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A NOTE ABOUT A THEOREM OF R. HARTE

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Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $T\colon \mathcal{A} \to \mathcal{B}$ be a unital continuous homomorphism. Put $\mathcal{J} = \operatorname{Ker} T$. Let $\operatorname{Fred}_T(\mathcal{A}) = \{x \in \mathcal{A} | T(x) \text{ is invertible in } \mathcal{B}\}$ and $\operatorname{Fred}^0_T(\mathcal{A}) = \{x + k | \, x \text{ is invertible in } \mathcal{A}, k \in \mathcal{J}\}$. In this note, we prove that if T has Property (F), then $\operatorname{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})} = \operatorname{Fred}^0_T(\mathcal{A})$ iff $\operatorname{ltsr}(\mathcal{J}) = 1$.

For a normed algebra $\mathcal{A}$ with unit 1, let $GL(\mathcal{A})$ (resp. $GL_0(\mathcal{A})$) denote the group of invertible elements in $\mathcal{A}$ (resp. the connected component of 1 in $GL(\mathcal{A})$). If $\mathcal{A}$ is non–unital, we set $GL(\mathcal{A}) = GL(\widetilde{\mathcal{A}})$ and $GL_0(\mathcal{A}) = GL_0(\widetilde{\mathcal{A}})$, where $\widetilde{\mathcal{A}} = \{\lambda 1 + a | \, \lambda \in \mathbb{C}, a \in \mathcal{A}\}$. For a Banach algebra $\mathcal{A}$, we view $\mathcal{A}^n$ as the set of all $n \times 1$ matrices over $\mathcal{A}$. According to [3], the left topological stable rank of the unital Banach algebra $\mathcal{A}$ is defined as follows:

$$\operatorname{ltsr}(\mathcal{A}) = \min\{\, n \in \mathbb{N} | \, \mathcal{A}^m \ \text{ is dense in } \operatorname{Lg}_m(\mathcal{A}), \forall m \geq n \,\}$$

where $\operatorname{Lg}_n(\mathcal{A})$ consists of the elements $(a_1, \cdots, a_n)^T$ in $\mathcal{A}^n$ with $\sum\limits_{i=1}^n b_i a_i = 1$ for some $b_1, \cdots, b_n \in \mathcal{A}$. If $\mathcal{A}$ is non–unital, we put $\operatorname{ltsr}(\mathcal{A}) = \operatorname{ltsr}(\widetilde{\mathcal{A}})$. We have $\operatorname{ltsr}(\mathcal{A}) = 1$ iff $GL(\mathcal{A})$ is dense in $\mathcal{A}$ (or $\widetilde{\mathcal{A}}$) (cf. [3]).

Let $\mathcal{A}$ be a unital Banach algebra. Write $\operatorname{Rg}(\mathcal{A}) = \{a \in \mathcal{A} | a \in a\mathcal{A}a\}$ and $\operatorname{Dr}(\mathcal{A}) = \{a \in \mathcal{A} | a \in a(GL(\mathcal{A}))a\}$ for all regular (generalized invertible) elements and decomposably regular elements of $\mathcal{A}$. Then $\operatorname{Dr}(\mathcal{A}) = \operatorname{Rg}(\mathcal{A}) \cap \overline{GL(\mathcal{A})}$ by [2, Theorem 1.1]). Now let $\mathcal{B}$ be a unital Banach algebra and $T\colon \mathcal{A} \to \mathcal{B}$ be a unital homomorphism (i.e., $T(1) = 1$). Put $\operatorname{Fred}_T(\mathcal{A}) = T^{-1}(GL(\mathcal{B}))$, $\operatorname{Fred}^0_T(\mathcal{A}) = GL(\mathcal{A}) + \operatorname{Ker} T$. The elements in $\operatorname{Fred}_T(\mathcal{A})$ are called to be T–Fredholm and in $\operatorname{Fred}^0_T(\mathcal{A})$ are called to be T–Weyl (cf. [1]).

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Let $\mathcal{A},\mathcal{B}$ be unital Banach algebras and T be a unital continuous homomorphism of $\mathcal{A}$ to $\mathcal{B}$. R. Harte proved in [2] that if $\mathrm{Fred}_T(\mathcal{A})\subset \mathrm{Rg}\,(\mathcal{A})$ and $1+\mathrm{Ker}\,T\subset \mathrm{Dr}\,(\mathcal{A})$, then

$$\mathrm{Fred}^0_T(\mathcal{A})=int(\mathrm{Fred}^0_T(\mathcal{A}))=\mathrm{Fred}_T(\mathcal{A})\cap\overline{GL(\mathcal{A})}. \qquad (*)$$

by means of the equation $\mathrm{Dr}\,(\mathcal{A})=\mathrm{Rg}\,(\mathcal{A})\cap\overline{GL(\mathcal{A})}$. In this short note, We will show when $\mathrm{Fred}^0_T(\mathcal{A})$ is closed in $\mathrm{Fred}_T(\mathcal{A})$ and prove that if T has Property (F) (see Definition 1 below) the equation (*) holds iff $\mathrm{ltsr}\,(\mathrm{Ker}\,T)=1$.

Throughout the paper, $\mathcal{A}$, $\mathcal{B}$ are unital Banach algebras and $T\colon\mathcal{A}\to\mathcal{B}$ is a unital continuous homomorphism.

Definition 1. *We say T has Property (F) if for every $b\in T(\mathcal{A})$ with $\|1-b\|<1$, then $b^{-1}\in T(\mathcal{A})$.*

Obviously, if $T(\mathcal{A})$ is closed in $\mathcal{B}$, then $T(\mathcal{A})$ has Property (F). Also, we have

Proposition 2. *Let $\mathcal{A}$, $\mathcal{B}$ and T be as above.*

1. *If $\mathrm{Fred}_T(\mathcal{A})\subset\mathrm{Rg}\,(\mathcal{A})$, then T has Property (F);*
2. *If T has Property (F), then $\mathrm{Fred}^0_T(\mathcal{A})$ is closed in $\mathrm{Fred}_T(\mathcal{A})$.*

Proof. (1) Let $b\in T(\mathcal{A})$ such that $\|1-b\|<1$. Then $b\in GL(\mathcal{B})$. Choose $a\in\mathcal{A}$ such that $b=T(a)$. Since $a\in\mathrm{Fred}_T(\mathcal{A})\subset\mathrm{Rg}\,(\mathcal{A})$, there is $a_0\in\mathcal{A}$ such that $aa_0a=a$ and consequently, $b^{-1}=T(a_0)$.

(2) Let $a\in\mathrm{Fred}_T(\mathcal{A})$ and $\{a_n\}_1^\infty\subset\mathrm{Fred}^0_T(\mathcal{A})$ such that $\lim\limits_{n\to\infty}a_n=a$. Choose n_0 such that $\|T(a_{n_0})-T(a)\|<\dfrac{1}{2\|(T(a))^{-1}\|}$. Then $\|T(a_{n_0})(T(a))^{-1}-1\|<\dfrac{1}{2}$. Put $b=T(a_{n_0})(T(a))^{-1}\in GL(\mathcal{B})$. Then $\|b^{-1}\|<\dfrac{1}{1-\|1-b\|}<2$. Since $b^{-1}\in T(\mathcal{A})$ and $\|b^{-1}-1\|\le\|b^{-1}\|\|b-1\|<1$, it follows that there is $d\in\mathcal{A}$ such that $b=(b^{-1})^{-1}=T(d)$. Combining this with $b^{-1}\in T(\mathcal{A})$, we can find $c\in\mathcal{A}$ such that $k_1=ac-1$ and $k_2=ca-1$ are in $\mathrm{Ker}\,T$. Pick n_1 such that $\|a_{n_1}-a\|<\dfrac{1}{\|c\|}$. Then $\|1+k_1-a_{n_1}c\|=\|(a-a_{n_1})c\|<1$ so that $g=k_1-a_{n_1}c\in GL(\mathcal{A})$. Therefore

$$a=g^{-1}(k_1-a_{n_1}c)a=g^{-1}k_1a-g^{-1}a_{n_1}k_2-g^{-1}a_{n_1}\in\mathrm{Fred}^0_T(\mathcal{A}).$$

□

Theorem 3. *Let $\mathcal{A}$, $\mathcal{B}$ be unital Banach algebras and $T\colon\mathcal{A}\to\mathcal{B}$ be a unital hormormphism with Property (F). Then*

$$\mathrm{Fred}^0_T(\mathcal{A})=int(\mathrm{Fred}^0_T(\mathcal{A}))=\mathrm{Fred}_T(\mathcal{A})\cap\overline{GL(\mathcal{A})}$$

iff $\mathrm{ltsr}\,(\mathrm{Ker}\,T)=1$.

Proof. Since $GL(\mathcal{A})$ is open in $\mathcal{A}$, $GL(\mathcal{A})+k$ is open in $\mathcal{A}$ for each $k \in \operatorname{Ker} T$. Thus $\mathrm{Fred}_T^0(\mathcal{A}) = \{GL(\mathcal{A}) + k \,|\, k \in \operatorname{Ker} T\}$ is open in $\mathcal{A}$ and hence is open in $\mathrm{Fred}_T(\mathcal{A})$, i.e., $\mathrm{Fred}_T^0(\mathcal{A}) = int(\mathrm{Fred}_T^0(\mathcal{A}))$.

By Proposition 2, when T has Property (F), $\mathrm{Fred}_T^0(\mathcal{A})$ is closed in $\mathrm{Fred}_T(\mathcal{A})$. Noting that $GL(\mathcal{A}) \subset \mathrm{Fred}_T^0(\mathcal{A})$ and $\mathrm{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})}$ is the closure of $GL(\mathcal{A})$ in $\mathrm{Fred}_T(\mathcal{A})$, thus $\mathrm{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})} \subset \mathrm{Fred}_T^0(\mathcal{A})$.

We now prove that $\mathrm{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})} \supset \mathrm{Fred}_T^0(\mathcal{A})$ iff $\mathrm{ltsr}\,(\operatorname{Ker} T) = 1$.

Suppose that $\mathrm{ltsr}\,(\operatorname{Ker} T) = 1$, then for any $a \in GL(\mathcal{A})$ and $k \in \operatorname{Ker} T$,

$$a + k = a(1 + a^{-1}k) \in a\,(\overline{GL(\widetilde{\operatorname{Ker} T})}) \subset \overline{GL(\mathcal{A})},$$

i.e., $\mathrm{Fred}_T^0(\mathcal{A}) \subset \mathrm{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})}$.

Conversely, for any $k \in \operatorname{Ker} T$ and any $\epsilon \in (0,1)$, there is $x_\epsilon \in GL(\mathcal{A})$ such that $\|1 + k - x_\epsilon\| < \dfrac{\epsilon}{4(1 + \|1+k\|)}$ $(< \dfrac{1}{2})$. Put $a_\epsilon = x_\epsilon - k$. Then $a_\epsilon \in GL(\mathcal{A})$ and $\|a_\epsilon^{-1}\| < \dfrac{1}{1 - \|1 - a_\epsilon\|} < 2$. Set $z_\epsilon = a_\epsilon^{-1}x_\epsilon$. Then $z_\epsilon \in GL(\mathcal{A})$, $T(z_\epsilon) = T(z_\epsilon^{-1}) = 1$, i.e., $z_\epsilon \in GL(\widetilde{\operatorname{Ker} T})$ and furthermore,

$$\|1 + k - z_\epsilon\| \le \|1 + k - x_\epsilon\| + \|a_\epsilon^{-1}\|\|1 - a_\epsilon\|\|x_\epsilon\| < \epsilon.$$

Now let $x = \lambda 1 + z \in \widetilde{\operatorname{Ker} T}$. If $\lambda = 0$, we put $x_\epsilon = \epsilon 1 + z = \epsilon(1 + \epsilon^{-1}z)$. Then $\|x - x_\epsilon\| < \epsilon$ and $x_\epsilon \in \overline{GL(\widetilde{\operatorname{Ker} T})}$. So $x \in \overline{GL(\widetilde{\operatorname{Ker} T})}$. If $\lambda \ne 0$, then $x = \lambda(1 + \lambda^{-1}z) \in \overline{GL(\widetilde{\operatorname{Ker} T})}$. Therefore, $\mathrm{ltsr}\,(\operatorname{Ker} T) = 1$. □

We conclude the paper with following two examples:

Example 4. Let X be a Banach space and let $B(X)$ (resp. $K(X)$) denote the Banach algebra of all bounded linear operators (resp. compact operators) on X. Let T be the canonical homomorphism of $B(X)$ onto $B(X)/K(X)$. Then $\operatorname{Ker} T = K(X)$. Using the fact that every nonzero point in the spectrum of a compact operator is isolated, we can deduce that $\mathrm{ltsr}\,(K(X)) = 1$. So by Theorem 3, $\mathrm{Fred}_T^0(B(X)) = int(\mathrm{Fred}_T^0(B(X)) = \mathrm{Fred}_T(B(X)) \cap \overline{GL(B(X))}$.

Example 5. Let $\mathcal{A} = C(\overline{\mathbf{D}})$ and $\mathcal{B} = (\mathbf{S}^1)$. Let T be the homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ given by restriction $T(f)(z) = f(z)$, $\forall\, z \in \mathbf{S}^1$, $f \in C(\overline{\mathbf{D}})$. Since $\operatorname{Ker} T \cong C_0(\mathbb{R}^2)$ and $\widetilde{\operatorname{Ker} T} \cong C(\mathbf{S}^2)$, it follows from [3, Proposition 1.7] that $\mathrm{ltsr}\,(\operatorname{Ker} T) = 2$. By Theorem 3, $\mathrm{Fred}_T^0(\mathcal{A})$ is both open and closed in $\mathrm{Fred}_T(\mathcal{A})$ and $\mathrm{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})} \subsetneqq \mathrm{Fred}_T^0(\mathcal{A})$.

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