

## COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS VIA IMPLICIT RELATIONS

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### Abstract

We prove common fixed point theorems for four mappings satisfying implicit relations in symmetric spaces using the concept of occasionally weakly compatible mappings. Our Theorems generalize results of [1], [3], [4] and [7].

## 1 Introduction and Preliminaries

Let  $A$  and  $S$  be self-mappings of a metric space  $(X, d)$  and  $C(A, S)$  the set of coincidence points of  $A$  and  $S$ .

**Definition 1.1** [5].  $A$  and  $S$  are said to be weakly compatible if  $SAu = ASu$  for all  $u \in C(A, S)$ .

**Definition 1.2** [2].  $A$  and  $S$  are said to be occasionally weakly compatible if  $SAu = ASu$  for some  $u \in C(A, S)$ .

**Remark 1.3** [2] If  $A$  and  $S$  are weakly compatible, then they are occasionally weakly compatible, but the following Example shows that the converse is not true in general.

**Example 1.4.** Let  $X = [1, \infty)$  with the usual metric. Define  $A, S : X \rightarrow X$  by:  $Ax = 3x - 2$  and  $Sx = x^2$ . We have  $Ax = Sx$  iff  $x = 1$  or  $x = 2$  and  $AS(1) = SA(1) = 1$ , but  $AS(2) \neq SA(2)$ . Therefore,  $A$  and  $S$  are occasionally weakly compatible, but they are not weakly compatible.

**Lemma 1.5** [6]. If  $A$  and  $S$  have a unique coincidence point  $w = Ax = Sx$ , then  $w$  is the unique common fixed point of  $A$  and  $S$ .

**Definition 1.6.** Let  $X$  be a set. A symmetric on  $X$  is a mapping  $d : X \times X \rightarrow [0, \infty)$  such that

$$d(x, y) = 0 \text{ iff } x = y \text{ and } d(x, y) = d(y, x) \text{ for all } x, y \in X.$$

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Let  $K_6$  the family of all continuous mappings  $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  with  $t_3 + t_4 \neq 0$  satisfying the following conditions:

- $(K_1)$  :  $K$  is decreasing in variables  $t_5$  and  $t_6$
  - $(K_2)$  : there exists  $0 \leq h < 1$  such that for all  $u, v, w \geq 0$  with
  - $(K_a)$  :  $F(u, v, v, u, u + v, 0) \leq 0$  or
  - $(K_b)$  :  $F(u, v, u, v, 0, u + v) \leq 0$
- we have  $u \leq hv$ .

Let  $F_6$  the family of all continuous mappings  $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  with  $t_3 + t_4 \neq 0$  satisfying the following condition:

- $(F_1)$  : there exists  $0 \leq h < 1$  such that for all  $u, v, w \geq 0$  with
  - $(F_a)$  :  $F(u, v, v, u, w, 0) \leq 0$  or
  - $(F_b)$  :  $F(u, v, u, v, 0, w) \leq 0$
- we have  $u \leq hv$ .

Let  $H_6$  the family of all continuous mappings  $H(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  with  $t_5 + t_6 \neq 0$  satisfying the following conditions.

- $(H_1)$  : there exists  $0 \leq h < 1$  such that for all  $u, v, w \geq 0$  with
  - $(H_a)$  :  $H(u, v, v, u, w, 0) \leq 0$  or
  - $(H_b)$  :  $H(u, v, u, v, 0, w) \leq 0$
- we have  $u \leq hv$ .

- $(H_2)$  :  $H(u, u, 0, 0, u, u) > 0$  for all  $u > 0$ .

Let  $G_6$  the family of all continuous mappings  $G(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  with  $t_2 + t_4 \neq 0$  satisfying the following conditions:

- $(G_1)$  : there exists  $0 \leq h < 1$  such that for all  $u, v \geq 0$  with
  - $(G_a)$  :  $G(u, v, v, u, w, 0) \leq 0$  or
  - $(G_b)$  :  $G(u, v, u, v, 0, w) \leq 0$
- we have  $u \leq hv$ .

- $(G_2)$  :  $G(u, u, 0, 0, u, u) > 0$  for all  $u > 0$ .

The following Theorems were proved by [1].

**Theorem 1.7.** *Let  $f, g, S$  and  $T$  be self-mappings of a metric space  $(X, d)$  satisfying the following conditions:*

$$S(X) \subset g(X) \text{ and } T(X) \subset f(X) \quad (1.1)$$

$$F(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0$$

for all  $x, y \in X$  if  $d(fx, Sx) + d(gy, Ty) \neq 0$ , where  $F \in F_6$  satisfies  $(F_1)$ , or

$$d(Sx, Ty) = 0 \text{ if } d(fx, Sx) + d(gy, Ty) = 0.$$

Suppose that one of  $S(X), T(X), f(X)$  and  $g(X)$  is a complete subspace of  $X$  and the pairs  $(S, f)$  and  $(T, g)$  are weakly compatible. Then,  $f, g, S$  and  $T$  have a unique common fixed point in  $X$ .

**Theorem 1.8.** *Let  $f, g, S$  and  $T$  be self-mappings of a metric space  $(X, d)$  satisfying (1.1) and*

$$H(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0$$

for all  $x, y \in X$  if  $d(fx, Ty) + d(Sx, gy) \neq 0$ , where  $H \in H_6$  satisfies  $(H_1)$  and  $(H_2)$ , or

$$d(Sx, Ty) = 0 \text{ if } d(fx, Ty) + d(Sx, gy) = 0.$$

Suppose that one of  $S(X), T(X), f(X)$  and  $g(X)$  is a complete subspace of  $X$  and the pairs  $(S, f)$  and  $(T, g)$  are weakly compatible. Then,  $f, g, S$  and  $T$  have a unique common fixed point in  $X$ .

**Theorem 1.9.** Let  $f, g, S$  and  $T$  be self-mappings of a metric space  $(X, d)$  satisfying (1.1) and

$$G(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0$$

for all  $x, y \in X$  if  $d(fx, gy) + d(gy, Ty) \neq 0$ , where  $G \in C_6$  satisfies  $(G_1)$  and  $(G_2)$ , or

$$d(Sx, Ty) = 0 \text{ if } d(fx, gy) + d(gy, Ty) = 0.$$

Suppose that one of  $S(X), T(X), f(X)$  and  $g(X)$  is a complete subspace of  $X$  and the pairs  $(S, f)$  and  $(T, g)$  are weakly compatible. Then,  $f, g, S$  and  $T$  have a unique common fixed point in  $X$ .

The following Theorems was proved by [7].

**Theorem 1.10.** Let  $f, g, S$  and  $T$  be self-mappings of a metric space  $(X, d)$  satisfying (1.1) and

$$K(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0$$

for all  $x, y \in X$  if  $d(fx, Sx) + d(gy, Ty) \neq 0$ , where  $K \in K_6$  satisfies  $(K_1)$  and  $(K_2)$ , or

$$d(Sx, Ty) = 0 \text{ if } d(fx, Sx) + d(gy, Ty) = 0.$$

Suppose that one of  $S(X), T(X), f(X)$  and  $g(X)$  is a complete subspace of  $X$  and the pairs  $(S, f)$  and  $(T, g)$  are weakly compatible. Then,  $f, g, S$  and  $T$  have a unique common fixed point in  $X$ .

It is our purpose in this paper to prove common fixed point theorems for occasionally weakly compatible mappings satisfying implicit relations in symmetric spaces. Our Theorems generalize results of [1], [3], [4] and [7].

## 2 Implicit relations

Let  $F_6$  the family of all functions  $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  with  $t_3 + t_4 \neq 0$

**Example 2.1.**  $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_2 t_3}{t_3 + t_4} - b \frac{t_4 t_5}{t_5 + t_6 + 1}, a, b > 0.$

**Example 2.2.**  $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_2 t_4}{t_3 + t_4} - b \frac{t_3 t_6}{t_5 + t_6 + 1}, a, b > 0.$

**Example 2.3.**  $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_3 t_5 + t_4 t_6}{t_3 + t_4} - bt_2, a, b > 0.$

**Example 2.4.**  $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{at_3 t_4 + bt_5 t_6}{t_3 + t_4} - ct_2, a, b, c > 0.$

Let  $H_6$  the family of all functions  $H(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  with  $t_5 + t_6 \neq 0$  satisfying the following condition:

$$(H_1) : H(u, u, 0, 0, u, u) > 0 \text{ for all } u > 0.$$

**Example 2.5.**  $H(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_3 t_6 + t_4 t_5}{t_5 + t_6} - b t_2$ ,  $a, b > 0$  and  $b < 1$ .

**Example 2.6.**  $H(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{a t_3 t_5 + b t_4 t_6}{t_5 + t_6} - c t_2$ ,  $a, b, c > 0$  and  $c < 1$ .

Let  $G_6$  the family of all functions  $G(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  with  $t_2 + t_4 \neq 0$  satisfying the following condition:

$$(G_1) : G(u, u, 0, 0, u, u) > 0 \text{ for all } u > 0.$$

**Example 2.7.**  $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a_1 \frac{t_2 t_5}{t_2 + t_4} - a_2(t_3 + t_4) - a_3(t_5 + t_6) - a_4 t_2$ ,  $a_1, a_2, a_3, a_4 > 0$  and  $a_1 + 2a_3 + a_4 < 1$ .

**Example 2.8.**  $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_2 t_4}{t_2 + t_4} - b \frac{t_3 t_5}{t_5 + t_6 + 1}$ ,  $a, b > 0$ .

**Example 2.9.**  $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_2 t_4}{t_2 + t_4} - b \frac{t_3 t_6}{t_5 + t_6 + 1}$ ,  $a, b > 0$ .

### 3 Main Results

**Theorem 3.1.** *Let  $f, g, S$  and  $T$  be self-mappings of a symmetric space  $(X, d)$  satisfying the following condition:*

$$F(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0 \quad (3.1)$$

for all  $x, y \in X$  if  $d(fx, Sx) + d(gy, Ty) \neq 0$ , where  $F \in F_6$ , or

$$d(Sx, Ty) = 0 \text{ if } d(fx, Sx) + d(gy, Ty) = 0. \quad (3.2)$$

Suppose that the pairs  $(S, f)$  and  $(T, g)$  are occasionally weakly compatible. Then,  $f, g, S$  and  $T$  have a unique common fixed point in  $X$ .

**Proof.** Since the pairs  $(S, f)$  and  $(T, g)$  are occasionally weakly compatible, there exist  $u, v \in X$  such that  $fu = Su$  and  $gv = Tv$ . As  $d(fu, Su) + d(gv, Tv) = 0$ , it follows from (3.2) that  $Su = Tv$  and so  $fu = Su = gv = Tv$ . Moreover, if there is another point  $u'$  such that  $fu' = Su'$ , using (3.2) it follows that  $fu' = Su' = gv = Tv$ . Therefore,  $z = fu = Su$  is the unique point of coincidence of  $f$  and  $S$ . By Lemma 1.5,  $z$  is the unique common fixed point of  $f$  and  $S$ . Similarly,  $z'$  is the unique common fixed point of  $g$  and  $T$ . On the other hand,  $d(fz, Sz) + d(gz', Tz') = 0$  implies that  $d(Sz, Tz') = 0$ , hence  $z = fz = Sz = gz' = Tz' = z'$ . Therefore,  $z$  is the unique common fixed point of  $f, g, S$  and  $T$ .

**Corollary 3.2.** (Theorem 1.7).

**Proof.** By Theorem 1.7, there exists  $u, v$  in  $X$  such that  $z = Tv = gv = Su = fu$ . Since weakly compatible mappings are occasionally weakly compatible, then the conclusion follows from Theorem 3.1.

**Corollary 3.3.** (Theorem 1.10).

**Theorem 3.4.** *Let  $f, g, S$  and  $T$  be self-mappings of a symmetric space  $(X, d)$  satisfying: the following condition:*

$$H(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0 \quad (3.3)$$

for all  $x, y \in X$  if  $d(fx, Ty) + d(Sx, gy) \neq 0$ , where  $H \in H_6$  satisfies  $(H_1)$ , or

$$d(Sx, Ty) = 0 \text{ if } d(fx, Ty) + d(Sx, gy) = 0. \quad (3.4)$$

Suppose that the pairs  $(S, f)$  and  $(T, g)$  are occasionally weakly compatible. Then,  $f, g, S$  and  $T$  have a unique common fixed point in  $X$ .

**Proof.** Since the pairs  $(S, f)$  and  $(T, g)$  are occasionally weakly compatible, there exist  $u, v \in X$  such that  $fu = Su$  and  $gv = Tv$ . Assume that  $Su \neq Tv$ . As  $d(fu, Tv) + d(Su, gv) \neq 0$ , using (3.3) we have

$$H(d(Su, Tv), d(Su, Tv), 0, 0, d(Su, Tv), d(Su, Tv)) \leq 0$$

which is a contradiction of  $(H_1)$  and so  $fu = Su = gv = Tv$ . The rest of the proof follows as in Theorem 3.1.

**Corollary 3.5** (Theorem 1.8).

**Theorem 3.6.** *Let  $f, g, S$  and  $T$  be self-mappings of a symmetric space  $(X, d)$  satisfying the following condition:*

$$G(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0 \quad (3.5)$$

for all  $x, y \in X$  if  $d(fx, gy) + d(gy, Ty) \neq 0$ , where  $G \in G_6$  satisfies  $(G_1)$ , or

$$d(Sx, Ty) = 0 \text{ if } d(fx, gy) + d(gy, Ty) = 0. \quad (3.6)$$

Suppose that the pairs  $(S, f)$  and  $(T, g)$  are occasionally weakly compatible. Then,  $f, g, S$  and  $T$  have a unique common fixed point in  $X$ .

**Proof.** It follows as in Theorem 3.4.

**Corollary 3.7** (Theorem 1.9).

**Corollary 3.8** (Theorem of [4]).

If  $T = S$  and  $g = f$  in Theorems 3.1, 3.3 and 3.6, we obtain the following Corollaries which generalize Corollaries of Theorems 1.7, 1.8 and 1.9.

**Corollary 3.9.** *Let  $f$  and  $S$  be self-mappings of a symmetric space  $(X, d)$  satisfying the following condition:*

$$F(d(Sx, Sy), d(fx, fy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(Sx, fy)) \leq 0$$

for all  $x, y \in X$  if  $d(fx, Sx) + d(fy, Sy) \neq 0$ , where  $F \in F_6$ , or

$$d(Sx, Sy) = 0 \text{ if } d(fx, Sx) + d(fy, Sy) = 0.$$

Suppose that the pair  $(S, f)$  is occasionally weakly compatible. Then,  $f$  and  $S$  have a unique common fixed point in  $X$ .

**Corollary 3.10.** *Let  $f$  and  $S$  be self-mappings of a symmetric space  $(X, d)$  satisfying the following condition:*

$$H(d(Sx, Sy), d(fx, fy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(Sx, fy)) \leq 0$$

for all  $x, y \in X$  if  $d(fx, Sy) + d(Sx, fy) \neq 0$ , where  $H \in H_6$  satisfies  $(H_1)$ , or

$$d(Sx, Sy) = 0 \text{ if } d(fx, Sy) + d(Sx, fy) = 0.$$

Suppose that the pair  $(S, f)$  is occasionally weakly compatible. Then,  $f$  and  $S$  have a unique common fixed point in  $X$ .

**Corollary 3.11.** *Let  $f$  and  $S$  be self-mappings of a symmetric space  $(X, d)$  satisfying the following condition:*

$$G(d(Sx, Sy), d(fx, fy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(Sx, fy)) \leq 0$$

for all  $x, y \in X$  if  $d(fx, fy) + d(fy, Sy) \neq 0$ , where  $G \in G_6$  satisfies  $(G_1)$ , or

$$d(Sx, Sy) = 0 \text{ if } d(fx, fy) + d(fy, Sy) = 0.$$

Suppose that the pair  $(S, f)$  is occasionally weakly compatible. Then,  $f$  and  $S$  have a unique common fixed point in  $X$ .

Now, we give Examples to support our Theorems.

**Example 3.12.** Let  $X = [1, \infty)$ ,  $d(x, y) = (x - y)^2$ ,  $f, g, S$  and  $T$  are self mappings of  $X$  defined by:

$Sx = 3x - 2$ ,  $fx = x^2$ ,  $Tx = 3x^2 - 2$ ,  $gx = x^4$  and  $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_3 t_5 + t_4 t_6}{t_3 + t_4} - bt_2$ ,  $a, b > 0$ . It is easy to see that the pairs  $(f, S)$  and  $(g, T)$  are occasionally weakly compatible, but they are not weakly compatible. We have for all  $x, y \in X$

$$\begin{aligned} d(fx, gy) &= (x^2 - y^4)^2 \\ &= (x - y^2)^2 (x + y^2)^2 \\ &\geq 4(x - y^2)^2 \\ &= \frac{4}{9} d(Sx, Ty). \end{aligned}$$

Therefore

$$\begin{aligned} d(Sx, Ty) &\leq \frac{9}{4} d(fx, gy) \\ &\leq a \frac{d(fx, Sx) d(fx, Ty) + d(gy, Ty) d(Sx, gy)}{d(fx, Sx) + d(gy, Ty)} \\ &\quad + \frac{9}{4} d(fx, gy), \quad a > 0, \\ \text{if } d(fx, Sx) + d(gy, Ty) &\neq 0. \\ d(Sx, Ty) &= 0 \text{ if } d(fx, Ty) + d(Sx, gy) = 0 \end{aligned}$$

and so for all  $x, y \in X$

$$F(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0$$

$$\text{if } d(fx, Sx) + d(gy, Ty) \neq 0.$$

and

$$d(Sx, Ty) = 0 \text{ if } d(fx, Sx) + d(gy, Ty) = 0.$$

Then, all conditions of Theorem 3.1 hold and 1 is the unique common fixed point of  $f, g, S$  and  $T$ .

**Example 3.13.** Let  $X = \{1\} \cup [\sqrt[3]{3}, \infty)$ ,  $d(x, y) = (x - y)^2$ ,  $f, g, S$  and  $T$  are self-mappings of  $X$  defined by:

$$Sx = \begin{cases} x^3 + 1 & \text{if } x \in \{1\} \cup [\sqrt[3]{3}, \infty) \text{ and } x \neq 4 \\ 4 & \text{if } x = 4 \end{cases},$$

$$fx = \begin{cases} 2x^6 & \text{if } x \in \{1\} \cup [\sqrt[3]{3}, \infty) \text{ and } x \neq 4 \\ 4 & \text{if } x = 4 \end{cases},$$

$$Tx = \begin{cases} x^2 + 1 & \text{if } x \in \{1\} \cup [\sqrt[3]{3}, \infty) \text{ and } x \neq 4 \\ 4 & \text{if } x = 4 \end{cases},$$

$$gx = \begin{cases} 2x^4 & \text{if } x \in \{1\} \cup [\sqrt[3]{3}, \infty) \text{ and } x \neq 4 \\ 4 & \text{if } x = 4 \end{cases}$$

$$\text{and } F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_3 t_6 + t_4 t_5}{t_5 + t_6} - b t_2, \text{ } a, b > 0 \text{ and } b < 1.$$

It is easy to see that the pairs  $(f, S)$  and  $(g, T)$  are occasionally weakly compatible, but they are not weakly compatible

If  $x = y = 4$  or  $x = y = 1$ , we have

$$d(Sx, Ty) = 0 \text{ since } d(fx, Ty) + d(Sx, gy) = 0.$$

If  $x \in [\sqrt[3]{3}, \infty)$ ,  $x \neq 4$  and  $y \in \{1\} \cup [\sqrt[3]{3}, \infty)$ ,  $y \neq 4$ , we get

$$d(fx, gy) = 4(x^6 - y^4)^2$$

$$= 4(x^3 - y^2)^2(x^3 + y^2)^2$$

$$\geq 64d(Sx, Ty).$$

Therefore

$$d(Sx, Ty) \leq \frac{1}{64}d(fx, gy).$$

If  $x \in [\sqrt[3]{3}, \infty)$ ,  $x \neq 4$  and  $y = 4$ , we get

$$d(fx, gy) = 4(x^6 - 2)^2$$

and

$$d(Sx, Ty) = (x^3 - 3)^2.$$

It follows that

$$\begin{aligned}\frac{d(fx, gy)}{d(Sx, Ty)} &= 4\left(x^3 + 3 + \frac{7}{x^3 - 3}\right)^2 \\ &> 64 \text{ if } x \neq \sqrt[3]{3}.\end{aligned}$$

Hence

$$d(Sx, Ty) < \frac{1}{64}d(fx, gy).$$

Similarly, if  $x = 4$  and  $y \in [\sqrt[3]{3}, \infty)$ ,  $y \neq 4$  we get

$$d(Sx, Ty) < \frac{1}{64}d(fx, gy).$$

Then, for all  $x, y \in X$

$$\begin{aligned}d(Sx, Ty) &\leq \frac{1}{64}d(fx, gy) + \\ a \frac{d(fx, Sx)d(fx, Ty) + d(gy, Ty)d(Sx, gy)}{d(fx, Sx) + d(gy, Ty)}, a &> 0 \text{ if } d(fx, Ty) + d(Sx, gy) \neq 0. \\ d(Sx, Ty) &= 0 \text{ if } d(fx, Ty) + d(Sx, gy) = 0.\end{aligned}$$

and so for all  $x, y \in X$

$$\begin{aligned}H(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) &\leq 0 \\ &\text{if } d(fx, Ty) + d(Sx, gy) \neq 0.\end{aligned}$$

and

$$d(Sx, Ty) = 0 \text{ if } d(fx, Ty) + d(Sx, gy) = 0.$$

Then, all conditions of Theorem 3.4 hold and 4 is the unique common fixed point of  $f, g, S$  and  $T$ .

Taking Example 3.13, It can be verified that for all  $x, y \in X$

$$\begin{aligned}G(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) &\leq 0 \\ &\text{if } d(fx, gy) + d(gy, Ty) \neq 0.\end{aligned}$$

and

$$d(Sx, Ty) = 0 \text{ if } d(fx, gy) + d(gy, Ty) = 0,$$

where  $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a_1 \frac{t_2 t_5}{t_2 + t_4} - a_2(t_3 + t_4) - a_3(t_5 + t_6) - a_4 t_2$ ,  $a_1, a_2, a_3, a_4 > 0$  and  $a_1 + 2a_3 + a_4 < 1$ .

Then, all conditions of Theorem 3.6 hold and 4 is the unique common fixed point of  $f, g, S$  and  $T$ .

**Remark 3.14.** Theorems of [1], [3], [4] and [7] can not be applicable since the pairs  $(f, S)$  and  $(g, T)$  are not weakly compatible and the function  $d$  defined in Examples 3.12 and 3.13 is not a metric.

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