*, if a^n is regular for any $n \geq m$, and S is called an *m-regular* (resp. *$\overline{m}$-regular*) *semigroup* (*ring*) if any its element is m-regular (resp. $\overline{m}$-regular). Clearly, an element a is π-regular if and only if it is m-regular for some $m \in \mathbb{N}$. If for an element a of a semigroup (ring) S there exists $m \in \mathbb{N}$ such that a is $\overline{m}$-regular, we then say that a is *$\overline{\pi}$-regular*, and a semigroup (ring) whose any element is $\overline{\pi}$-regular is called a *$\overline{\pi}$-regular semigroup* (*ring*). If a is an element of a semigroup (ring) S and a^m is left (resp. right, completely) regular for some $m \in \mathbb{N}$, then a^n is left (resp. right, completely) regular for any $n \geq m$.

Some relationships between the π-regularity, left π-regularity, right π-regularity, complete π-regularity and intra-π-regularity were investigated by many authors. We give here the most important results concerning these relationships. The first theorem that we give was proved by Bogdanović and Ćirić in [**55**], 1996:

Theorem 2.5. *A semigroup S is left π-regular if and only if it is intra-π-regular and* $\mathrm{Intra}(S) = \mathrm{LReg}(S)$.

By this theorem we obtain the following interesting result:

Theorem 2.6. *If S is a completely π-regular semigroup, then*

$$\mathrm{Gr}(S) = \mathrm{LReg}\, S) = \mathrm{RReg}(S) = \mathrm{Intra}(S) \subseteq \mathrm{Reg}(S).$$

Note that there exists a completely π-regular semigroup S in which $\mathrm{Gr}(S)$ is a proper subset of $\mathrm{Reg}(S)$. Completely π-regular semigroups whose regular part coincide with the group part will be considered in Section 5.

Another theorem gives some connections between the complete π-regularity, π-regularity and left (or right) π-regularity:

Theorem 2.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely π-regular;*
(ii) *S is left and right π-regular;*
(iii) *S is π-regular and left (or right) π-regular;*
(iv) *for any $a \in S$ there exists $n \in \mathbb{N}$ such that a^n is regular and left (or right) regular.*

The equivalence of conditions (i) and (iv) was proved by Hongan in [**143**], 1986, and of (i) and (iii) by Bogdanović and Ćirić in [**44**], 1992.

For rings a more rigorous theorem holds:

Theorem 2.8. *The following conditions on a ring R are equivalent:*

(i) *R is left π-regular;*
(ii) *R is right π-regular;*
(iii) *R is completely π-regular.*

This very important theorem was proved by Dischinger in [**108**], 1976, and another proof was given by Hirano in [**139**], 1978.

Clearly, any completely π-regular ring is π-regular. Various conditions under which a π-regular ring is completely regular were investigated by many authors. The best known results from this area are the results obtained by Azumaya in [**14**], 1954. He investigated rings in which the indices of nilpotency of all nilpotent elements are bounded, called the *rings of bounded index* and he proved the following two theorems:

Theorem 2.9. *If R is a ring of bounded index, then*

$$\mathrm{RReg}(R) = \mathrm{LReg}(R) = \mathrm{Gr}(R).$$

Theorem 2.10. *Let R be a ring of bounded index. Then R is π-regular if and only if it is completely π-regular.*

In connection with the π-regularity, rings of bounded index were also investigated by Tominaga in [**329**], 1955, and Hirano in [**140**], 1990.

As known, Moore in [**215**], 1936, Penrose in [**234**], 1955, and Rado in [**264**], 1956, introduced the notion of a generalized inverse of a matrix. Namely, by a result obtained by Moore, but stated in a more convenient form by Penrose, for any square complex matrix a there exists a unique complex matrix x such that $axa = a$, $xax = x$ and both ax and xa are hermitian. Such a matrix x is called the *generalized inverse*, or the *Moore-Penrose inverse*, of a. In order to give a further generalization of generalized inverses, Drazin introduced in [**110**], 1958, the following notion: Given a semigroup (ring) S and an element $a \in S$. An element $x \in S$ is called the *pseudo-inverse*, or the *Drazin inverse*, of a, if $ax = xa$, $x^2a = x$ and there exists $m \in \mathbb{N}$ such that $a^m = a^{m+1}x$. An element having a pseudo-inverse is called *pseudo-invertible*, and also, a semigroup (ring) whose any element is pseudo-invertible is called a *pseudo-invertible semigroup* (*ring*). As was shown by Drazin, a pseudo-inverse of an element a, if it exists, is unique. He also proved the following:

Theorem 2.11. *An element a of a semigroup (ring) S is pseudo-invertible if and only if it is completely π-regular.*

Let us note that an element a of a semigroup S is completely π-regular if and only if there exists $n \in \mathbb{N}$ such that the power a^n lies in some subgroup of S (see Theorem 2.4). The next theorem, proved by Drazin in [**110**], 1958, and in a slightly simplified form by Munn in [**218**], 1961, and known in Theory of semigroups as the *Munn's lemma*, gives an interesting property of such elements:

Theorem 2.12. *Let a be an element of a semigroup S such that for some $n \in \mathbb{N}$, a^n belongs to some subgroup G of S, and let e be the identity of this group. Then $ea = ae \in G_e$ and $a^m \in G_e$, for each integer $m \geq n$.*

Using the previous two theorems, pseudo-inverses can be represented in another way. Namely, if a is a pseudo-invertible, or equivalently, a completely π-regular element of a semigroup S, then $a^n \in G_e$, for some $n \in \mathbb{N}$ and $ae \in G_e$, and then the pseudo-inverse x of a is given by $x = (ae)^{-1}$, i.e. x is the *group inverse* of the element ae in the group G_e. If a is an element of a completely π-regular semigroup S and $a^n \in G_e$, for some $n \in \mathbb{N}$ and $e \in E(S)$, then a^0 denotes the identity of G_e, i.e. $a^0 = e$.

An interesting characterization of completely π-regular rings was given by Ôhori in [**229**], 1985. Before we exhibit this result, we must introduce some new notions. These notions were introduced by Hirano, Tominaga and Yaqub in [**142**], 1988, but they are given here in a slightly modified form. Let A and B be two subsets of a ring R. We say that R is (A, B)-*representable* if for any $x \in R$ there exist $a \in A$ and $b \in B$ such that $x = a + b$, and that it is *uniquely* (A, B)-*representable* if for any $x \in X$ there exist unique $a \in A$ and $b \in B$ such that $x = a + b$. Similarly, we say that R is $[A, B]$-*representable* if for any $x \in R$ there exist $a \in A$ and $b \in B$ such that $x = a + b$ and $ab = ba$, and that it is *uniquely* $[A, B]$-*representable* if for any $x \in R$ there exist unique $a \in A$ and $b \in B$ such that $x = a + b$ and $ab = ba$. Clearly, any uniquely (A, B)-representable ring is (A, B)-representable, any uniquely $[A, B]$-representable ring is $[A, B]$-representable, and all these rings are (A, B)-representable.

The characterization of completely π-regular rings given by Ôhori in [**229**], 1985, is the following:

Theorem 2.13. *A ring R is completely π-regular if and only if it is* $[\mathrm{Nil}(R), \mathrm{Gr}(R)]$*-representable.*

In order to generalize the concept of an inverse semigroup, Galbiati and Veronesi defined in [**120**], 1980, a semigroup (and also ring) to be π-*inverse* if it is π-regular and any its regular element has a unique inverse. A further generalization of these concept was given by Bogdanović in [**35**], 1984, who defined a semigroup (or ring) S to be *left* (resp. *right*) π-*inverse*if it is π-regular and for all $a, x, y \in S$, $a = axa = aya$ implies $ax = xa$ (resp. $a = axa = aya$ implies $xa = ya$).

Similarly, a semigroup (ring) S is called *completely* π-*inverse*(resp. *left completely* π-*inverse*, *right completely* π-*inverse*) if it is completely π-regular and π-inverse (resp. left π-inverse, right π-inverse).

The following theorem, which characterizes left π-inverse semigroups, was proved by Bogdanović in [**35**], 1984:

Theorem 2.14. *The following conditions on a semigroup S are equivalent:*

(i) *S is left π-inverse;*
(ii) *S is π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (ef)^n e$;*
(iii) *S is π-regular and for any pair $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n \mathcal{L} (fe)^n$;*
(iv) *for any $a \in S$ there exists $n \in \mathbb{N}$ such that $(a^n)_L$ has a unique idempotent generator.*

A consequence of the previous theorem and its dual is the following result obtained by Galbiati and Veronesi in [**120**], 1980, and Bogdanović in [**33**], 1982, and [**35**], 1984.

Theorem 2.15. *The following conditions on a semigroup S are equivalent:*

(i) *S is π-inverse;*
(ii) *S is left and right π-inverse;*
(iii) *S is π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (fe)^n$.*
(iv) *S is π-regular and for any $a \in S$ there exists $n \in \mathbb{N}$ such that $(a^n)_L$ and $(a^n)_R$ have unique idempotent generators.*

Left completely π-inverse semigroups were studied by Bogdanović and Ćirić in [**44**], 1992, where the following result was obtained:

Theorem 2.16. *A semigroup S is left completely π-inverse if and only if it is π-regular and for all $a \in S$, $e \in E(S)$, there exists $n \in \mathbb{N}$ such that $(ea)^n = (ea)^n e$.*

Finally, completely π-inverse semigroups are characterized by the following theorem, due to Galbiati and Veronesi [**124**], 1984.

Theorem 2.17. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely π-inverse;*
(ii) *S is left and right completely π-inverse;*
(iii) *S is π-regular and for all $a \in S$, $e \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ea)^n = (ae)^n$.*

2.3. Periodic semigroups and rings. Periodic semigroups and rings are among the most important special types of completely π-regular semigroups and rings. They are defined as semigroups (rings) in which for any element a there exist different $m, n \in \mathbb{N}$ such that $a^m = a^n$, or equivalently, as semigroups (rings) in which for any element a, some power of a is an idempotent.

Periodic semigroups and rings have many very interesting properties. For example, the property "being periodic" is a hereditary property, both for semigroups and rings, and many subclasses of the class of periodic semigroups (rings) can be characterized in terms of variable identities, as we will see in Section 5. Clearly, the whole class of periodic semigroups is definable by a variable identity $\{x^m = x^n \mid m, n \in \mathbb{N}, m \neq n\}$ over the one-element alphabet. Also, all finite semigroups and rings are periodic, and the periodicity was often investigated as a generalization of the finiteness.

An element a of a semigroup (ring) S having the property that $a^m = a^n$, for some different $m, n \in \mathbb{N}$, will be called a *periodic element*. An interesting type of periodic elements of a semigroup (ring) are potent elements defined as follows: an element a of a semigroup (ring) S is *potent* if $a = a^n$, for some $n \in \mathbb{N}$, $n \geq 2$. The set of all potent elements of S is denoted by $P(S)$ and called the *potent part* of S.

Periodic rings have especially interesting properties. The next theorem, which is due to Chacron [**68**], 1969, gives a criterion of periodicity of rings, known as the *Chacron's criterion of the periodicity*.

Theorem 2.18. *A ring R is periodic if and only if for any $a \in R$ there exists $n \in \mathbb{N}$ and a polynomial $p(x)$ with integer coefficients such that $a^n = a^{n+1}p(a)$.*

Another proof of this theorem can be found in Bell [**19**], 1980.

The following properties of periodic rings were found by Bell in [**18**], 1977.

Theorem 2.19. *Let R be a periodic ring. Then the following conditions hold:*

(a) *for any $a \in R$ there exists $n \in \mathbb{N}$ such that $a - a^n \in \mathrm{Nil}(R)$;*
(b) *R is $(\mathrm{Nil}(R), P(R))$-representable;*
(c) *if I is an ideal of R and $a+I$ is a non-zero nilpotent of R/I, then R contains a nilpotent element u such that $a \equiv u \pmod{I}$.*

By Grosen, Tominaga and Yaqub in [**129**], 1990, rings satisfying the condition (b) of the above theorem were called *weakly periodic rings*. Therefore, the Bell's theorem asserts that any periodic ring is weakly periodic. The converse does not hold, but Ôhori in [**229**], 1985, found the conditions under which a weakly periodic rings is periodic, and this result is given here as the following theorem:

Theorem 2.20. *A ring R is periodic if and only if it is $[P(R), \mathrm{Nil}(R)]$-representable.*

3. On completely Archimedean semigroups

The topic of this paper are uniformly π-regular semigroups and rings, i.e. semigroups and rings decomposable into a semilattice of completely Archimedean semigroups, or equivalently, into a semilattice of nil-extensions of completely simple semigroups, so we must present the main properties of completely Archimedean and completely simple semigroups.

3.1. Completely simple semigroups. As known, a semigroup S having no an ideal different than the whole S is called a *simple semigroup*, and similarly, a semigroup S having no a left (resp. right) ideal different than the whole S is called a *left simple* (resp. *right simple*) *semigroup*. In other words, a semigroup S is simple (resp. left simple, right simple) if and only if $a \mid b$ (resp. $a \underset{l}{\mid} b$, $a \underset{r}{\mid} b$), for all $a, b \in S$. The first papers from Theory of semigroups were devoted exactly to these semigroups, because they are the closest generalization of groups. Namely, a semigroup is a group if and only if it is both left and right simple. By Sushkevich in [**304**], 1928, and [**305**], 1937, and Rees in [**271**], 1940, finite simple semigroups and other significant special types of simple semigroups were investigated. In this section we talk about the most important special types of these semigroups.

Semigroups which are both simple and left (resp. right) regular were called by Bogdanović and Ćirić in [**55**], 1996, *left* (resp. *right*) *completely simple*. Some characterizations of these semigroups are given by the following theorem:

Theorem 3.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is left completely simple;*
(ii) *S is simple and left π-regular;*
(iii) *S is simple and has a minimal left ideal;*
(iv) *S is a union of its minimal left ideals;*
(v) *S is a disjoint union of its principal left ideals;*
(vi) *any principal left ideal of S is a left simple subsemigroup of S;*
(vii) *any left ideal of S is right consistent;*
(viii) *S is a matrix of left simple semigroups;*
(ix) *S is a right zero band of left simple semigroups;*
(x) *$\underset{l}{\mid}$ is a symmetric relation on S;*
(xi) *$S/\mathcal{L}$ is a discrete partially ordered set;*
(xii) *$\mathcal{LI}d(S)$ is a Boolean algebra;*
(xiii) *$(\forall a, b \in S)\ a \in Sba$.*

The equivalence of the conditions (iii), (iv), (vii) and (xiii) was proved by Croisot in [**107**], 1953, of (vi), (ix) and (xiii) by Bogdanović in [**33**], 1982, and of (i), (ii), (viii), (ix), (x), (xi) and (xiii) by Bogdanović and Ćirić in [**55**], 1996. The equivalence of the conditions (vii), (ix) and (xii) is an immediate consequence of the results of Bogdanović and Ćirić from [**53**], 1995, concerning so-called right sum decomposition of semigroups with zero. In the book of Clifford and Preston [**106**], 1967, semigroups satisfying the condition (xiii) of the above theorem were called *left stratified semigroups*.

Another important type of simple semigroups are simple semigroups having a primitive idempotent, called *completely simple semigroups*. Recall that an idempotent e of a semigroup S is called *primitive* if it is minimal in the partially ordered set of idempotents on S, i.e. if for $f \in E(S)$, $ef = fe = f$ implies $e = f$. Completely simple semigroups were first studied also by Sushkevich in [**304**], 1928, and [**305**], 1937, and Rees in [**271**], 1940, who gave the following fundamental representation theorem for these semigroups:

Theorem 3.2. *Let G be a group, let I and Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G. Define a multiplication on $S = G \times I \times \Lambda$ by:*

$$(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu).$$

Then S with so defined multiplication is a completely simple semigroup.

Conversely, any completely simple semigroup is isomorphic to some semigroup constructed in this way.

The semigroup constructed in accordance with this recipe is called the *Rees matrix semigroup of type $\Lambda \times I$ over a group G with the sandwich matrix P*, and is denoted by $M(G; I, \Lambda, P)$. The previous theorem is usually called the *Rees-Sushkevich theorem*.

Some other characterizations of completely simple semigroups are given by the following theorem:

Theorem 3.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely simple;*
(ii) *S is simple and completely π-regular;*
(iii) *S is simple and completely regular;*
(iv) *S is simple and has a minimal left ideal and a minimal right ideal;*
(v) *S is simple and has a minimal quasi-ideal;*
(vi) *S is a union of its minimal quasi-ideals;*
(vii) *S is left and right completely simple;*
(viii) *S is left* (*or right*) *completely simple and has an idempotent;*
(ix) *S is regular and all its idempotents are primitive;*
(x) *S is regular and $a = axa$ implies $x = xax$;*
(xi) *S is regular and weakly cancellative;*
(xii) $(\forall a, b \in S)\, a \in aSba$*;*
(xii') $(\forall a, b \in S)\, a \in abSa$*;*
(xiii) $\underset{t}{|}$ *is a symmetric relation on S;*
(xiv) *$S/\mathcal{H}$ is a discrete partially ordered set.*

The equivalence of conditions (i) and (iv) is from Clifford [**100**], 1948. The assertion (i) $\Leftrightarrow$ (ii) was proved by Munn in [**218**], 1961, and is known as the *Munn theorem*. For periodic semigroups this assertion was proved by Rees in [**271**], 1940. The equivalence of the conditions (iv) and (v) is a result of Schwarz from [**279**], 1951, and the equivalence of the conditions (v) and (vi) is derived from the results of Steinfeld from [**296**], 1956 (see also his book [**301**]). For the proof of the equivalence of conditions (i), (ix), (x) and (xi) we refer to the book of Petrich [**241**], 1973. The equivalence of the conditions (vii), (xii), (xii'), (xiii) and (xiv) is an immediate consequence of Theorem 3.1 and its dual.

Special types of completely simple semigroups are left, right and rectangular groups. A semigroup S is called a *rectangular group* if it is a direct product of a rectangular band and a group, and is called a *left group* (resp. *right group*) if it is a direct product of a left zero band (resp. right zero band) and a group. Rectangular groups and left groups are characterized by the following two theorems:

Theorem 3.4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a rectangular group;*
(ii) *S is completely simple and $E(S)$ is a subsemigroup of S;*
(iii) *S is regular and $E(S)$ is a rectangular band;*
(iv) *$S \cong M(G; I, \Lambda, P)$ with $p_{\lambda i}^{-1} p_{\lambda j} = p_{\mu i}^{-1} p_{\mu j}$, for all $i, j \in I$, $\lambda, \mu \in \Lambda$.*

For the proof of this theorem we refer to the book of Petrich [**241**], 1973.

Theorem 3.5. *The following conditions on a semigroup S are equivalent:*

(i) *S is a left group;*
(ii) *S is left simple and right cancellative;*
(iii) *S is left simple and has an idempotent;*
(iv) *S has a right identity e and $e \in Sa$, for any $a \in S$;*
(v) *S is regular and right cancellative;*
(vi) *S is regular and $E(S)$ is a right zero band;*
(vii) *for all $a, b \in S$, the equation $xa = b$ has a unique solution in S;*
(viii) *for any $a \in S$, the equation $xa^2 = a$ has a unique solution in S;*
(ix) *S is a left zero band of groups;*
(x) *$(\forall a, b \in S)\, a \in aSb$;*
(xi) *$S \cong M(G; I, \Lambda, P)$ with $|I| = 1$.*

The equivalence of conditions (i), (ii) and (iii) was proved by Sushkevich in [**304**], 1928, for finite semigroups, and in [**305**], 1937, in the general case, and it was also formulated (without proofs) by Clifford in [**98**], 1933. The assertion (i) $\Leftrightarrow$ (iv) was proved by Clifford in [**98**], 1933, (i) $\Leftrightarrow$ (v) is an unpublished result of Munn, and (i) $\Leftrightarrow$ (x) was proved by Bogdanović and Stamenković in [**66**], 1988.

Now, in terms of left groups, right groups and groups, completely simple semigroups can be characterized as follows:

Theorem 3.6. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely simple;*
(ii) *S is a left zero band of right groups;*
(iii) *S is a right zero band of left groups;*
(iv) *S is a matrix of groups.*

The above theorem is an immediate consequence of the Rees-Sushkevich representation theorem for completely simple semigroups, and also, of Theorem 3.1, its dual and Theorem 3.3.

Note finally that the multiplicative semigroup of a non-trivial ring may not be simple, since a semigroup with zero is simple only if it is trivial. But, simple semigroups can appear in Theory of rings as subsemigroups of multiplicative semigroups of rings, as we will see later. On the other hand, in investigations of semigroups with zero one introduces other more suitable concepts. For example, one defines a semigroup $S = S^0$ to be a *0-simple semigroup* if $S^2 \neq 0$ and it has no an ideal different than 0 and the whole S. Similarly, *completely 0-simple semigroups* one defines as 0-simple semigroups having a *0-primitive idempotent*, by which we mean a minimal element in the partially ordered set of all non-zero idempotents of S. It is interesting to note that these semigroups have also a representation theorem of the Rees-Sushkevich type, through so-called Rees matrix semigroups over a group with zero adjoined. More information on completely 0-simple semigroups can be found in the books: Clifford and Preston [**105**], 1961, and [**106**], 1967, Howie [**144**], 1976, Steinfeld [**301**], 1978, Bogdanović and Ćirić [**48**], 1993, and others.

In theory of rings, a ring R having no an ideal different than 0 and the whole ring R is called a *simple ring*. More information about them and on so-called *Rees matrix rings over a division ring* can be found in the Petrich's book [**243**], 1974.

3.2. Completely Archimedean semigroups. By a natural generalization of semigroups considered in the previous section, the following semigroups one obtains: A semigroup S is called an *Archimedean semigroup* if $a \longrightarrow b$, for all $a, b \in S$, and similarly, S is called a *left Archimedean* (resp. *right Archimedean*) *semigroup* if $a \xrightarrow{l} b$ (resp. $a \xrightarrow{r} b$), for all $a, b \in S$. A semigroup which is both left and right Archimedean is called *two-sided Archimedean*, or shortly, a *t-Archimedean semigroup*.

The structure of Archimedean semigroups is quite complicated, but when an Archimedean semigroup is supplied by some additional property, such as the π-regularity, intra-, left, right or complete π-regularity, then its structure can be described more precisely, as we will see in the further text.

First we present the following two theorems, due mostly to Putcha [**255**], 1973.

Theorem 3.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a simple semigroup;*
(ii) *S is Archimedean and intra-π-regular;*
(iii) *S is Archimedean and has an intra-regular element;*
(iv) *S is Archimedean and has a kernel;*
(v) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in Sb^{2n}S$.

Theorem 3.8. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a left simple semigroup;*
(ii) *S is left Archimedean and intra-π-regular;*
(iii) *S is left Archimedean and left π-regular;*
(iv) *S is left Archimedean and has an intra-regular element;*
(v) *S is left Archimedean and has a left regular element;*
(vi) *S is left Archimedean and has a kernel;*
(vii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in Sb^{n+1}$.

By Theorem 3.7 it follows that a semigroup S is Archimedean and π-regular if and only if it is a nil-extension of a regular simple semigroup.

Left (resp. right) π-regular Archimedean semigroups were studied under the name *left* (resp. *right*) *completely Archimedean semigroups* by Bogdanović and Ćirić in [**59**], where the following theorem was proved:

Theorem 3.9. *The following conditions on a semigroup S are equivalent:*

(i) *S is left completely Archimedean;*
(ii) *S is a nil-extension of a left completely simple semigroup;*
(iii) *S is Archimedean and has a minimal left ideal;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in Sba^n$.

In analogy with completely simple semigroups, Archimedean semigroups having a primitive idempotents was called by Bogdanović in [**36**], 1985, *completely Archimedean semigroups*. The structure of these semigroups is described by the following theorem:

Theorem 3.10. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely Archimedean;*
(ii) *S is a nil-extension of a completely simple semigroup;*
(iii) *S is Archimedean and completely π-regular;*
(iv) *S is Archimedean and has a minimal left ideal and a minimal right ideal;*
(v) *S is Archimedean and has a minimal quasi-ideal;*
(vi) *S is left and right completely Archimedean;*
(vii) *S is left* (*or right*) *completely Archimedean and has an idempotent;*
(viii) *S is π-regular and all its idempotents are primitive;*
(ix) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in a^nSba^n$;
(ix') $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in a^nbSa^n$.

The equivalence of the conditions (ii), (viii), (ix) and (ix') was proved by Bogdanović and Milić in [**64**], 1984, the assertion (i) $\Leftrightarrow$ (iii) due to Galbiati and Veronesi [**123**], 1984, while (i) $\Leftrightarrow$ (ii) is an immediate consequence of Theorems 3.7 and 3.3.

A representation theorem of the Rees-Sushkevich type for completely Archimedean semigroups was given by Shum and Ren in [**289**], 1995.

Before we give a theorem which characterizes nil-extensions of rectangular groups, we must introduce the following notion: Let S and T be semigroups and let a semigroup H be a common homomorphic image of S and T, with respect to homomorphisms φ and ψ, respectively. Then

$$P = \{(a, b) \in S \times T \mid a\varphi = b\psi\},$$

is a subsemigroup of the direct product $S \times T$ of semigroups S and T, and is called a *spined product* of S and T with respect to H. It is known that P is a subdirect product of S and T. In Universal algebra this notion is known as a *pullback product*. It was introduced by Fuchs in [**117**], 1952, and since studied by Fleischer in [**116**], 1955, and Wenzel in [**343**], 1968. In Theory of semigroups these products have been intensively studied by Kimura, Yamada, Ćirić and Bogdanović and others, and the name "spined product" was introduced by Kimura in [**156**], 1958.

Theorem 3.11. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a rectangular group;*
(ii) *S is completely Archimedean and $E(S)$ is a subsemigroup of S;*
(iii) *S is π-regular and $E(S)$ is a rectangular band;*
(iv) *S is π-regular and Archimedean and for any $e \in E(S)$, the mapping $\varphi_e : x \mapsto exe$ is a homomorphism of S onto eSe;*
(v) *S is a subdirect product of a group and a nil-extension of a rectangular band;*
(vi) *S is a subdirect product of a group, a nil-extension of a left zero band and a nil-extension of a right zero band;*
(vii) *S is a spined product of a nil-extension of a left group and a nil-extension of a right group with respect to a nil-extension of a group.*

The equivalence of the conditions (i), (v) and (vi) was established by Putcha in [**255**], 1973, and of (i), (iii), (iv) and (vii) by Ren, Shum and Guo in [**273**]. Ren, Shum and Guo also gave a representation theorem of the Rees-Sushkevich type for these semigroups.

The next theorem, which characterizes nil-extensions of left groups, is mostly due to Bogdanović and Milić [**64**], 1984.

Theorem 3.12. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a left group;*
(ii) *S is left Archimedean and π-regular;*
(iii) *S is left Archimedean and right π-regular;*
(iv) *S is left Archimedean and completely π-regular;*
(v) *S is left Archimedean and has an idempotent;*
(vi) *S is π-regular and $E(S)$ is a left zero band;*
(vii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ a^n \in a^n S a^n b$.

A Rees-Sushkevich type representation theorem for nil-extensions of left groups was given by Shum, Ren and Guo in [**290**].

The previous theorem and its dual give the following:

Theorem 3.13. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a group;*
(ii) *S is π-regular and has a unique idempotent;*
(iii) *S is Archimedean and has a unique idempotent;*
(iv) *S is t-Archimedean and intra-π-regular;*
(v) *S is t-Archimedean and π-regular;*
(vi) *S is t-Archimedean and has an intra-regular element;*
(vii) *S is t-Archimedean and has an idempotent.*

The equivalence of the conditions (i) and (iii) was established by Tamura in [**318**], 1982.

Note finally that a semigroup with zero may be Archimedean if and only if it is a nil-semigroup, so Ćirić and Bogdanović introduced in [**89**], 1996, a concept more convenient for semigroups with zero, which generalizes both 0-simple and Archimedean semigroups. Namely, they defined a semigroup $S = S^0$ to be a *0-Archimedean semigroup* if $a \longrightarrow b$, for all $a, b \in S - 0$. These semigroups and some their special types were also studied by Ćirić and Bogdanović in [**86**], 1996, and Ćirić, Bogdanović and Bogdanović in [**97**].

4. Completely regular semigroups and rings

Although in Section 2 we have already discussed intra-, left, right and completely regular semigroups and rings, here we present their precise structure.

4.1. Completely regular semigroups. We start with intra-regular semigroups.

Theorem 4.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is intra-regular;*
(ii) *S is a union of simple semigroups;*
(iii) *any $\mathcal{J}$-class of S is a subsemigroup;*
(iv) *S is a semilattice of simple semigroups;*
(v) *any ideal of S is completely semiprime;*
(vi) $(\forall a, b \in S)\, (a) \cap (b) = (ab)$;
(vii) *$A \cap B \subseteq AB$, for any left ideal A and any right ideal B of S.*

The equivalence of conditions (i) and (vii) was proved by Lajos and Szász in [**192**], 1975. The rest of the theorem due to Croisot [**107**], 1953, and Anderson [**7**], 1952.

Combining the previous theorem with Theorem 2.1, the following theorem was obtained:

Theorem 4.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is regular and intra-regular;*
(ii) *S is a semilattice of regular simple semigroups;*
(iii) *$A \cap B = AB \cap BA$, for any left ideal A and any right ideal B of S;*
(iv) *$A \cap B \subseteq AB$, for all bi-ideals* (*or quasi-ideals*) *A and B of S;*
(v) *any quasi-ideal of S is globally idempotent.*

The equivalence of conditions (i) and (v) was established by Lajos in [**177**], 1972, and of (i) and (iv) by Lajos and Szász in [**192**], 1975. By Lajos in [**187**], 1991, the proof of (i) $\Leftrightarrow$ (iii) was attributed to Pondeliček. Finally, (i) $\Leftrightarrow$ (ii) is an immediate consequence of Theorem 4.1.

Structure of left regular semigroups was described by Croisot, 1953, and Bogdanović and Ćirić, 1996, who proved the following:

Theorem 4.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is left regular;*
(ii) *S is intra-regular and left π-regular;*
(iii) *S is a union of left simple semigroups;*
(iv) *any $\mathcal{L}$-class of S is a subsemigroup;*
(v) *S is a semilattice of left completely simple semigroups;*
(vi) *any left ideal of S is completely semiprime.*

The equivalence of conditions (i), (ii) and (v) was proved by Bogdanović and Ćirić in [**107**], 1996, and the rest is from Croisot [**55**], 1953.

For an element a of a semigroup (ring) S we say that it is *left duo* (*right duo*) if the principal left (right) ideal generated by a is a two-sided ideal, and that a is *duo* if it is both left and right duo. Similarly, a semigroup (ring) S is called *left duo* (*right duo*) if any left (right) ideal of S is a two-sided ideal, and is called *duo* if it is both left and right duo. The notion of a duo ring (semigroup) was introduced by Feller in [**114**], 1958, and Thierrin in [**325**], 1960, the corresponding definition for elements was given first by Steinfeld in [**300**], 1973, and left and right duo semigroups, rings and elements were first defined and studied by Lajos in [**181**] and [**182**], 1974. Between these notions the following relationship holds:

Theorem 4.4. *A semigroup (ring) is duo (resp. left duo, right duo) if and only if any its element is duo (resp. left duo, right duo).*

The previous theorem was proved by Kertész and Steinfeld in [**154**], 1974, and Steinfeld in [**300**], 1973, for the case of duo semigroups and rings.

Note also that the following holds:

Theorem 4.5. *An element a of a semigroup (ring) S is duo (resp. left duo, right duo) if and only if $(a)_L = (a)_R$ (resp. $(a)_R \subseteq (a)_L$, $(a)_L \subseteq (a)_R$).*

Recall that $(a)_L$ and $(a)_R$ denote the principal left and the principal right ideal of S generated by a, respectively.

Now we are ready to give the following characterization of semilattices of left simple semigroups.

Theorem 4.6. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left simple semigroups;*
(ii) *S is left (or intra-) regular and left duo;*
(iii) *S is left quasi-regular and left duo;*
(iv) *$A \cap B = AB$, for all left ideals A and B of S.*

The equivalence of conditions (i) and (ii) was proved by Petrich in [**236**], 1964, the proof of (i) $\Leftrightarrow$ (iv) was given by Saitô in [**274**], 1973, and the equivalence of (ii) and (iii) is an immediate consequence of Theorem 4.3 and Theorem 1 from the paper of Lajos and Szász [**192**], 1975.

Now we go to the completely regular semigroups. Various characterizations of these semigroups are collected in the following theorem:

Theorem 4.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely regular;*
(ii) *S is regular and left (or right) regular;*
(iii) *S is a union of groups;*
(iv) *any $\mathcal{H}$-class of S is a subsemigroup;*
(v) *S is a semilattice of completely simple semigroups;*
(vi) *any one-sided ideal of S is completely semiprime;*
(vii) *any left (or right, bi-) ideal of S is a regular semigroup;*
(viii) *any principal bi-ideal of S has an idempotent generator.*

The equivalence of conditions (i), (iii) and (v) was established by Clifford in [**99**], 1941, of (i), (ii) and (vi) by Croisot in [**107**], 1953, of (i) and (vii) by Lajos in [**184**], 1983. As was noted by Lajos in [**187**], 1991, (i) $\Leftrightarrow$ (viii) was proved in his paper from 1976. Note that the analogue of the condition (vii) for two-sided ideals is valid in any regular semigroup and ring (see Kaplansky [**151**], 1969, or Steinfeld [**301**], 1978).

For various constructions of completely regular semigroups we refer to Lallement [**194**], 1967, Petrich [**244**], 1974, and [**245**], 1977, Clifford [**104**], 1976, Warne [**341**], 1973, and Yamada [**346**], 1971.

Next we present the structure descriptions of the most important special types of completely regular semigroups.

Theorem 4.8. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of rectangular groups;*
(ii) *S is regular and $a = axa$ implies $a = ax^2a^2$;*
(iii) *S is completely regular and $E(S)$ is a subsemigroup.*
(iv) *S is completely regular and any inverse of any idempotent of S is an idempotent.*

The equivalence of conditions (iii) and (iv) is an immediate consequence of the result of Reilly and Scheiblich from [**269**], 1967, by which in any regular semigroup S, the idempotents of S form a subsemigroup if and only if any inverse of any idempotent of S is an idempotent. For the proof of the rest of the theorem we refer to Petrich [**241**], 1973.

Theorem 4.9. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left groups;*
(ii) *S is regular and $a = axa$ implies $ax = ax^2a$;*
(iii) *S is completely regular and $E(S)$ is a left regular band;*
(iv) *S is regular (or right regular) and left duo;*
(v) *S is quasi-regular (or right quasi-regular) and left duo;*
(vi) *$A \cap B = BAB$, for any left ideal A and any right ideal B of S;*
(vii) *$A \cap B = AB$, for any bi-ideal A and any right ideal B of S;*
(viii) *$A \cap B = AB$, for any bi-ideal A and any two-sided ideal B of S;*
(ix) *$A \cap B = BA$, for any left ideal A and any quasi-ideal B of S.*

The equivalence of conditions (i) and (iv) was established by Lajos in [**178**], 1972, and [**182**], 1974, of (i) and (viii) by Lajos in [**177**], 1972, and (i) $\Leftrightarrow$ (v) is a consequence of Theorem 4.6 and Theorem 1 from the paper of Lajos and Szász [**192**], 1975. The proofs of (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) can be found in Petrich [**241**], 1973. Finally, the conditions (vi), (vii) and (ix) are assumed from the survey paper of Lajos [**187**], 1991.

As we said before, completely regular semigroups were first investigated by Clifford in [**99**], 1941, and in some origins these semigroups were called the Clifford semigroups. But, some other authors, for example Howie in [**144**], 1976, used this name for another class of semigroups, studied first also by Clifford in [**99**], 1941, and following the terminology of these authors, in this paper by a *Clifford semigroup* (*ring*) we mean a regular semigroup (ring) whose all idempotents are central. These semigroups are characterized by the following theorem:

Theorem 4.10. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of groups;*
(ii) *S is a strong semilattice of groups;*
(iii) *S is a Clifford semigroup;*
(iv) *S is regular and $a = axa$ implies $ax = xa$;*
(v) *S is completely regular and $E(S)$ is a semilattice;*
(vi) *S is completely regular and inverse;*
(vii) *S is regular* (*or left, right, intra regular*) *and duo;*
(viii) *S is quasi-regular* (*or left, right quasi-regular*) *and duo;*
(ix) *$A \cap B = AB$, for any left ideal A and any right ideal B of S;*
(x) *$A \cap B = AB$, for all bi-ideals A and B of S;*
(xi) *$A \cap B = AB$, for all quasi-ideals A and B of S;*
(xii) *S is regular and a subdirect product of groups with a zero possibly adjoined.*

By Clifford in [**99**], 1941, the equivalence of conditions (i), (ii) and (iii) was proved, the equivalence of conditions (iv), (v) and (vi) is an immediate consequence of Theorem 2.2, (i) $\Leftrightarrow$ (vii) was proved by Petrich in [**236**], 1964, and (i) $\Leftrightarrow$ (xii) by the same author in [**242**], 1973. The equivalence of the condition (i) or (vii) and the conditions (ix), (x) and (xi) was established by Lajos in [**167**] and [**168**], 1969, [**170**] and [**171**], 1970, and [**174**] and [**175**], 1971.

The previous theorem, applied to commutative semigroups, gives the following their characterizations:

Theorem 4.11. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of Abelian groups;*
(ii) *S is a strong semilattice of Abelian groups;*
(iii) *S is regular and commutative;*
(iv) *S is quasi-regular and commutative;*
(v) *S is regular and a subdirect product of Abelian groups with a zero possibly adjoined.*

Various other characterizations of the semigroups considered here in terms of two-sided, one-sided, bi- and quasi-ideals we refer to the book of Steinfeld [**301**], 1978, the survey paper of Lajos [**187**], 1991, and other their papers given in the list of references.

4.2. Completely regular rings.

In this section we will see that many of the concepts from Theory of semigroups considered in the previous section coincide in Theory of rings and are equivalent to the complete regularity. But, in Theory of rings we have many interesting special types of completely regular rings, such as Jacobson rings, p-rings, Boolean rings etc, whose main properties will be presented here.

The first theorem that we quote here gives various equivalents of the complete regularity of rings.

Theorem 4.12. *The following conditions on a ring R are equivalent:*

(i) *R is completely regular;*
(ii) *R is left* (*right*) *regular;*
(iii) *R is regular and intra-regular;*
(iv) *R is inverse;*
(v) *R is a Clifford ring;*
(vi) *R is regular and has no non-zero nilpotents;*
(vii) *R is a regular* (*left, right*) *duo ring;*
(viii) *R is an intra-regular* (*left, right*) *duo ring;*
(ix) *R is a* (*left, right*) *quasi-regular* (*left, right*) *duo ring;*
(x) *R is regular and a subdirect sum of division rings;*
(xi) *any left* (*right, bi-*) *ideal of R is a regular ring;*
(xii) *$A \cap B = AB$, for any left ideal A and any right ideal B of R;*
(xiii) *$A \cap B = AB$, for all left* (*right*) *ideals A and B of R;*
(xiv) *$A \cap B = AB$, for all quasi-ideals A and B of R;*
(xv) *any quasi-ideal of R is globally idempotent.*

The equivalence of conditions (v), (vi) and (vii) was proved by Schein in [**276**], 1966, although (vi) $\Rightarrow$ (vii) was first stated by Calais in [**67**], 1961. The equivalence of (vi) and (xvi) is due to Kovács [**160**], 1956, of (vii), (xii) and (xiii) is due to Lajos [**166**], 1969, and [**169**], 1970, while (i) $\Leftrightarrow$ (xiii) is due to Andrunakievich [**8**], 1964, (vii) $\Leftrightarrow$ (xiv) was proved by Lajos in [**175**], 1971, and Steinfeld in [**299**], 1971, (vi) $\Leftrightarrow$ (x) by Forsythe and Mc Coy in [**117**], 1946, (ii) $\Leftrightarrow$ (xi) by Lajos in [**184**], 1983, (ii) $\Leftrightarrow$ (v) is from Lajos and Szász [**189**] and [**190**], 1970. The proof of (iii) $\Leftrightarrow$ (xv) can be found in Steinfeld [**301**], 1978. Finally, the equivalence of (i) and (ii) is an immediate consequence of the result of Azumaya given in Section 2 as Theorem 2.9.

Let us hold our attention on the equivalence of the conditions (vi) and (x) of the above theorem. This result can be viewed as a consequence of a more general result obtained by Andrunakievich and Ryabuhin in [**9**], 1968, given by the following theorem:

Theorem 4.13. *A ring R has no non-zero nilpotent elements if and only if it is a subdirect sum of rings without zero divisors.*

A proof of this theorem can be found also in their book [**10**], 1979 (see also Thierrin [**327**], 1967). In the commutative case this theorem was proved by Krull in [**161**], 1929, and [**162**], 1950.

An analogue of the previous theorem holds in Theory of semigroups. It was proved by Park, Kim and Sohn in [**233**], 1988, and it follows directly from the theorem that asserts that any completely semiprime ideal of a semigroup is an intersection of some family of their completely prime ideals. The proofs of this theorem given by Petrich in [**241**], 1973, and Park, Kim and Sohn in [**233**], 1988, include an essential use of the Zorn lemma, but Ćirić and Bogdanović showed in [**87**] and [**91**], 1996, that its proof can be derived from the general theory of semilattice decompositions of semigroups, without recourse to transfinite methods.

Let us also note that direct sums of division rings were characterized by Gerchikov in [**126**], 1940, by the following theorem:

Theorem 4.14. *A ring R is a direct sum of division rings if and only if it has no non-zero nilpotent elements and it satisfies minimum conditions on left* (*or right*) *ideals.*

In the case of commutative rings we have

Theorem 4.15. *The following conditions on a ring R are equivalent:*

(i) *R is regular and commutative;*
(ii) *R is quasi-regular and commutative;*
(iii) *R is regular and a subdirect sum of fields.*

As we said before, several special types of completely regular rings are of the great importance in Theory of rings. The first of these types are Jacobson rings, which one defines in the following way: A ring R is called a *Jacobson ring* if for any $a \in R$ there exists $n \in \mathbb{N}$, $n \geq 2$ such that $a^n = a$. This condition is known as the *Jacobson's $a^n = a$ condition*. This condition has appeared in investigations of algebraic algebras without nilpotent elements over a finite field, carried out by Jacobson in [**148**], 1945. In this paper Jacobson proved that such algebras are commutative and as a consequence he obtained the following very important result:

Theorem 4.16. (Jacobson's $\boldsymbol{a^n = a}$ theorem) *Any Jacobson ring is commutative.*

This theorem can be viewed as a generalization of the celebrated Wedderburn's theorem from [**342**], 1905, which asserts that any finite division ring must be a field.

A complete characterization of Jacobson rings, in few ways, is given by the next theorem, which is an immediate consequence of the Jacobson's $a^n = a$ theorem and Theorem 4.12.

Theorem 4.17. *The following conditions on a ring R are equivalent:*

(i) *R is a Jacobson ring;*

(ii) *R is commutative, regular and periodic;*
(iii) *R is completely regular and periodic;*
(iv) *R is regular and a subdirect sum of periodic fields;*
(v) *$\mathcal{M}R$ is a semilattice of periodic groups;*
(vi) *$\mathcal{M}R$ is a semilattice of periodic Abelian groups.*

A special case of Jacobson rings are the rings satisfying the semigroup identity of the form $x^n = x$, where $n \geq 2$ is an integer. Such rings were studied by Ayoub and Ayoub in [**13**], 1965, Luh in [**203**] and [**204**], 1967, and others. Luh characterized in [**203**], 1967, these rings in terms of p^k-rings, which are introduced by Mc Coy and Montgomery in [**211**], 1937, in the following way: A ring R is called a *p^k-ring* if there exists a prime p and a positive integer k such that R has the characteristic p and it satisfies the identity $x^{p^k} = x$. Rings defined in such a way with $k = 1$ are known as *p-rings*. The theorem proved by Luh in [**203**], 1967, is the following:

Theorem 4.18. *The following conditions on a ring R are equivalent:*
(i) *R satisfies the identity $x^n = x$, for some integer $n \geq 2$;*
(ii) *R satisfies the identity $x^p = x$, for some prime p;*
(iii) *R is a direct sum of finitely many p^k-rings.*

Particularly, p-rings are characterized by the following theorem:

Theorem 4.19. *Let p be a prime. A ring R is a p-ring if and only if it is a subdirect sum of fields of integers modulo p.*

Let us emphasize that p-rings, and consequently p^k-rings, trace one's origin to the famous *Boolean rings*, defined as rings whose any element is an idempotent. The following theorem characterizes these rings:

Theorem 4.20. *The following conditions on a ring R are equivalent:*
(i) *R is a Boolean ring;*
(ii) *R is a 2-ring;*
(iii) *R is a subdirect sum of fields of integers modulo 2;*
(iv) *$\mathcal{M}R$ is a band;*
(v) *$\mathcal{M}R$ is a semilattice.*

For more information on Boolean rings, and especially on their connections with Boolean algebras, we refer to the book of Abian [**1**], 1976, the paper of Stone [**302**], 1936, and others.

Various subdirect and direct sums whose summands are division rings or integral domains were studied by Kovácz in [**160**], 1956, Sussman in [**306**], 1958, Abian in [**2**], 1970, Chacron in [**69**], 1971, Wong in [**345**], 1976 and others.

5. Uniformly π-regular semigroups and rings

In Section 2 we seen that the left regular, right regular, intra-regular and group part of a completely π-regular semigroup (ring) coincide, but in the general case, they form a proper subset of the regular part of S. This motivates as to give the following definition: a π-regular semigroup (ring) S is called *uniformly π-regular* if every its regular element is completely regular. Similarly, a π-regular semigroup (ring) whose any regular element is left (resp. right) regular will be called *left* (resp. *right*) *uniformly π-regular*. We will see later that all of these notions coincide.

The subject of this section are some general structural properties of uniformly π-regular semigroups and rings. We will also consider *uniformly π-inverse* (resp. *left uniformly π-inverse*, *right uniformly π-inverse*) *semigroups* (*rings*), defined as uniformly π-regular semigroups (rings) which are also π-inverse (resp. left π-inverse, right π-inverse), and *uniformly periodic semigroups* (*rings*), defined as semigroups (rings) which are both uniformly π-regular and periodic.

5.1. Uniformly π-regular semigroups. One of the celebrated results in Theory of semigroups is the theorem of Tamura from [**314**], 1956, which asserts that any semigroup has a greatest semilattice decomposition, whose components are semilattice indecomposable semigroups. The smallest semilattice congruence on a semigroup, which corresponds to this decomposition, has various characterizations, but two of these characterization, given by Tamura in [**317**], 1972, and Putcha in [**257**], 1974, are especially interesting. Namely, T. Tamura proved that the transitive closure of the relation $\longrightarrow$ on a semigroup S is a quasi-order on S whose symmetric opening, i.e. its natural equivalence, equals the smallest semilattice congruence on S. On the other hand, M. S. Putcha started from the relation $—$ on S, defined as the symmetric opening of $\longrightarrow$, i.e. $— = \longrightarrow \cap (\longrightarrow)^{-1}$, and he proved that the smallest semilattice congruence on S equals the transitive closure of $—$. In the special case when the relations $\longrightarrow$ and $—$ are transitive, we obtain exactly semigroups having a decomposition into a semilattice of Archimedean semigroups, as demonstrated by the following theorem:

Theorem 5.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of Archimedean semigroups;*
(ii) $(\forall a,b \in S)\ a \longrightarrow b \ \Rightarrow\ a^2 \longrightarrow b$;
(iii) $(\forall a,b,c \in S)\ a \longrightarrow b \ \&\ b \longrightarrow c \ \Rightarrow\ a \longrightarrow c$;
(iv) $(\forall a,b,c \in S)\ a \longrightarrow c \ \&\ b \longrightarrow c \ \Rightarrow\ ab \longrightarrow c$;
(v) $(\forall a,b \in S)\ a — b \ \Rightarrow\ a^2 — b$;
(vi) $(\forall a,b,c \in S)\ a — b \ \&\ b — c \ \Rightarrow\ a — c$;
(vii) $(\forall a,b,c \in S)\ a — c \ \&\ b — c \ \Rightarrow\ ab — c$;
(viii) $(\forall a,b \in S)\ a^2 \longrightarrow ab$;
(viii)' $(\forall a,b \in S)\ b^2 \longrightarrow ab$;
(ix) *$\sqrt{A}$ is an ideal (or left ideal, right ideal) of S, for any ideal A of S;*
(x) *$\sqrt{SabS} = \sqrt{SaS} \cap \sqrt{SbS}$, for all $a,b \in S$.*

The first characterization of semilattices of Archimedean semigroups was given by Putcha in [**255**], 1973, who proved the equivalence of conditions (i) and (ii) of the above theorem. The equivalence of the conditions (ii), (iii) and (iv) was proved by Tamura in [**316**], 1972. The condition (ii) is known as the *power property*, (iii) is the transitivity and (iv) is known as the *common multiple property*, or shortly *cm-property* of a relation. The equivalence of the conditions (i), (v), (vi) and (vii) was established by Bogdanović, Ćirić and Popović in [**62**], Ćirić and Bogdanović in [**83**], 1993, showed the equivalence of the conditions (i), (viii), (viii)' and (ix), while Kmeť in [**157**], 1988, proved (i) $\Leftrightarrow$ (ix). The condition (x) is obtained from a more general result given by Ćirić and Bogdanović in [**87**], 1996.

Semilattices of left Archimedean semigroups are characterized by the following theorem:

Theorem 5.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ a \longrightarrow b \ \ \Rightarrow \ \ a \xrightarrow{\ l\ } b$;
(iii) $(\forall a, b \in S)\ a \xrightarrow{\ l\ } ab$;
(iv) *$\sqrt{L}$ is an ideal (or right ideal) of S, for any left ideal L of S;*
(v) *$\sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb}$, for all $a, b \in S$.*

The equivalence (i) $\Leftrightarrow$ (ii) was proved by Putcha in [**258**], 1981, (i) $\Leftrightarrow$ (iii) by Bogdanović in [**34**], 1984, and the equivalence of (i), (iv) and (v) was established by Bogdanović and Ćirić in [**43**], 1992.

By the previous theorem and its dual one obtains:

Theorem 5.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of t-Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ a \longrightarrow b \ \ \Rightarrow \ \ a \xrightarrow{\ t\ } b$;
(iii) $(\forall a, b \in S)\ a \xrightarrow{\ l\ } ab\ \&\ b \xrightarrow{\ r\ } ab$;
(iv) *$\sqrt{B}$ is an ideal of S, for any bi-ideal B of S.*

Supplying the above considered semigroups with the intra-π-regularity we obtain the following two theorems:

Theorem 5.4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of simple semigroups;*
(ii) *S is a semilattice Archimedean semigroups and it is intra π-regular;*
(iii) *S is intra-π-regular and any $\mathcal{J}$-class of S containing an intra-regular element is a subsemigroup;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in S(ba)^n(ab)^nS$.

The equivalence of conditions (i), (ii) and (iii) was given by Putcha in [**255**], 1973.

Theorem 5.5. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left simple semigroups;*
(ii) *S is a semilattice left Archimedean semigroups and it is intra-π-regular;*
(iii) *S is a semilattice left Archimedean semigroups and it is left π-regular;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in S(ab)^na$.

The next theorem, which characterizes semilattices of left completely Archimedean semigroups, was proved by Bogdanović and Ćirić in [**59**].

Theorem 5.6. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left completely Archimedean semigroups;*
(ii) *S is a semilattice of Archimedean semigroups and it is left π-regular;*
(iii) *S is left π-regular and each $\mathcal{L}$-class of S containing a left regular element is a subsemigroup;*
(iv) *S is left π-regular and each $\mathcal{J}$-class of S containing a left regular element is a subsemigroup;*
(v) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in Sa(ab)^n$.

Finally, we go to the uniformly π-regular semigroups. These semigroups are characterized by the following theorem:

Theorem 5.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is uniformly π-regular;*
(ii) *S is left* (*or right*) *uniformly π-regular;*
(iii) *S is a semilattice of completely Archimedean semigroups;*
(iv) *S is a semilattice of Archimedean semigroups and it is completely π-regular;*
(v) *S is a semilattice of left completely Archimedean semigroups and it is right π-regular;*
(vi) *S is a semilattice of left completely Archimedean semigroups and it is π-regular;*
(vii) *S is π-regular and any $\mathcal{L}$-class of S containing an idempotent is a subsemigroup;*
(viii) *S is completely π-regular and any $\mathcal{J}$-class of S containing an idempotent is a subsemigroup;*
(ix) *S is completely π-regular and any $\mathcal{D}$-class of S containing a regular element is a subsemigroup;*
(x) *S is completely π-regular and $\mathbb{A}_2$ and $\mathbb{B}_2$ don't divide S through completely π-regular subsemigroups of S;*
(xi) $(\forall a, b \in S)(\exists n \in \mathbb{N})(ab)^n \in (ab)^n bS(ab)^n$;

The first characterization of semilattices of completely Archimedean semigroups was given by Putcha in [**255**], 1973, who proved that the conditions (iii), (iv) and (viii) are equivalent. The equivalence of the conditions (i), (iii), (ix) and (x) was stated without proofs by Shevrin in [**282**], 1977, and [**284**], 1981, and it was proved in [**285**], 1994. Some of these conditions, and also some other conditions equivalent to the uniform π-regularity of semigroups, were independently found by Veronesi in [**339**], 1984. The conditions (ii), (v), (vi) and (vii) were given by Bogdanović and Ćirić in [**59**], while (xi) is from Bogdanović [**38**], 1987.

Next we give the results obtained by Bogdanović in [**34**], 1984, Bogdanović and Ćirić in [**54**], 1995, and Shevrin in [**286**], 1994.

Theorem 5.8. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of rectangular groups;*

(ii) *S is π-regular and for all $a, x \in S$, $a = axa$ implies $a = ax^2a^2$;*
(iii) *S is uniformly π-regular and any inverse of any idempotent of S is an idempotent;*
(iv) *S is uniformly π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (ef)^{n+1}$;*
(v) *S is completely π-regular and $(ab)^0 = (ab)^0(ba)^0(ab)^0$, for all $a, b \in S$.*

The results collected in the next theorem were also obtained by Bogdanović in [**34**], 1984, Bogdanović and Ćirić in [**54**], 1995, and Shevrin in [**286**], 1994.

Theorem 5.9. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of left groups;*
(ii) *S is a semilattice of left Archimedean semigroups and it is π-regular* (*or right π-regular, completely π-regular*)*;*
(iii) *S is left uniformly π-inverse;*
(iv) *S is π-regular and for all $a, x \in S$, $a = axa$ implies $ax = xa^2x$;*
(v) *S is completely π-regular and $(ab)^0 = (ab)^0(ba)^0$, for all $a, b \in S$.*
(vi) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in (ab)^n S(ba)^n$.

Finally, semilattices of nil-extensions of groups are characterized by the following theorem:

Theorem 5.10. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of groups;*
(ii) *S is a semilattice of t-Archimedean semigroups and it is π-regular* (*or intra-π-regular, left π-regular, right π-regular, completely π-regular*)*;*
(iii) *S is π-regular and for all $a, x \in S$, $a = axa$ implies $ax = xa$;*
(iv) *S is uniformly π-inverse;*
(v) *S is completely π-regular and $(ab)^0 = (ba)^0$, for all $a, b \in S$.*
(vi) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in (ba)^n S(ba)^n$.

The equivalence of the conditions (i) and (iv) was established by Veronesi in [**339**], 1984, of (i), (ii), (iii) and (vi) by Bogdanović in [**34**], 1984, and of (i) $\Leftrightarrow$ (v) was proved by Shevrin in [**286**], 1994, and Bogdanović and Ćirić in [**54**], 1995.

5.2. Uniformly π-regular rings. Uniformly π-regular rings have also a very interesting structure characterization, given by the following theorem:

Theorem 5.11. *The following conditions on a ring R are equivalent:*
(i) *R is uniformly π-regular;*
(ii) *R is π-regular and $\mathrm{Nil}(R)$ is an ideal od $\mathcal{M}R$;*
(iii) *R is π-regular and $\mathrm{Nil}(R)$ is an ideal od R;*
(iv) *R is π-regular and an ideal extension of a nil-ring by a Clifford ring;*
(v) *$\mathcal{M}R$ is a semilattice of completely Archimedean semigroups;*
(vi) *$\mathcal{M}R$ is a semilattice of left* (*or right*) *completely Archimedean semigroups;*
(vii) *$\mathcal{M}R$ is a semilattice of Archimedean semigroups and R is π-regular.*

The equivalence of the conditions (v), (ii) and (iii) was proved by Putcha in [**258**], 1981, for completely π-regular rings, and the same proof was translated to π-regular rings by Ćirić and Bogdanović in [**81**], 1992, where also it was proved that (i), i.e. (v), is equivalent to (iv) and (vii). The equivalence of the conditions (v) and (vi) follows by Theorems 2.8 and 5.7.

Note that an analogue of the equivalence (v) $\Leftrightarrow$ (vii) is not valid in Theory of semigroups. For example, bicyclic semigroups are regular and simple, but these are no uniformly π-regular.

As we will see later, the condition (iv) has a great importance, since it gives a possibility to represent uniformly π-regular rings by Everett's sums.

Problem. Can the equivalence of the conditions (i), (ii), and (iii) of the previous theorem be proved if in (ii) and (iii) we omit the assumption that R is π-regular?

Some special cases of uniformly π-regular rings are also interesting. First we give

Theorem 5.12. *The following conditions on a ring R are equivalent:*

(i) *R is uniformly π-regular and for all $e, f \in E(R)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (ef)^{n+1}$;*
(ii) *R is uniformly π-regular and $(ef)^2 = (ef)^3$, for all $e, f \in E(R)$;*
(iii) *$\mathcal{M}R$ is a semilattice of nil-extensions of rectangular groups.*

Rings whose multiplicative semigroups can be decomposed into a semilattice of nil-extensions of left groups were investigated by Bogdanović and Ćirić in [**44**], 1992, who proved the following theorem:

Theorem 5.13. *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a semilattice of nil-extensions of left groups;*
(ii) *R is left π-inverse;*
(iii) *R is left completely π-inverse;*
(iv) *R is π-regular and $ea = eae$, for any $a \in R$ and $e \in E(R)$;*
(v) *R is π-regular and $E(R)$ is a left regular band.*

The previous theorem and its dual yield the next theorem, proved also by Bogdanović and Ćirić in [**44**], 1992.

Theorem 5.14. *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a semilattice of nil-extensions of groups;*
(ii) *R is π-inverse;*
(iii) *R is completely π-inverse;*
(iv) *R is uniformly π-inverse;*
(v) *R is π-regular and the idempotents of R are central;*
(vi) *R is π-regular and $E(R)$ is a semilattice.*

In the case of completely π-regular rings with an identity, Putcha in [**258**], 1981, showed that the above considered concepts coincide. Namely, he proved the following:

Theorem 5.15. *The following conditions on a completely π-regular ring R with the identity are equivalent:*

(i) $\mathcal{M}R$ *is a semilattice of left Archimedean semigroups;*
(ii) $\mathcal{M}R$ *is a semilattice of right Archimedean semigroups;*
(iii) $\mathcal{M}R$ *is a semilattice of t-Archimedean semigroups;*
(iv) *the idempotents of R are central.*

In the mentioned paper, M. S. Putcha gave an example of a ring that is a semilattice of right Archimedean semigroups, but it is not a semilattice of left Archimedean semigroups.

5.3. Uniformly periodic semigroups and rings. There are examples that the property "being a semilattice of Archimedean semigroups" is not a hereditary property. Semigroups on which this property is hereditary were investigated by Bogdanović, Ćirić and Mitrović in [**60**], 1995, where the following theorem was given:

Theorem 5.16. *The following conditions on a semigroup S are equivalent:*

(i) *any subsemigroup of S is a semilattice of Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ ab \uparrow a^2$;
(ii)' $(\forall a, b \in S)\ ab \uparrow b^2$;
(iii) *S satisfies one of the following variable identities over A_2:*
  (a) $\{(xy)^n = w \mid w \in A_2^* x^2 A_2^* \cup A_2^* x,\ n \in \mathbb{N}\}$;
  (b) $\{(xy)^n = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^*,\ n \in \mathbb{N}\}$;
  (c) $\{(xy)^n x = w \mid w \in A_2^* x^2 A_2^*,\ n \in \mathbb{N}\}$;
  (d) $\{(xy)^n x = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^* \cup A_2^* y,\ n \in \mathbb{N}\}$.

Semigroups in which the property "being uniformly π-regular" is hereditary are exactly the uniformly periodic semigroups. This is demonstrated by the next theorem, proved in the same paper of S. Bogdanović, M. Ćirić and M. Mitrović.

Theorem 5.17. *The following conditions on a semigroup S are equivalent:*

(i) *S is uniformly periodic;*
(ii) *S is a semilattice of nil-extensions of periodic completely simple semigroups;*
(iii) *S is periodic and a semilattice of Archimedean semigroups;*
(iv) *any subsemigroup of S is uniformly π-regular;*
(v) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n = (ab)^n\left((ba)^n(ab)^n\right)^n$;
(v)' $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n = \left((ab)^n(ba)^n(ab)^n\right)^n$;
(vi) *S satisfies one of the following variable identities over A_2:*
  (a) $\{(xy)^n = w \mid w \in A_2^* x^2 A_2^* \cup A_2^* x,\ |w| \neq 2n,\ n \in \mathbb{N}\}$;
  (b) $\{(xy)^n = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^*,\ |w| \neq 2n,\ n \in \mathbb{N}\}$;
  (c) $\{(xy)^n x = w \mid w \in A_2^* x^2 A_2^*,\ |w| \neq 2n+1,\ n \in \mathbb{N}\}$;
  (d) $\{(xy)^n x = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^* \cup A_2^* y,\ |w| \neq 2n+1,\ n \in \mathbb{N}\}$.

The next theorem, which describes the structure of uniformly periodic rings, is due to the authors.

Theorem 5.18. *The following conditions on a ring R are equivalent:*

(i) *R is uniformly periodic;*
(ii) *R is an ideal extension of a nil-ring by a Jacobson's ring;*
(iii) *any subring of R is uniformly π-regular;*
(iv) *any subsemigroup of $\mathcal{M}R$ is uniformly π-regular;*
(v) *$\mathcal{M}R$ is a semilattice of nil-extensions of periodic completely simple semigroups.*

A special type of the above considered rings, namely the rings which are ideal extensions of a nil-ring by a Boolean ring, were studied by Hirano, Tominaga and Yaqub in [**142**], 1988, where the following theorem was proved:

Theorem 5.19. *The following conditions on a ring R are equivalent:*

(i) *R is an ideal extension of a nil-ring by a Boolean ring;*
(ii) $(\forall a \in R)\ a - a^2 \in \mathrm{Nil}(R)$*;*
(iii) *R is $[E(R), \mathrm{Nil}(R)]$-representable;*
(iv) *R is uniquely $[E(R), \mathrm{Nil}(R)]$-representable.*

In the same paper, Y. Hirano, H. Tominaga and A. Yaqub also considered the condition of the form

$$(\#)_n \qquad (\forall a \in R)\ x - x^n \in \mathrm{Nil}(R),$$

where $n \in \mathbb{N}$, $n \geq 2$. By the above theorem, $\mathrm{Nil}(R)$ form an ideal of R, whenever a ring R satisfies $(\#)_2$, but this does not holds for all $n \in \mathbb{N}$. Necessary and sufficient conditions for n, under which $\mathrm{Nil}(R)$ is an ideal of R, for any ring R satisfying $(\#)_n$, are determined by the following theorem, proved also in the above mentioned paper.

Theorem 5.20. *Let $n \in \mathbb{N}$, $n \geq 2$. Then the following conditions are equivalent:*

(i) $\mathrm{Nil}(R)$ *is an ideal of R, for any ring R which satisfies $(\#)_n$;*
(ii) $n \not\equiv 1 \pmod 3$ *and* $n \not\equiv 1 \pmod 8$*;*
(iii) *for each prime p, $n \not\equiv 1 \pmod{p^2 - 1}$;*
(iv) *for each prime p, $M_2(GF(p))$ fails to satisfy $(\#)_n$.*

5.4. Nil-extensions of unions of groups.

The subject of this section are semigroups decomposable into a nil-extension of a union of groups. In other words, these are π-regular semigroups in which the group part form an ideal, and they are one of the most significant special cases of uniformly π-regular semigroups.

Except the mentioned semigroups, here we also consider certain their special types, such as retractive, nilpotent and retractive nilpotent extensions of unions of groups. Recall that we say that a semigroup S is a *retractive extension* of a semigroup K if S is an ideal extension of K and there exists a retraction of S onto K. These are extensions which can be more easily constructed than many other kinds of extensions, and this make important their investigation.

For $n \in \mathbb{N}$, a retractive $(n+1)$-nilpotent extension S of a semigroup K was called by Bogdanović and Milić in [**65**], 1987, an *n-inflation* of K. These authors also gave a general construction for such extensions. It is important to note that 1-inflations are called simply *inflations*, while 2-inflations are also known as *strong inflations*. Inflations of semigroups were first defined and studied by Clifford in [**102**], 1950, and strong inflations by Petrich in [**238**], 1967.

The first theorem which we quote here was proved by Bogdanović and Ćirić in [**41**], 1991, and it describes nil-extensions of regular semigroups.

Theorem 5.21. *A semigroup S is a nil-extension of a regular semigroup if and only if for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^ny \in xa^nySxa^ny$.*

An immediate consequence of the previous theorem is the following:

Theorem 5.22. *A semigroup S is a nil-extension of a union of groups if and only if for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^ny \in xa^nyxSxa^ny$.*

Nil-extensions of semilattices of left groups are characterized similarly:

Theorem 5.23. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of left groups;*
(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^ny \in xa^nySya^nx$;
(iii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^ny \in xSx$.*

The equivalences (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv) are from Bogdanović and Ćirić [**41**], 1991, and [**46**], 1992, respectively.

When we deal with retractive nil-extensions of regular semigroups, the following theorem has a crucial role:

Theorem 5.24. *A semigroup S is a retractive nil-extension of a regular semigroup if and only if it is a subdirect product of a nil-semigroup and a regular semigroup.*

The above theorem was proved by Bogdanović and Ćirić in [**45**], 1992. The same authors in another paper [**46**], 1992, proved the following:

Theorem 5.25. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a union of groups;*
(ii) *S is a subdirect product of a nil-semigroup and a union of groups;*
(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^ny \in x^2Sy^2$.

A very interesting property of retractive nil-extensions of unions of groups was found by Bogdanović and Ćirić in [**41**], 1991, who gave:

Theorem 5.26. *Let a semigroup S be a nil-extension of a union of groups K. Then an arbitrary retraction φ of S onto K has the following representation:*

$$a\varphi = ea \quad \text{if } a \in \sqrt{G_e},\ \text{for } e \in E(S) \qquad (a \in S).$$

In view of the Munn's lemma (Theorem 2.12), if the above condition is fulfilled, then $ae = ea \in G_e$, so also $a\varphi = ae$.

The next theorem was proved by Bogdanović and Ćirić in [**46**], 1992:

Theorem 5.27. *A semigroup S is a retractive nil-extension of a semilattice of left groups if and only if it is π-regular and the following condition holds:*

$$(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y \in x^2 S x.$$

Nil-extensions of Clifford semigroups (semilattices of groups) one considers in the following theorem:

Theorem 5.28. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of groups;*
(ii) *S is a retractive nil-extension of a semilattice of groups;*
(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y \in xa^n y S y a^n x \cap y a^n x S x a^n y$;
(iv) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^n y \in xSx \cap ySy$.*

The equivalence (i) $\Leftrightarrow$ (ii) was proved by Bogdanović and Ćirić in [**41**], 1991. The remaining conditions are derived from Theorem 5.23 and its dual.

Now we pass from nil-extensions to the nilpotent ones. For an arbitrary $n \in \mathbb{N}$, a semigroup S which is an $(n+1)$-nilpotent extension of a regular semigroup (resp. union of groups) can be characterized by a simple condition $S^{n+1} \subseteq \mathrm{Reg}(S)$ (resp. $S^{n+1} \subseteq \mathrm{Gr}(S)$). But, $(n+1)$-nilpotent extensions of semilattices of left groups have a more interesting characterization, given by the following theorem:

Theorem 5.29. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of left groups;*
(ii) *S is π-regular (or right π-regular) and $xS^n = xS^n x$, for any $x \in S$;*
(iii) $(\forall x_1, x_2, \dots, x_{n+1} \in S)\ x_1 x_2 \cdots x_{n+1} \in x_1 x_2 \cdots x_{n+1} S x_1$.

The equivalence of the conditions (i) and (iii) was proved by Bogdanović and Stamenković in [**66**], 1988.

As was proved by Bogdanović and Ćirić in [**45**], 1992, retractive nilpotent extensions of regular semigroups can be also characterized in terms of subdirect products:

Theorem 5.30. *Let $n \in \mathbb{N}$. A semigroup S is an n-inflation of a regular semigroup K if and only if it is a subdirect product of K and an $(n+1)$-nilpotent semigroup.*

Applying this theorem to n-inflations of unions of groups we obtain the following:

Theorem 5.31. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an n-inflation of a union of groups;*
(ii) *S is a subdirect product of a union of groups and an $(n+1)$-nilpotent semigroup;*
(iii) $(\forall x, y \in S)\ xS^{n-1}y = x^2S^ny^2$.

The equivalence of conditions (i) and (ii) was proved by Bogdanović and Milić in [**65**], 1987. In the case $n = 1$ this was shown by Bogdanović in [**37**], 1985.

Next we quote

Theorem 5.32. *Let $n \in \mathbb{N}$. A semigroup S is an n-inflation of a semilattice of left groups if and only if the following condition holds:*

$$(\forall x \in S) xS^n = x^2S^nx.$$

This theorem was proved by Bogdanović and Stamenković in [**66**], 1988 (see also Bogdanović and Ćirić [**46**], 1992), and by Bogdanović in [**38**], 1987, in the case $n = 1$.

This section we finish giving the following theorem:

Theorem 5.33. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of groups;*
(ii) *S is an n-inflation of a semilattice of groups;*
(iii) $(\forall x, y \in S)\ xa^ny \in y^2S^nx$.

These results are due to Bogdanović and Milić [**65**], 1987. Inflations of semilattices of groups were described in a similar way by Bogdanović in [**37**], 1985.

5.5. Nil-extensions of unions of periodic groups. The class of semigroups which are nil-extensions of unions of groups, and certain its subclasses, have very nice characterizations in terms of variable identities, which will be presented in this section. Except the results whose origins we quote explicitly, all remaining results are unpublished results of the authors.

For a given $n \in \mathbb{N}$, $n \geq 3$, at the start of the section we deal with semigroup identities over A_n of the form

$$x_1u(x_2, \dots, x_{n-1})x_n = w(x_1, x_2, \dots, x_n), \tag{1}$$

and the following conditions concerning them:

(A1) for a fixed $i \in \{1, \dots, n\}$, x_i appears once on one side of (1) and at most twice on another side;
(B1) $|w| \neq |u| + 2$;

(C1.1) $x_1 \not|_l\, w$; (C1.2) $h^{(2)}(w) = x_1^2$; (C1.3) $h(w) \neq x_1$;

(D1.1) $x_n \not|_r\, w$; (D1.2) $t^{(2)}(w) = x_n^2$; (D1.3) $t(w) \neq x_n$;

and we also deal with identities of the form

$$x_1 u(x_2, \ldots , x_n) = v(x_1, \ldots , x_{n-1})x_n, \tag{2}$$

and the following conditions concerning them:

(A2) for a fixed $i \in \{1, \ldots , n\}$, x_i appears once on one side of (2) and at most twice on another side;

(B2) $|u| \neq |v|$;

(C2.2) $h^{(2)}(v) = x_1^2$; (C2.3) $h(v) \neq x_1$;

(D2.2) $t^{(2)}(u) = x_n^2$; (D2.3) $t(u) \neq x_n$.

Let us observe that the following implications hold: (C1.2) $\Rightarrow$ (C1.1) & (A.1), (C1.3) $\Rightarrow$ (C1.1), (D1.2) $\Rightarrow$ (D1.1) & (A.1), (D1.3) $\Rightarrow$ (D1.1), (C2.2) $\Rightarrow$ (A2) and (D2.2) $\Rightarrow$ (A2).

The next five theorems are due to the authors:

Theorem 5.34. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a union of periodic groups;*

(ii) $(\forall x, a, y \in S)(\exists m, n \in \mathbb{N})\ xa^n y = (xa^n y)^{m+1}$;

(iii) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (A1), (B1), (C1.1) *and* (D1.1).

Theorem 5.35. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a union of periodic groups;*

(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = x^{n+1}a^n y^{n+1}$;

(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y^{n+1} = x^{n+1}a^n y$;

(iv) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (B1), (C1.2) *and* (D1.2)*;*

(v) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (2) *having the properties* (B.2), (C2.2) *and* (D2.2).

Theorem 5.36. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of periodic left groups;*

(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = xa^n yx^n$;

(iii) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (A1), (B1), (C1.1) *and* (D1.3)

Theorem 5.37. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a semilattice of periodic left groups;*

(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = x^{n+1}a^n yx^n$;

(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n ya^n = x^{n+1}a^n y$;

(iv) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (B1), (C1.2) *and* (D1.3)*;*

(v) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (2) *having the properties* (B2), (C2.2) *and* (D2.3).

Theorem 5.38. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of periodic groups;*
(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = y^n x a^n y x^n$;
(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y^{n+1} a^n = a^n x^{n+1} a^n y$;
(iv) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (A1), (B1), (C1.3) *and* (D1.3);
(v) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (2) *having the properties* (A2), (B2), (C2.3) *and* (D2.3).

For $n \in \mathbb{N}$, now we deal with semigroup identities over A_{n+1} of the form

$$x_1 x_2 \cdots x_{n+1} = w(x_1, x_2, \dots, x_{n+1}), \tag{3}$$

and the following conditions concerning them:

(A3) for a fixed $i \in \{1, \dots, n+1\}$, x_i appears once on one side of (3) and at most twice on another side;
(B3) $|w| \geq n+2$;
(C3.1) $x_1 \underset{l}{\nparallel} w$; (C3.2) $h^{(2)}(w) = x_1^2$; (C3.3) $h(w) \neq x_1$;
(D3.1) $x_{n+1} \underset{r}{\nparallel} w$; (D1.2) $t^{(2)}(w) = x_{n+1}^2$; (D1.3) $t(w) \neq x_{n+1}$;

The next theorems characterize various types of nilpotent extensions of unions of groups.

Theorem 5.39. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a union of periodic groups;*
(ii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1 x_2 \cdots x_{n+1} = (x_1 x_2 \cdots x_{n+1})^{m+1}$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (A3), (B3), (C3.1) *and* (D3.1).

The equivalence (i) $\Leftrightarrow$ (ii) was proved by Bogdanović and Milić in [**65**], 1987, and for $n = 1$ by Bogdanović in [**37**], 1985. A condition similar to (iii) was given by Putcha and Weissglass in [**263**], 1972 (they required that the condition (A3) holds for all $i \in \{1, 2, \dots, n+1\}$).

Theorem 5.40. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an n-inflation of a union of periodic groups;*
(ii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1 x_2 \cdots x_{n+1} = x_1^{m+1} x_2 \cdots x_n x_{n+1}^{m+1}$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (B3), (C3.2) *and* (D3.2).

The equivalence of the conditions (i) and (ii) was established by Bogdanović and Milić in [**65**], 1987, whereas in the case $n = 1$ this was shown by Bogdanović in [**37**], 1985.

Theorem 5.41. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of periodic left groups;*
(ii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1x_2\cdots x_{n+1} = x_1x_2\cdots x_{n+1}x_1^m$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (A3), (B3), (C3.1) *and* (D3.3).

A condition equivalent to (i), and similar to (ii), was given by Bogdanović and Stamenković in [**66**], 1988, and by Bogdanović in [**38**], 1987, for the case $n = 1$. These remarks hold also for the next theorem.

Theorem 5.42. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an n-inflation of a semilattice of periodic left groups;*
(ii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1x_2\cdots x_{n+1} = x_1^{m+1}x_2\cdots x_{n+1}x_1^m$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (B3), (C3.2) *and* (D3.3).

Theorem 5.43. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of periodic groups;*
(ii) *S is an n-inflation of a semilattice of periodic groups;*
(iii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1x_2\cdots x_{n+1} = x_{n+1}^m x_1\cdots x_{n+1}x_1^m$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (A3), (B3), (C3.3) *and* (D3.3).

The equivalence (i) $\Leftrightarrow$ (iii) was proved by Bogdanović and Milić in [**65**], 1987, while (i) $\Leftrightarrow$ (iv) was shown by Putcha and Weissglass in [**263**], 1972. The related results concerning the case $n = 1$ were given by Bogdanović in [**37**], 1985, and Putcha and Weissglass in [**262**], 1971. The equivalence of the conditions (i) and (ii) was obtained as a consequence of Theorem 5.28.

The theorems characterizing nilpotent and nil-extensions of bands, left regular bands and semilattices, and their retractive analogues, are very similar to the previous ones, so they will be omitted. We only note that the variable identities describing these semigroups consist of the corresponding identities from the above theorems, having an additional property:

(A1–3)* for a fixed $i \in \{1, \dots, x_n\}$ (resp. $i \in \{1, \dots, x_n\}$, $i \in \{1, \dots, x_{n+1}\}$), x_i appears once on one side of (1) (resp. (2), (3)), and exactly twice on another side.

This condition forces all subgroups of a semigroup to be one-element.

5.6. Direct sums of nil-rings and Clifford rings. In Section 5.2 we have seen that the set of nilpotents of a π-regular ring is a ring ideal if and only if it is a semigroup ideal. Here we show that this property also holds for the group part of such a ring, i.e. that the group part of a π-regular ring is a ring ideal if and only if it is a semigroup ideal. In this case we get a decomposition of this ring into a direct sum of a nil-ring and a Clifford ring, as it is demonstrated by the following theorem:

Theorem 5.44. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a nil-ring and a Clifford ring;*
(ii) *R is a subdirect sum of a nil-ring and a Clifford ring;*
(iii) *R is a strong extension of a nil-ring by a Clifford ring;*
(iv) *R is uniquely $(\mathrm{Gr}(R), \mathrm{Nil}(R))$-representable;*
(v) *R is π-regular and uniquely $(\mathrm{LReg}(R), \mathrm{Nil}(R))$-representable;*
(vi) *R is π-regular and $E(R)$ is contained in a reduced ideal of R;*
(vii) *$\mathcal{M}R$ is a nil-extension of a completely regular* (*or a Clifford*) *semigroup;*
(viii) *$\mathcal{M}R$ is a retractive nil-extension of a completely regular* (*or a Clifford*) *semigroup;*
(ix) *$\mathcal{M}R$ is a subdirect product of a nil-semigroup and a completely regular* (*or a Clifford*) *semigroup;*
(x) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a completely regular* (*or a Clifford*) *semigroup.*

The equivalence of conditions (i), (v) and (vi) was proved by Hirano and Tominaga in [**141**], 1985, and of (i) and (ii) by Bell and Tominaga in [**23**], 1986, Tominaga [**332**]. For some related results see also Tominaga [**331**]. Ćirić and Bogdanović in [**80**], 1990, showed that the conditions (iii), (vii) and (viii) are equivalent, and in [**90**], 1996, they established the equivalence of the conditions (i), (vii) and (x). The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (ix) are obvious, while (ix) $\Rightarrow$ (viii) is an immediate consequence of Theorem 5.24.

For some related results see also Bell and Yaqub [**24**], 1987, and Abu-Khuzam and Yaqub [**4**], 1985. Certain more general decompositions can be found in Hirano and Tominaga [**141**], 1985, and Bell and Tominaga [**23**], 1986.

On the other hand, a special case of the above decompositions are decompositions into a direct sum of a nil-ring and a Jacobson ring. The results concerning these decompositions are collected in the following theorem:

Theorem 5.45. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a nil-ring and a Jacobson ring;*
(ii) *R is a subdirect sum of a nil-ring and a Jacobson ring;*
(iii) *R is uniquely $(P(R), \mathrm{Nil}(R))$-representable;*
(iv) *$E(R) \cdot N_2(R) = N_2(R) \cdot E(R) = 0$ and R is $[P(R), \mathrm{Nil}(R)]$-representable;*
(v) *R is periodic and $E(R)$ is contained in a reduced ideal of R;*
(vi) *$\mathcal{M}R$ is a nil-extension of a union* (*or a semilattice*) *of periodic groups;*
(vii) *$\mathcal{M}R$ is a retractive nil-extension of a union* (*or a semilattice*) *of periodic groups;*
(viii) *$\mathcal{M}R$ is a subdirect product of a nil-semigroup and a union* (*or a semilattice*) *of periodic groups;*
(ix) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a union* (*or a semilattice*) *of periodic groups.*

The conditions (i), (iii) and (iv) are equivalent by theorems proved by Bell and Tominaga in [**23**], 1986, and Hirano, Tominaga and Yaqub in [**142**], 1988, although (i) $\Leftrightarrow$ (iii) was proved by Bell in [**20**], 1985, and Hirano and Tominaga in [**141**], 1985, under the assumption that R is periodic. The condition (v) is assumed from Hirano and Tominaga [**141**], 1985. The equivalence of the conditions (i), (vii), (viii) and (ix) was established by Ćirić and Bogdanović in [**90**], 1996.

By the next theorem we describe direct sums of nil-rings and Boolean rings.

Theorem 5.46. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a nil-ring and a Boolean ring;*
(ii) *R is a subdirect sum of a nil-ring and a Boolean ring;*
(iii) *R is a strong extension of a nil-ring by a Boolean ring;*
(iv) *$E(R)\cdot N_2(R) = N_2(R)\cdot E(R) = 0$ and R satisfies one of the conditions of Theorem 5.19;*
(v) *$E(R)\cdot N_2(R) = N_2(R)\cdot E(R) = 0$ and R is $(E(R), \mathrm{Nil}(R))$-representable;*
(vi) *$E(R)\cdot N_2(R) = N_2(R)\cdot E(R) = 0$ and R is uniquely $(E(R), \mathrm{Nil}(R))$-representable;*
(vii) *$\mathcal{M}R$ is a nil-extension of a band (or a semilattice);*
(viii) *$\mathcal{M}R$ is a retractive nil-extension of a band (or a semilattice);*
(ix) *$\mathcal{M}R$ is a subdirect product of a nil-semigroup and a band (or a semilattice);*
(x) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a band (or a semilattice).*

Hirano, Tominaga and Yaqub in [**142**], 1988, proved that (i), (iv), (v) and (vi) are equivalent. The remaining conditions were given by Ćirić and Bogdanović in [**80**], 1990, and [**90**], 1996.

In the rest of the section we present the results characterizing direct sums of nilpotent rings and of Clifford, Jacobson and Boolean rings.

Theorem 5.47. *Let $n \in \mathbb{N}$. Then the following conditions on a ring R are equivalent:*

(i) *R is a direct sum of an $(n+1)$-nilpotent ring and a Clifford ring;*
(ii) *R is a subdirect sum of an $(n+1)$-nilpotent ring and a Clifford ring;*
(iii) $(\forall a \in R)\ aR^n = aR^n a$;
(iv) $(\forall a \in R)\ aR^n \subseteq R^n a^2$;
(v) $(\forall a \in R)\ aR^n = a^2R^n \ \&\ R^n a = R^n a^2$;
(vi) $R^{n+1} \subseteq \mathrm{LReg}(R)\ \big(\text{or } R^{n+1} \subseteq \mathrm{RReg}(R)\big)$;
(vii) *$\mathcal{M}R$ is an $(n+1)$-nilpotent extension of a completely regular (or a Clifford) semigroup;*
(viii) *$\mathcal{M}R$ is an n-inflation of a completely regular (or a Clifford) semigroup;*
(ix) *$\mathcal{M}R$ is a subdirect product of an $(n+1)$-nilpotent semigroup and a completely regular (or a Clifford) semigroup;*
(x) *$\mathcal{M}R$ is a direct product of an $(n+1)$-nilpotent semigroup and a completely regular (or a Clifford) semigroup.*

The conditions (iii) and (iv) are equivalent to their left-right analogues.

The equivalence of the conditions (i), (iii), (iv) and (v) was proved by Chiba and Tominaga in [**75**], 1976, the condition (vi) is assumed from Komatsu and Tominaga [**138**], 1989, and the remaining conditions are from Ćirić and Bogdanović [**80**], 1990, and [**90**], 1996. Note that the assertion (i) $\Leftrightarrow$ (iii) is a consequence of Theorems 2.8 and 5.29.

Theorem 5.48. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a null-ring and a Clifford ring;*
(ii) *R is a subdirect sum of a null-ring and a Clifford ring;*
(iii) $(\forall a \in R)\ aR = aRa$;
(iv) $(\forall a \in R)\ aR \subseteq Ra^2$;
(v) $(\forall a \in R)\ aR = a^2R\ \&\ Ra = Ra^2$;
(vi) *$\mathcal{M}R$ is a null-extension of a completely regular* (*or a Clifford*) *semigroup;*
(viii) *$\mathcal{M}R$ is an inflation of a completely regular* (*or a Clifford*) *semigroup;*
(ix) *$\mathcal{M}R$ is a subdirect product of a null-semigroup and a completely regular* (*or a Clifford*) *semigroup;*
(x) *$\mathcal{M}R$ is a direct product of a null-semigroup and a completely regular* (*or a Clifford*) *semigroup.*

The conditions (ii) *and* (iv) *are equivalent to their left-right analogues.*

Rings satisfying (iii) were first studied by Szász in [**309**], 1972, and they are known as *P_1-rings*. The equivalence of the conditions (i) and (iii) was established by Ligh and Utumi in [**301**], 1974, and of (i), (iv) and (v) by Chiba and Tominaga in [**74**], 1975.

Theorem 5.49. *Let $n \in \mathbb{N}$. Then the following conditions on a ring R are equivalent:*

(i) *R is a direct sum of an $(n+1)$-nilpotent ring and a Jacobson ring;*
(ii) *R is a subdirect sum of an $(n+1)$-nilpotent ring and a Jacobson ring;*
(iii) $R^{n+1} \subseteq P(R)$;
(iv) *$\mathcal{M}R$ satisfies a variable identity consisting of all identities of the form:*

$$x_1x_2\cdots x_{n+1} = (x_1x_2\cdots x_{n+1})^2u,$$

with $u \in A_{n+1}^+$;
(v) *$\mathcal{M}R$ is an $(n+1)$-nilpotent extension of a union* (*or a semilattice*) *of periodic groups;*
(viii) *$\mathcal{M}R$ is an n-inflation of a union* (*or a semilattice*) *of periodic groups;*
(ix) *$\mathcal{M}R$ is a subdirect product of an $(n+1)$-nilpotent semigroup and a union* (*or a semilattice*) *of periodic groups;*
(x) *$\mathcal{M}R$ is a direct product of an $(n+1)$-nilpotent semigroup and a union* (*or a semilattice*) *of periodic groups.*

Characterizations of direct sums of $(n+1)$-nilpotent rings and Jacobson rings through the conditions (iii) and (iv) were given by H. Komatsu and H. Tominaga, while the remaining conditions are due to the first two authors of this paper.

Theorem 5.50. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a null-ring and a Jacobson ring;*
(ii) *R is a subdirect sum of a null-ring and a Jacobson ring;*
(iii) $(\forall a, b \in R)(\exists p(x,y) \in \mathbb{Z}\langle x,y\rangle)\ ab = (ab)^2 p(a,b)$*;*
(iv) $(\forall a, b \in R)(\exists p(x,y) \in \mathbb{Z}\langle x,y\rangle)\ ab = (ba)^2 p(a,b)$*;*
(v) *$\mathcal{M}R$ satisfies a variable identity consisting of all identities of the form $xy = (xy)^{m+1}$, with $m \in \mathbb{N}$;*
(vi) *$\mathcal{M}R$ is a null-extension of a union (or a semilattice) of periodic groups;*
(vii) *$\mathcal{M}R$ is an inflation of a union (or a semilattice) of periodic groups;*
(viii) *$\mathcal{M}R$ is a subdirect product of a null-semigroup and a union (or a semilattice) of periodic groups;*
(ix) *$\mathcal{M}R$ is a direct product of a null-semigroup and a union (or a semilattice) of periodic groups.*

The equivalence (i) $\Leftrightarrow$ (v) was proved by Ligh and Luh in [**200**], 1989, and (iii) and (iv) are assumed from Bell and Ligh [**22**], 1989.

Note that the above considered rings are commutative. An elementary proof of the commutativity of rings satisfying the condition (v) was given by Ó Searcóid and Mac Hale in [**232**], 1986.

Note that all direct sums of nil-, nilpotent and null-rings and Jacobson rings considered above can be characterized in terms of variable identities, using the semigroup-theoretical results presented in the previous section. The next three theorems, which were proved by Bell in [**18**], 1977, follow immediately from such obtained characterizations.

Theorem 5.51. *Let R be a ring satisfying one of the following variable identities over A_2:*

(a) $\{xy = w \mid |x|_w \geq 2,\ |y|_w \geq 2\}$*;*
(b) $\{xy = w \mid w = yx^n,\ n \in \mathbb{N}, n \geq 2\}$*;*
(c) $\{xy = w \mid w = y^n x,\ n \in \mathbb{N}, n \geq 2\}$*;*
(d) $\{xy = w \mid |y|_w = 0,\ |w| \geq 3\}$*;*
(e) $\{xy = w \mid |x|_w = 0,\ |w| \geq 3\}$*;*
(f) $\{xy = w \mid w = x^m y x^n,\ m, n \in \mathbb{N}\}$*;*
(g) $\{xy = w \mid w = y^m x y^n,\ m, n \in \mathbb{N}\}$*.*

Then R is commutative.

Theorem 5.52. *If a periodic ring R satisfies a variable identity $xy = w(x,y)$, with $w = yx$, or $h(w) = y$ and $|x|_w \geq 2$, then R is commutative.*

Theorem 5.53. *If a ring R satisfies a variable identity $xy = w(x,y)$, with $h(w) = y$ and $|x|_w \geq 2$, then R is commutative.*

6. Semigroups and rings satisfying certain semigroup identities

There are many semigroup identities for which it was observed that they induce certain structural properties on semigroups on which they are satisfied. But, the general problem of finding all semigroup identities inducing a given structural property was first stated by Clarke in [**78**], 1981, and in a more general form in the Ph. D. thesis of Ćirić [**79**], 1991, and in the paper of Ćirić and Bogdanović [**83**], 1993. This problem was formulated in the following way:

(P1) for a given class $\mathcal{X}$ of semigroups, find all semigroup identities $u=v$ having the property $[u=v]\subseteq\mathcal{X}$.

It was also stated one similar problem:

(P2) for given classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of semigroups, find all semigroup identities $u=v$ having the property $[u=v]\cap\mathcal{X}_1\subseteq\mathcal{X}_2$.

Identities having the property $[u=v]\subseteq\mathcal{X}$ are called $\mathcal{X}$*-identities*, and identities having the property $[u=v]\cap\mathcal{X}_1\subseteq\mathcal{X}_2$ are called $\mathcal{X}_1\triangleright\mathcal{X}_2$*-identities*.

In other words, (P1) is the problem of finding all identities having the property that every semigroup satisfying them must be in $\mathcal{X}$, and (P2) is the problem of finding all identities having the property that every semigroup from $\mathcal{X}_1$ satisfying them must be in $\mathcal{X}_2$. Problems of this type were treated only in the mentioned papers of Clarke, Ćirić and Bogdanović, and also by Ćirić and Bogdanović in [**84**], 1994, and [**88**], 1996. The results obtained in these papers, which characterize all identities that induce decompositions of semigroups into a semilattice of Archimedean semigroups and nil-extensions into a union of groups, will be presented in Sections 1 and 2. In Section 3 we show how these results can be applied in Theory of rings.

As was proved by Chrislock in [**77**], 1969, any semigroup which satisfies a heterotype identity is a nil-extensions of a periodic completely simple semigroup, and hence, any ring satisfying a heterotype semigroup identity is a nil-ring. Therefore, studying of heterotype semigroup identities is not so interesting, and in this section we aim our attention only to homotype semigroup identities. Our topic under question will be identities of the form

$$u(x_1,x_2,\dots,x_n)=v(x_1,x_2,\dots,x_n), \tag{1}$$

where $u,v\in A_n^+$ and $c(u)=c(v)=A_n$, $n\in\mathbb{N}, n\geq 2$. We will also treat a particular case

$$u(x,y)=v(x,y), \tag{2}$$

where $u,v\in A_2^+$ and $c(u)=c(v)=A_2$.

Before we give the results promised above, we introduce the following notations:

Notation	Class of semigroups	Notation	Class of semigroups
$\mathcal{A}$	*Archimedean*	$\mathcal{CA}$	*completely Archimedean*
$\mathcal{LA}$	*left Archimedean*	$\mathcal{LG}$	*left groups*
$\mathcal{TA}$	*t-Archimedean*	$\mathcal{G}$	*groups*
$\mathcal{S}$	*semilattices*	$\mathcal{N}$	*nil-semigroups*
$\pi\mathcal{R}$	π*-regular*	$\mathcal{N}_k$	$k+1$*-nilpotent*
$\mathcal{CS}$	*completely simple*	$\mathcal{UG}$	*unions of groups*
$\mathcal{M}\times\mathcal{G}$	*rectangular groups*		

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be classes of semigroups. By $\mathcal{X}_1 \circ \mathcal{X}_2$ we denote the *Maljcev's product* of classes $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ρ such that S/ρ is in $\mathcal{X}_2$ and every ρ-class which is a subsemigroup is in $\mathcal{X}_1$. This product was introduced by Mal'cev in [**206**], 1967. The related decomposition is called an $\mathcal{X}_1 \circ \mathcal{X}_2$-*decomposition.* It is clear that $\mathcal{X} \circ \mathcal{S}$ is the class of all semilattices of semigroups from the class $\mathcal{X}$. If $\mathcal{X}_2$ is a subclass of the class $\mathcal{N}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is a class of all semigroups which are ideal extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$. Also, in such a case, by $\mathcal{X}_1 \circledast \mathcal{X}_2$ we denote a class of all semigroups which are retract extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

6.1. On $\mathcal{A}\circ\mathcal{S}$-identities. Various types of $\mathcal{A}\circ\mathcal{S}$-identities have been investigated by many authors. The commutativity identity $xy = yx$ is an identity for which it has been first proved that it is an $\mathcal{A}\circ\mathcal{S}$-identity. This was done by Tamura and Kimura in [**319**], 1954. After that, the same property was established by Chrislock in [**76**], 1969, for the *medial identity*: $x_1x_2x_3x_4 = x_1x_3x_2x_4$, by Tamura and Shafer in [**321**], 1972, Tamura and Nordahl in [**320**], 1972, and Nordahl in [**225**], 1974, for the *exponential identity*: $(xy)^n = x^ny^n$, $n \in \mathbb{N}$, $n \geq 2$, by Schutzenberger in [**277**], 1976, for the identity $(xy)^n = \big((xy)^n(yx)^n(xy)^n\big)^n$, $n \in \mathbb{N}$, by Sapir and Suhanov in [**275**], 1985, for the identity $(xy)^m = \big((xy)^m(yx)^m\big)^n(xy)^m$, $m, n \in \mathbb{N}$, and identities of the form $x_1x_2\cdots x_{n+1} = w(x_1, x_2, \dots , x_{n+1})$, $n \in \mathbb{N}$, etc. But, the first general characterization of all $\mathcal{A}\circ\mathcal{S}$-identities was given by Ćirić and Bogdanović in [**83**], 1993, who proved the following theorem:

Theorem 6.1. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on the semigroup* $\mathbb{B}_2$*;*
(iii) *there exists a homomorphism* $\varphi : A_n^+ \to A_2^+$ *and a permutation* π *of a set* $\{u, v\}$ *such that one of the following conditions hold:*
  (A1) $(u\pi)\varphi \in (xy)^+$ *and* $(v\pi)\varphi \notin (xy)^+$*;*
  (A2) $(u\pi)\varphi \in (xy)^+x$ *and* $(v\pi)\varphi \notin (xy)^+x$*;*
(iv) *there exists* $k \in \mathbb{N}$ *and* $w \in A_2^*x^2A_2^* \cup A_2^*y^2A_2^*$ *such that*

$$[u = v] \subseteq [(xy)^k = w].$$

One description of all identities which are satisfied on the semigroup $\mathbb{B}_2$ was given by Mashevitskiĭ in [**208**], 1979, but it is quite complicated.

Using the above theorem, for many other significant semigroup identities it can be proved that they are $\mathcal{A}\circ\mathcal{S}$-identities. For example, this can be proved for *permutation identities*, by which we mean identities of the form $x_1x_2\cdots x_n = x_{1\sigma}x_{2\sigma}\cdots x_{n\sigma}$, where σ is a non-identical permutation of the set $\{1, 2, \dots , n\}$, for *quasi-permutation identities*, which have the form

$$x_1\cdots x_{k-1}yx_k\cdots x_n = x_{1\sigma}\cdots x_{(l-1)\sigma}y^2x_{l\sigma}\cdots x_{n\sigma},$$

for some permutation σ of the set $\{1,2,\dots,n\}$ and some $k,l\in\{2,\dots,n\}$, and other.

In the mentioned paper of Ćirić and Bogdanović [**83**], from 1993, the authors also investigated some special types of $\mathcal{A}\circ\mathcal{S}$-identities, and all theorems from 6.4 to 6.9 were proved in this paper.

The next theorem says that the set of all $\mathcal{A}\circ\mathcal{S}$-identities coincides with the set of all identities which forces all π-regular semigroups to be uniformly π-regular.

Theorem 6.2. *The identity* (1) *is a* $\pi\mathcal{R}\triangleright\mathcal{CA}\circ\mathcal{S}$*-identity if and only if* (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity.*

The following theorem, which is a consequence of the previous two theorems, give an answer to one problem stated by Shevrin and Suhanov in [**288**], 1989, concerning semigroup varieties consisting of semilattices of Archimedean semigroups.

Theorem 6.3. *Let* $\mathcal{X}$ *be a variety of semigroups. Then the following conditions are equivalent:*

(i) $\mathcal{X}\subseteq\mathcal{A}\circ\mathcal{S}$;
(ii) $\mathcal{X}$ *does not contain the semigroup* $\mathbb{B}_2$;
(iii) *any regular semigroup from* $\mathcal{X}$ *is completely regular;*
(iv) *any completely 0-simple semigroup from* $\mathcal{X}$ *has no zero divisors;*
(v) *in any semigroup with zero from* $\mathcal{X}$ *the set of all nilpotents is a subsemigroup;*
(vi) *in any semigroup with zero from* $\mathcal{X}$ *the set of all nilpotents is an ideal.*

More information on semigroup varieties contained in $\mathcal{A}\circ\mathcal{S}$ can be found in Schutzenberger [**277**], 1976, Sapir and Suhanov [**275**], 1985, Shevrin and Volkov [**287**], 1985, and Shevrin and Suhanov [**288**], 1989.

The next two theorems characterize identities which induce decompositions of π-regular semigroups into a semilattice of left Archimedean semigroups and into a semilattice of t-Archimedean semigroups.

Theorem 6.4. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{LG}\circ\mathcal{N})\circ\mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$ *and* $\mathbb{R}_2$;
(iii) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity and* $t(u)\neq t(v)$.

Theorem 6.5. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{G}\circ\mathcal{N})\circ\mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{R}_2$ *and* $\mathbb{L}_2$;
(iii) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity,* $h(u)\neq h(v)$ *and* $t(u)\neq t(v)$.

Using the previous theorems, it can be proved that the identities of the form $x_1\cdots x_nx_{m+1}\cdots x_{m+n}=x_{m+1}\cdots x_{m+n}x_1\cdots x_n$, called the (m,n)-*commutativity identities*, are $\mathcal{TA}\circ\mathcal{S}$-identities. These identities were intensively studied by Babcsanyi in [**15**], 1991, Babcsanyi and Nagy in [**16**], 1993, Lajos in [**185**], 1990, and

[**186**], 1991, and by Nagy in [**221**], 1992, and [**222**], [**223**], 1993. The assertion that these identities are $\mathcal{TA} \circ \mathcal{S}$-identities was proved by Lajos in [**185**], 1990.

The next two theorems were also given by Ćirić and Bogdanović in [**83**], 1993:

Theorem 6.6. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{CS} \circledast \mathcal{N}) \circ \mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_{3,1}$*;*
(iii) (1) *is an* $\mathcal{A} \circ \mathcal{S}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t^{(2)}(u) \neq t^{(2)}(v)$.

Theorem 6.7. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{LG} \circledast \mathcal{N}) \circ \mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_2$*;*
(iii) (1) *is an* $\mathcal{A} \circ \mathcal{S}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t(u) \neq t(v)$.

Identities over the two-element alphabet were systematically investigated by Ćirić and Bogdanović in [**88**], 1996. In this paper it was shown that $\mathcal{A} \circ \mathcal{S}$-identities over the two-element alphabet have a more simple characterization, given by the following theorem:

Theorem 6.8. *The identity* (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity if and only if it is p-equivalent to one of the following identities:*

(B1) $xy = w(x, y)$, *where* $w \neq xy$;
(B2) $(xy)^k = w(x, y)$, *where* $k \in \mathbb{N}$, $k \geq 2$ *and* $w \notin (xy)^+$;
(B3) $(xy)^k x = w(x, y)$, *where* $k \in \mathbb{N}$ *and* $w \notin (xy)^+ x$;
(B4) $xy^k = w(x, y)$, *where* $k \in \mathbb{N}$, $k \geq 2$ *and* $w \notin xy^+$;
(B5) $x^k y = w(x, y)$, *where* $k \in \mathbb{N}$, $k \geq 2$ *and* $w \notin x^+ y$.

In the same paper the authors proved the following two theorems:

Theorem 6.9. *The following conditions for the identity* (2) *are equivalent:*

(i) (2) *is a* $\mathcal{LA} \circ \mathcal{S}$*-identity;*
(ii) (2) *is not satisfied on semigroups* $\mathbb{B}_2$ *and* $\mathbb{R}_2$*;*
(iii) (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity and* $t(u) \neq t(v)$.

Theorem 6.10. *The following conditions for the identity* (2) *are equivalent:*

(i) (2) *is a* $\mathcal{TA} \circ \mathcal{S}$*-identity;*
(ii) (2) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{R}_2$ *and* $\mathbb{L}_2$*;*
(iii) (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity,* $t(u) \neq t(v)$ *and* $h(u) \neq h(v)$.

Note that there are not any characterizations of $\mathcal{LA} \circ \mathcal{S}$-identities and $\mathcal{TA} \circ \mathcal{S}$-identities over the alphabet with more than two letters.

The next two theorems were also proved in [**88**]:

Theorem 6.11. *The identity* (2) *is a* $\mathcal{CS} \triangleright \mathcal{M} \times \mathcal{G}$*-identity if and only if one of the following conditions holds:*

(C1) $h(u) \neq h(v)$ *or* $t(u) \neq t(v)$;

(C2) (1) *is p-equivalent to some identity of the form*

$$x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots x^{m_h}y^{n_h} = x^{k_1}y^{l_1}x^{k_2}y^{l_2}\cdots x^{k_s}y^{l_s}$$

$m_i, n_i, k_j, l_j \in \mathbb{N}$, *with* $\gcd(p_x, p_y, h-s) = 1$, *where* $p_x = \Sigma_{i=1}^{h} m_i - \Sigma_{j=1}^{s} k_j$ *and* $p_y = \Sigma_{i=1}^{h} n_i - \Sigma_{j=1}^{s} l_j$.

(C3) (1) *is p-equivalent to some identity of the form*

$$x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots x^{m_h}y^{n_h}x^{m_{h+1}} = x^{k_1}y^{l_1}x^{k_2}y^{l_2}\cdots x^{k_s}y^{l_s}x^{k_{s+1}}$$

$m_i, n_i, k_j, l_j \in \mathbb{N}$, *with* $\gcd(p_x, p_y, h-s) = 1$, *where* $p_x = \Sigma_{i=1}^{h+1} m_i - \Sigma_{j=1}^{s+1} k_j$ *and* $p_y = \Sigma_{i=1}^{h} n_i - \Sigma_{j=1}^{s} l_j$.

Theorem 6.12. *The identity* (2) *is a* $\pi\mathcal{R} \triangleright (\mathcal{M} \times \mathcal{G} \circ \mathcal{N}) \circ \mathcal{S}$*-identity if and only if* (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity and a* $\mathcal{CS} \triangleright \mathcal{M} \times \mathcal{G}$*-identity.*

6.2. On $\mathcal{UG} \circ \mathcal{N}$-identities.

There are many papers in which some types of $\mathcal{UG} \circ \mathcal{N}$-identities have been investigated. The identity $xy = y^m x^m$, for $m, n \in \mathbb{N}$, $m + n \geq 3$, was studied by Tully in [**334**], the identity $xy = (xy)^m$, $m \in \mathbb{N}$, $m \geq 2$, by Gerhard in [**127**], 1977, the *distributive identities* $xyz = xyxz$ and $xyz = xzyz$ by Petrich in [**239**], 1969, etc. Various $\mathcal{UG} \circ \mathcal{N}$-identities of the form $x_1x_2\cdots x_{n+1} = w(x_1, x_2, \dots, x_{n+1})$ were investigated by Bogdanović and Stamenković in [**66**], 1988, Ćirić and Bogdanović in [**80**], 1990, Tishchenko in [**328**], 1991, and others. Tamura in [**310**], 1969, stated the general problem of describing structure of semigroups satisfying an identity of the form $xy = w(x, y)$, where $|w| \geq 3$, known as *Tamura's problem*. Various cases appearing in this problem were treated in the mentioned paper of Tamura, and also by Lee in [**196**], 1973, Clarke in [**78**], 1981, and Bogdanović in [**38**], 1987. Complete solutions of all possible cases of the Tamura's problem were given by Ćirić and Bogdanović in [**88**], 1996. More information on problems of Tamura's type can be found in another survey paper of Bogdanović and Ćirić [**49**], 1993.

A complete description of all $\mathcal{UG} \circ \mathcal{N}$-identities was given by Ćirić and Bogdanović in [**84**], 1994, by the following theorem:

Theorem 6.13. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity;*

(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$ *and* $\mathbb{C}_{2,1}$*;*

(iii) $\Pi(u) \neq \Pi(v)$ *and* (1) *is p-equivalent to some identity of one of the following forms:*

(*D*1) $$x_1u'(x_2, \dots, x_n) = v'(x_1, \dots, x_{n-1})x_n,$$
where $x_1 \nmid\mid_l v'$ *and* $x_n \nmid\mid_r u'$*;*

(*D*2) $$x_1u'x_n = v',$$
where $x_1, x_n \nmid u', x_1 \nmid\mid_l v'$ *and* $x_n \nmid\mid_r v'$*;*

(*D*3) $$x_1u'(x_2, \dots, x_n) = v'(x_2, \dots, x_n)x_1.$$

In the same paper the next two theorems were obtained:

Theorem 6.14. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $(\mathcal{LG} \circ \mathcal{S}) \circ \mathcal{N}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$ *and* $\mathbb{R}_2$*;*
(iii) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity and* $t(u) \neq t(v)$.

Theorem 6.15. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $(\mathcal{G} \circ \mathcal{S}) \circ \mathcal{N}$*-identity;*
(ii) (1) *is a* $(\mathcal{G} \circ \mathcal{S}) \circledast \mathcal{N}$*-identity;*
(iii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{R}_2$ *and* $\mathbb{L}_2$*;*
(iv) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $t(u) \neq t(v)$ *and* $h(u) \neq h(v)$.

Identities which induce retractive nil-extensions of a union of groups were characterized in the following way:

Theorem 6.16. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circledast \mathcal{N}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_{3,1}$*;*
(iii) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t^{(2)}(u) \neq t^{(2)}(v)$.

Theorem 6.17. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $(\mathcal{LG} \circ \mathcal{S}) \circledast \mathcal{N}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_2$*;*
(iii) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t(u) \neq t(v)$.

Further we consider identities which induce nilpotent and retractive nilpotent extensions of a union of groups. These identities are described by the next two theorems which are also due to Ćirić and Bogdanović [**84**], 1994.

Theorem 6.18. *Let* $k \in \mathbb{N}$*. Then the following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circ \mathcal{N}_k$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{D}_N$ *and* $\mathbb{N}_{k+1}$*;*
(iii) $n \leq k+1$ *and* (1) *is p-equivalent to some identity of the form*

$$x_1 x_2 \dots x_n = w,$$

where $|w| \geq n+1, x_1 \underset{l}{\nmid} w$ *and* $x_n \underset{l}{\nmid} w$.

Theorem 6.19. *Let* $k \in \mathbb{N}$*. Then the following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circledast \mathcal{N}_k$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{L}_{3,1}$, $\mathbb{R}_{3,1}$, $\mathbb{D}_N$ *and* $\mathbb{N}_{k+1}$*;*
(iii) (1) *is p-equivalent to some identity of the form*

$$x_1 x_2 \dots x_n = w,$$

where $|w| \geq n+1$, $h^{(2)}(u) \neq x_1 x_2$ *and* $t^{(2)}(v) \neq x_{n-1} x_n$.

Applying the above results to the case of identities over the two-element alphabet, Ćirić and Bogdanović obtained in [**88**], 1996, the following two theorems:

Theorem 6.20. *The identity* (2) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity if and only if it is p-equivalent to an identity of one of the following forms:*

(F1) $xy = w(x, y)$, *where* $w \neq yx$, $w \notin xy^+$ *and* $w \notin x^+y$;
(F2) $xy^m = x^n y$, *where* $m, n \in \mathbb{N}$, $m, n \geq 2$.

Theorem 6.21. *The identity* (2) *is a* $\mathcal{UG} \circledast \mathcal{N}$*-identity if and only if it is p-equivalent to an identity of one of the following forms:*

(G1) $xy = w$, *where* $w \in A_2^+$, $|w| \geq 3$, $h^{(2)}(w) \neq xy$ *and* $t^{(2)}(w) \neq xy$;
(G2) $xy^m = x^n y$, *where* $m, n \in \mathbb{N}$, $m, n \geq 2$.

Finally, a consequence of the previous theorem is the following theorem proved by Clarke in [**78**], 1981:

Theorem 6.22. *A semigroup identity determines a variety of inflations of unions of groups if and only if this identity has one of the following forms:*

(i) $x = w$, *where* $w \neq x$;
(ii) $xy = w$, *where* $w \neq yx$ *is a word which neither begins nor ends with* xy.

6.3. Rings satisfying certain semigroup identities. The results presented in the previous two sections, together with the results given in Section 5, make a possibility to give very nice descriptions of the structure of rings satisfying certain semigroup identities. These descriptions will be presented in this section. But, we first introduce some necessary notions.

For a semigroup identity $u = v$ over the alphabet A_n, $n \in \mathbb{N}$, $n \geq 2$, and for $i \in \{1, 2, \dots, n\}$, let $p_i = \big||x_i|_u - |x_i|_v\big|$. If there exists $i \in \{1, 2, \dots, n\}$ such that $p_i \neq 0$, then we say that $u = v$ is a *periodic identity*, and the number $p = \gcd(p_1, p_2, \dots, p_n)$ is called the *period* of this identity. When we deal with the two-element alphabet $A_2 = \{x, y\}$, then $p_x = \big||x|_u - |x|_v\big|$, $p_y = \big||y|_u - |y|_v\big|$ and $p = \gcd(p_x, p_y)$. Otherwise, if $p_i = 0$, for any $i \in \{1, 2, \dots, n\}$, then we say that the identity $u = v$ is *aperiodic*. In some origins periodic identities were called *unbalanced*, and aperiodic identities were called *balanced*. But, our terminology is justified by the following theorem:

Theorem 6.23. *The following conditions for a semigroup identity* $u = v$ *are equivalent:*

(i) $[u = v]$ *consists of* π*-regular semigroups;*
(ii) $[u = v]$ *consists of completely* π*-regular semigroups;*
(iii) $[u = v]$ *consists of periodic semigroups;*
(iv) $u = v$ *is a periodic identity.*

As was noted by Ćirić and Bogdanović in [**90**], 1996, any group satisfying a semigroup identity of the period p satisfies also the identity $x = x^{p+1}$, and any commutative semigroup satisfying the identity $x = x^{p+1}$, satisfies also any identity of the period p. Using these properties and Theorems 5.44 and 6.13, in the mentioned paper Ćirić and Bogdanović proved the following

Theorem 6.24. *A ring R satisfies an $\mathcal{UG}\circ\mathcal{N}$-identity of the period p if and only if R is a direct sum of a nil-ring that satisfies the same identity and a nil-ring that satisfies the identity $x = x^{p+1}$.*

As a consequence of this result, the same authors also obtained

Theorem 6.25. *Any ring which satisfies the identity $xy = w(x, y)$, with $w \notin xy^+ \cup x^+y$, is commutative.*

In the same paper the authors gave some examples which justify that the previous assertion does not hold for identities of the form $xy = xy^n$ and $xy = x^ny$, $n \in \mathbb{N}$.

Many well-known results in Theory of rings are consequences of the above quoted theorems. Here we present the results obtained by Abian and Mc Worter in [**3**], 1964, and Lee in [**196**], 1973.

Let p be a prime. A ring R is called a *pre p-ring* if it is a commutative ring of the characteristic p and it satisfies an identity $xy^p = x^py$. The structure of these rings was described by Abian and Mc Worter in [**3**], 1964, in the following way:

Theorem 6.26. *Let p be a prime. A ring R is a pre-p-ring if and only if it is a direct sum of a p-ring and a pre-p-nil-ring.*

On the other hand, Lee investigated in [**196**], 1973, rings satisfying a system of identities $(xy)^n = xy = x^ny^n$. He proved the following two theorems:

Theorem 6.27. *Let $n \in \mathbb{N}$, $n \geq 2$. A ring R satisfies a system of identities $(x + y)^n = x^n + y^n$, $(xy)^n = xy = x^ny^n$, if and only if it is a direct sum of a ring satisfying the identity $x = x^n$ and a null-ring.*

Theorem 6.28. *A ring R satisfies a system of identities $(xy)^2 = xy = x^2y^2$ if and only if it is a direct sum of a Boolean ring a null-ring.*

Except for the rings satisfying a $\mathcal{UG}\circ\mathcal{N}$-identity, very nice structural descriptions can be given for rings satisfying certain other $\mathcal{A}\circ\mathcal{S}$-identities, especially the periodic ones. The main tool used in these descriptions are Theorems 5.11 and 6.1, and the Everett's representations of rings which follow by these theorems.

Here we present results concerning the structure of rings satisfying a semigroup identity of the form

$$x_1 \cdots x_n = w(x_1, \dots, x_n), \tag{3}$$

where $n \in \mathbb{N}$, $n \geq 2$, $c(w) = A_n$ and $|w| \geq n + 1$. Identities of this form have been investigated by many authors, and the general result characterizing rings satisfying an arbitrary semigroup identity of this form was given by Ćirić, Bogdanović and Petković in [**94**], 1995.

The identity (3) is a periodic $\mathcal{A}\circ\mathcal{S}$-identity. Let p be its period and let

$$h = \max\Big(\{i \mid w = x_1\cdots x_i u(x_{i+1},\dots,x_n)\}\cup 0\Big),$$
$$t = \max\Big(\{l \mid w = u'(x_1,\dots,x_{n-l})x_{n-l+1}\cdots x_n\}\cup 0\Big).$$

The quadruplet (n,p,h,t) was called the *characteristic quadruplet* of the identity (3) [**94**]. Clearly, $h+t\le n-1$, and the following conditions hold:

$$x_1\cdots x_n = x_1\cdots x_h u(x_{h+1},\cdots,x_n),$$

with $x_{h+1} \underset{l}{\not\parallel} u$, if $h\ge 1$,

$$x_1\cdots x_n = u'(x_1,\dots,x_{n-t})x_{n-t+1}\dots x_n,$$

with $x_{n-t} \underset{r}{\not\parallel} u'$, if $t\ge 1$, and

$$x_1\cdots x_n = x_1\cdots x_h v(x_{h+1},\cdots,x_{n-t})x_{n-t+1}\cdots x_n,$$

with $x_{h+1} \underset{l}{\not\parallel} v$, $x_{n-t} \underset{r}{\not\parallel} v$, if $h\ge 1$ and $t\ge 1$.

Using the above notion, M. Ćirić, S. Bogdanović and T. Petković proved the following theorem:

Theorem 6.29. *Let* (3) *be an identity with the characteristic quadruplet* (n,p,h,t). *Then the following conditions for a ring* R *are equivalent:*

(i) R *satisfies* (3)*;*

(ii) R *is an ideal extension of an* n*-nilpotent ring* N *by a ring satisfying the identity* $x = x^{p+1}$ *and*

$$N^{h+1}\cdot E(R) = E(R)\cdot N^{t+1} = E(R)\cdot N\cdot E(R) = 0;$$

(iii) R *is an ideal extension of an* n*-nilpotent ring* N *by a ring satisfying the identity* $x = x^{p+1}$ *and*

$$N^{h+1}\cdot \mathrm{Reg}(R) = \mathrm{Reg}(R)\cdot N^{t+1} = \mathrm{Reg}(R)\cdot N\cdot \mathrm{Reg}(R) = 0;$$

(iv) $R = E(N,Q;\theta;[,];\langle,\rangle)$*, where* N *is an* n*-nilpotent ring,* Q *is a ring satisfying the identity* $x = x^{p+1}$*, and*

$$\theta^b N\theta^c = 0, \qquad \textit{for all } b,c\in Q.$$
$$N^{h+1}\theta^b = \theta^b N^{t+1} = 0, \qquad \textit{for each } b\in Q.$$

In the particular case when the characteristic quadruplet of (3) has the form $(n,p,0,0)$, the same authors obtained the following:

Theorem 6.30. *Let* (3) *be an identity with the characteristic quadruplet* $(n, p, 0, 0)$. *Then a ring R satisfies the identity* (3) *if and only if R is a direct sum of an n-nilpotent ring and a ring satisfying the identity $x = x^{p+1}$.*

The previous theorem is a consequence both of Theorems 6.24 and 6.29.

M. Ćirić, S. Bogdanović and T. Petković gave also a consequent classification of semigroup identities over the two-element and the three-element alphabet.

Theorem 6.31. *For the identity*

$$xy = w(x, y), \tag{4}$$

with $w \in A_2^+$, $|w| \geq 3$, there are exactly three possibilities:

(i) (4) *has the characteristic quadruplet* $(2, p, 0, 0)$, *and then a ring satisfies* (4) *if and only if it is a direct sum of a ring satisfying $x = x^{p+1}$ and a null-ring, and consequently these rings are commutative.*

(ii) (4) *has the characteristic quadruplet* $(2, p, 1, 0)$, *and this holds if and only if it is of the form $xy = xy^{p+1}$.*

(iii) (4) *has the characteristic quadruplet* $(2, p, 0, 1)$, *and this holds if and only if it is of the form $xy = x^{p+1}y$.*

Theorem 6.32. *For the identity*

$$xyz = w(x, y, z), \tag{5}$$

with $w \in A_3^+$, $|w| \geq 4$, there are exactly six possibilities:

(i) (5) *has the characteristic quadruplet* $(3, p, 0, 0)$, *and then a ring satisfies* (5) *if and only if it is a direct sum of a ring satisfying $x = x^{p+1}$ and a 3-nilpotent ring.*

(ii) (5) *has the characteristic quadruplet* $(3, p, 1, 0)$, *and this holds if and only if it is of the form $xyz = xu(y, z)$, $|u| \geq 3$.*

(iii) (5) *has the characteristic quadruplet* $(3, p, 0, 1)$, *and this holds if and only if it is of the form $xyz = v(x, y)z$, $|v| \geq 3$.*

(iv) (5) *has the characteristic quadruplet* $(3, p, 2, 0)$, *and this holds if and only if it is of the form $xyz = xyz^{p+1}$.*

(v) (5) *has the characteristic quadruplet* $(3, p, 0, 2)$, *and this holds if and only if it is of the form $xyz = x^{p+1}yz$.*

(vi) (5) *has the characteristic quadruplet* $(3, p, 1, 1)$, *and this holds if and only if it is of the form $xyz = xy^{p+1}z$.*

Important particular types of the identities of the form (3) are the identities $xyz = xyxz$ and $xyz = xzyz$. Rings satisfying the first one are known as *left distributive* (or *left self distributive*) *rings*, and the rings satisfying another identity are *right distributive* (or *right self distributive*). These rings have an important role when we study rings whose any additive endomorphism is also multiplicative (see Birkenmeier and Heatherly [**27**], 1990). Left distributive rings were investigated by Birkenmaier, Heatherly and Kepka in [**29**], 1992. Using Theorem 6.29, these rings can be characterized as follows:

Theorem 6.33. *A ring R is left distributive if and only if it is an ideal extension of a 3-nilpotent ring N by a Boolean ring, and the following conditions hold:*

$$E(R)\cdot N\cdot E(R)=N\cdot E(R)=E(R)\cdot N^2=0.$$

Rings which are both left and right distributive are known as *distributive rings*. These rings are characterized by the following theorem proved by Petrich in [**239**], 1969.

Theorem 6.34. *A ring R is distributive if and only if it is a direct sum of a Boolean ring and a 3-nilpotent ring.*

One generalization of distributive rings was introduced by Ćirić and Bogdanović in [**80**], 1990, who defined a ring R to be *n-distributive*, where $n\in\mathbb{N}$, $n\geq 2$, if it satisfies the system of identities

$$x_1x_2\cdots x_{n+1}=(x_1x_2)(x_1x_3)\cdots(x_1x_{n+1}),$$
$$x_1x_2\cdots x_{n+1}=(x_1x_{n+1})(x_2x_{n+1})\cdots(x_nx_{n+1}).$$

These rings can be characterized as follows:

Theorem 6.35. *A ring R is n-distributive if and only if it is a direct sum of a ring satisfying the identity $x=x^n$ and a $(n+1)$-nilpotent ring.*

Note finally that rings satisfying identities of the form

$$x_1x_2\cdots x_n=w(x_1,x_2,\dots,x_n)$$

(without the assumption $|w|\geq n+1$) were studied by Putcha and Yaqub in [**260**], 1972. They proved that in such a ring R, the commutator ideal $C(R)$ is a nilpotent ideal, and there exists $m\in\mathbb{N}$ such that $R^mC(R)R^m=0$. Rings satisfying permutation identities were studied by Birkenmeier and Heatherly in [**26**] and [**28**].

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