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SOME SIMPLE CRITERIA OF STARLIKENESS FOR MEROMORPHIC FUNCTIONS

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Abstract

In this paper we will use the theory of differential subordinations to obtain a new condition for meromorphic functions, defined in the punctured disc $\dot{U} = U \setminus \{0\}$, $U = \{z \in \mathbb{C} : |z| < 1\}$, which are of the form $f(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + \dots$, to be starlike functions. The new condition for starlikeness is expressed by means of $|(1-\alpha) z f(z) + z^2 f'(z) + \alpha|$, where $\alpha \in [0, 1)$.

1 Introduction

For integer $n \geq 0$, denote by Σ_n the class of meromorphic functions, defined in the punctured disc $\dot{U} = U \setminus \{0\}$, which are of the form

$$f(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + \dots$$

and let $\Sigma = \Sigma_0$.

Let

$$\Sigma_n^* = \{f \in \Sigma_n : \text{Re} \left(-\frac{z f'(z)}{f(z)} \right) > 0, \quad z \in \dot{U}\}$$

the class of meromorphic starlike functions.

The following definitions and lemmas will be used in the next section.

Let $\mathcal{H}(U)$ denote the space of analytic functions in U. For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}_n = \{f \in \mathcal{H}(U) : \ f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots\}$$

and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

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If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 1.1. [2], [3] *Let $n \in \mathbb{N}^*$, $\mathbb{N}^* = \{1, 2, \ldots\}$, let $\alpha \in \mathbb{R}$ with $0 \leq \alpha < n+1$ and let $M > 0$.*

If $f \in \Sigma_n$ satisfies

$$|z^2 f'(z) + (1-\alpha) z f(z) + \alpha| < M$$

then

$$|zf(z) - 1| < \frac{M}{n+1-\alpha}$$

and this result is sharp.

Lemma 1.2. [2], [3] *Let $n \in \mathbb{N}^*$, $\mathbb{N}^* = \{1, 2, \ldots\}$, let $\alpha \in [0, 1]$, and let*

$$M_n(\alpha) = \frac{n+1-\alpha}{\sqrt{(n+1-\alpha)^2 + \alpha^2} + 1 - \alpha}.$$

If $f \in \Sigma_n$ satisfies the condition

$$|z^2 f'(z) + (1-\alpha) z f(z) + \alpha| < M_n(\alpha), \quad z \in U,$$

then $f \in \Sigma_n^*$.

2 Main results

Theorem 2.1. *Let $n \in \mathbb{N}^*$, $\mathbb{N}^* = \{1, 2, \ldots\}$, let $\alpha \in [0, 1)$ and let*

$$M(\alpha, n) = \max\{M_0(\alpha, n), M_1(\alpha, n)\}, \tag{2.1}$$

where

$$M_0(\alpha, n) = \frac{n+1-\alpha}{\sqrt{(n+1-\alpha)^2 + \alpha^2} + 1 - \alpha} \tag{2.2}$$

and

$$M_1(\alpha, n) = \frac{2(n+1-\alpha)(1-\alpha)}{(1-\alpha)(n-1) + \sqrt{(n+1-\alpha)^2(1-\alpha) + (n-1)^2(1-\alpha)^2}}. \tag{2.3}$$

If $f \in \Sigma_n$ satisfies the condition

$$|(1-\alpha) z f(z) + z^2 f'(z) + \alpha| < M, \tag{2.4}$$

where $0 < M \leq M(\alpha, n)$, with $M(\alpha, n)$ given by (2.1), then $f \in \Sigma_n^*$.

Proof. Let

$$0 < M \le M_1(\alpha, n)$$

and suppose that $f \in \Sigma_n$ satisfies (2.4).

If we set

$$P(z) = zf(z) - 1$$

then $P \in \mathcal{H}_{n+1}$ and by Lemma 1.1 we have

$$|P(z)| < \frac{M}{n+1-\alpha} \equiv R, \quad z \in U. \tag{2.5}$$

Hence if we let

$$p(z) = -\frac{zf'(z)}{f(z)}$$

then $p \in \mathcal{H}[1, n+1]$ and (2.4) can be written in the form

$$|p(z)(1 + P(z)) - (1-\alpha)P(z) - 1| < M.$$

We claim that this inequality implies Re $p(z) > 0$, $z \in U$. If this is false, then there exist a point $z_0 \in U$ such that $p(z_0) = i\rho$, where ρ is real.

We will show that at z_0 we have

$$|i\rho(1 + P(z_0)) - (1-\alpha)P(z_0) - 1| \ge M, \tag{2.6}$$

for all $\rho \in \mathbb{R}$.

If we let $P_0 = P(z_0)$, we have

$$|i\rho(1 + P_0) - (1-\alpha)P_0 - 1|^2 = \rho^2|1 + P_0|^2 + |(1-\alpha)P_0 + 1|^2 + 2\rho\alpha \text{Im } P_0$$

and we deduce that (2.6) is equivalent to

$$\rho^2|1 + P_0|^2 + |(1-\alpha)P_0 + 1|^2 + 2\alpha\rho \text{Im } P_0 - M^2 \ge 0$$

which holds for all real ρ is and only if

$$\alpha^2(\text{Im } P_0)^2 \le |1 + P_0|^2(|(1-\alpha)P_0 + 1|^2 - M^2), \quad z \in U. \tag{2.7}$$

Letting $P_0 = u + iv$ and $r^2 = u^2 + v^2$ then (2.7) is written as

$$[(1-\alpha)r^2 - (\alpha u - 1) + 2u]^2 - M^2(r^2 + 2u + 1) \ge 0.$$

The above inequality will hold if

$$[(1-\alpha)r^2 - (\alpha u - 1) + 2u]^2 - M^2(r + 1)^2 \ge 0 \tag{2.8}$$

and the inequality (2.8) holds if

$$H(r) \equiv (1-\alpha)r^2 - (\alpha u - 1) + 2u - M(r + 1) \ge 0.$$

From (2.5) we deduce that

$$u < \frac{M}{n+1-\alpha} \tag{2.9}$$

Since the discriminant of H is

$$\Delta = M^2 - 4 + 4\alpha + 4(1-\alpha)M - 4u(\alpha-1)(\alpha-2),$$

by using (2.9) and $M \leq M_1(\alpha, n)$, we deduce that $\Delta \leq 0$, which shows that $H(r) \geq 0$. Hence (2.6) holds and one obtains $\operatorname{Re} p(z) > 0$, i.e. $f \in \Sigma_n^*$. $\square$

Remark 2.1. Note that for $\alpha = 1$, $M_0(1, n)$ was obtained in [1] and for the special case $\alpha = 0$ the value of $M_0(0, n) = \dfrac{n+1}{n+2}$ was obtained in [3] and for this case $M_1(0, n) > M_0(0, n)$, $n = 1, 2, \ldots, 11$ and we get the following criterion of starlikeness for meromorphic functions.

Corollary 2.1. *Let $n \in \{1, 2, \ldots, 11\}$ and let*

$$M(n) = \frac{2(n+1)}{n-1+\sqrt{(n+1)^2+(n-1)^2}}. \tag{2.10}$$

If $f \in \Sigma_n$ satisfies the condition

$$|zf(z) + z^2 f'(z)| < M(n) \tag{2.11}$$

with $M(n)$ given by (2.10), then $f \in \Sigma_n^*$.

Since a function $f \in \Sigma_n$ can be written as

$$f(z) = \frac{1}{z} + g(z), \quad 0 < |z| < 1 \tag{2.12}$$

where $g \in \mathcal{H}_n$, Theorem 2.1 can be rewritten in the following equivalent form, that is useful for the other results.

Theorem 2.2. *Let $f \in \Sigma_n$, $n \geq 1$, have the form (2.12), where $g \in \mathcal{H}_n$. If*

$$|(1-\alpha)zg(z) + z^2 g'(z)| < M, \quad \alpha \in [0, 1)$$

where $0 < M \leq M(\alpha, n)$

$$M(\alpha, n) = \max\{M_0(\alpha, n), M_1(\alpha, n)\}$$

and $M_0(\alpha, n)$, $M_1(\alpha, n)$ given by (2.2) respectively (2.3), then $f \in \Sigma_n^*$.

Example 2.1. If we let

$$f(z) = \frac{1}{z} + \lambda(1 - \cos z)$$

In this case $f \in \Sigma_2$ and for $\alpha = 0$ we get

$$|zf(z) + z^2 f'(z)| < |\lambda|(e-1)$$

Hence by Theorem 2.1 if

$$|\lambda| < \frac{2(\sqrt{10}-1)}{3(e-1)} = 0.8389\ldots$$

then

$$\frac{1}{z} + \lambda(1-\cos z) \in \Sigma_2^*.$$

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