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# ON THE SIMULTANEOUS IMPROVING $K$ INCLUSION DISKS FOR POLYNOMIAL ZEROS*

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**Abstract**

A modification of the iterative method of Börsch-Supan type for the simultaneous inclusion of polynomial zeros is considered. The modified method provides the simultaneous inclusion of $k$ (of $n \geq k$) zeros, dealing with $k$ inclusion disks of these zeros and the point (unchangeable) approximations to the remaining $n-k$ zeros. It is proved that the $R$-order of convergence of the considered method is two if $k < n$ and three if $k = n$. Three numerical examples are given to illustrate convergence properties of the presented method.

## 1 Introduction

Iterative methods for the simultaneous determination of complex zeros of a given polynomial, realized in complex interval arithmetics, are very efficient device to error estimates for the given set of approximate zeros. In general, inclusion methods produce resulting disks or rectangles containing complex zeros. In this manner, the upper error bounds, given by the radii of disks or semidiagonals of rectangles, are obtained automatically. The price to be paid in order to achieve the above advantages of interval methods is the increase of numerical operations in each iterative step.

In some applications it is not necessary to calculate all zeros of a polynomial. The aim of this paper is to present a modification of the interval method for the iterative simultaneous inclusion of polynomial zeros, proposed in [3]. This modification enables the simultaneous inclusion of $k$ $(1 \leq k \leq n)$ zeros of a given polynomial of degree $n$ using the disks containing the wanted zeros and the initial "point" approximations of the remaining $n-k$ zeros. It is worth noting that these point approximations remain unchangeable in the course of iterative process.

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The main subject of the paper is the convergence analysis of the proposed modified method, including computationally verifiable initial conditions for the guaranteed convergence.

The presentation of the paper is organized as follows. The basic properties of circular complex arithmetic, necessary for the development and convergence analysis of the presented method are given at the end of Introduction. In Section 2 we derive the modified method, while the convergence analysis is given in Section 3. The numerical examples that illustrate convergence properties of the presented algorithm are given in Section 4.

A circular closed region (disk) $Z := \{z : |z-c| \le r\}$ with center $c := \text{mid } Z$ and radius $r := \text{rad } Z$ will be denoted by the parametric notation $Z := \{c; r\}$. The following properties are valid in circular complex arithmetic:

$$
\begin{aligned}
Z_1 \pm Z_2 &= \{c_1 \pm c_2; r_1 + r_2\}, \\
\alpha \cdot \{c; r\} &= \{\alpha c; |\alpha| r\} \quad (\alpha \in \mathbb{C}), \\
Z_1 \cdot Z_2 &= \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\}, \\
z \in \{c; r\} &\iff |z - c| \le r, \\
\{c_1; r_1\} \cap \{c_2; r_2\} = 0 &\iff |c_1 - c_2| > r_1 + r_2, \\
\{c_1; r_1\} \subseteq \{c_2; r_2\} &\iff |c_1 - c_2| \le r_2 - r_1.
\end{aligned}
$$

The inversion of a non-zero disk $Z = \{c; r\}$ is defined by the Möbius transformation

$$
Z^{-1} = \{c; r\}^{-1} = \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\} \quad (0 \notin Z), \tag{1.1}
$$

where the bar denotes the complex conjugate. Then the division of disks is given by

$$
Z_1 : Z_2 = Z_1 \cdot Z_2^{-1} \quad (0 \notin Z_2).
$$

If $F$ is a circular complex function and the implication

$$
Z_1 \subseteq Z_2 \Longrightarrow F(Z_1) \subseteq F(Z_2)
$$

holds, then $F$ is an *inclusion isotone* function. In particular, we have

$$
z \in Z \Longrightarrow F(z) \in F(Z). \tag{1.2}
$$

More details about circular arithmetic can be found in the books [1], [5] and [7]. Throughout this paper disks in the complex plane will be denoted by capital letters.

## 2 Derivation of the inclusion method

Let

$$
P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = \prod_{j=1}^{n} (z - \zeta_j)
$$

be a monic polynomial of degree $n \geq 3$ with simple real or complex zeros $\zeta_1, \ldots, \zeta_n$ and let $z_1, \ldots, z_n$ be approximations of these zeros. Let us consider the rational function

$$R(z) = \frac{P(z) - Q(z)}{Q(z)},$$

where

$$Q(z) = \prod_{j=1}^{n} (z - z_j).$$

The rational function $R(z)$ has the following development into elementary fractions

$$R(z) = \frac{P(z) - Q(z)}{Q(z)} = \sum_{j=1}^{n} \frac{W_j}{z - z_j}, \tag{2.1}$$

where

$$W_j = \frac{P(z_j)}{\prod_{\substack{\lambda=1 \\ \lambda \neq j}}^{n} (z_j - z_\lambda)} \quad (j \in \boldsymbol{I}_n := \{1, \ldots, n\}).$$

From (2.1), putting $z = \zeta_i$, we obtain for any zero

$$\sum_{j=1}^{n} \frac{W_j}{\zeta_i - z_j} = -1.$$

Hence we single out $\zeta_i$ and obtain the fixed point relation

$$\zeta_i = z_i - \frac{W_i}{1 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j}{\zeta_i - z_j}} \quad (i \in \boldsymbol{I}_n). \tag{2.2}$$

Assume that we have found disjoint disks $Z_i = \{z_i; r_i\}$ $(i \in \boldsymbol{I}_n)$ such that $\zeta_i \in Z_i$ for each $i \in \boldsymbol{I}_n$. Using the inclusion property (1.2), from the fixed point relation (2.2) we construct the iterative interval method

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j^{(m)}}{Z_i^{(m)} - z_j^{(m)}}} \quad (i \in \boldsymbol{I}_n) \tag{2.3}$$

for the simultaneous determination of the inclusive disks for all zeros of the polynomial $P$, proposed in [3]. This method can also be derived from the family of simultaneous methods of the third order presented in [4].

From the iterative formula (2.3) we observe that the determination of the new inclusion disk of the zero $\zeta_i$ requires only disk $Z_i$ and the point approximations of the remaining zeros. This enables us to construct the modified method for the simultaneous inclusion of $k$ $(k < n)$ zeros of $P$ using the disks containing the wanted zeros and the initial unchangeable point approximations of the rest $n-k$ zeros.

Assume now that we have found $k$ disjoint disks $Z_i^{(0)} = \{z_i^{(0)}; r_i^{(0)}\}$ $(i \in \boldsymbol{I}_k := \{1,\dots,k\})$ containing the zeros $\zeta_1,\dots,\zeta_k$ and the "point" approximations $z_{k+1}^{(0)},\dots,z_n^{(0)}$ to the zeros $\zeta_{k+1},\dots,\zeta_n$. From the fixed point relation (2.2) we obtain the iterative method for the simultaneous inclusion of $k$ zeros $\zeta_1,\dots,\zeta_k$ of the polynomial $P$

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{W_j^{(m)}}{Z_i^{(m)} - z_j^{(m)}} + \sum_{j=k+1}^{n} \frac{\widetilde{W}_j^{(m)}}{Z_i^{(m)} - z_j^{(0)}}}, \tag{2.4}$$

for $i \in \boldsymbol{I}_k$ and $m = 0,1,\dots,$ where

$$W_j^{(m)} = \frac{P(z_j^{(m)})}{\displaystyle\prod_{\substack{\lambda=1\\ \lambda\neq j}}^{k} \left(z_j^{(m)} - z_\lambda^{(m)}\right) \prod_{\lambda=k+1}^{n} \left(z_j^{(m)} - z_\lambda^{(0)}\right)} \quad (j = 1,\dots,k),$$

$$\widetilde{W}_j^{(m)} = \frac{P(z_j^{(0)})}{\displaystyle\prod_{\substack{\lambda=1\\ \lambda\neq j}}^{k} \left(z_j^{(0)} - z_\lambda^{(m)}\right) \prod_{\lambda=k+1}^{n} \left(z_j^{(0)} - z_\lambda^{(0)}\right)} \quad (j = k+1,\dots,n).$$

# 3 Convergence analysis

In this section we give the convergence analysis of the interval method (2.4). In the sequel we will always assume that $n \geq 3$.

Let us suppose that we have found $n$ disjoint disks $Z_i^{(0)} = \{z_i^{(0)}; r_i^{(0)}\}$ $(i \in \boldsymbol{I}_n)$ containing the zeros $\zeta_1,\dots,\zeta_n$. For all $m = 0,1,\dots$ let us introduce the following notation

$$\begin{aligned}
r^{(m)} &= \max_{1\leq i\leq k} r_i^{(m)}, \quad d = \max_{k+1\leq i\leq n} r_i^{(0)},\\
\varepsilon_j^{(m)} &= z_j^{(m)} - \zeta_j,\\
\rho_1^{(m)} &= \min_{\substack{1\leq i,j\leq k\\ i\neq j}} \left\{\left|z_j^{(m)} - z_i^{(m)}\right| - r_i^{(m)}\right\},\\
\rho_2^{(m)} &= \min_{\substack{1\leq i\leq k\\ k+1\leq j\leq n}} \left\{\left|z_j^{(0)} - z_i^{(m)}\right| - r_i^{(m)}\right\},\\
\rho^{(m)} &= \min\{\rho_1^{(m)}, \rho_2^{(m)}\},\\
\rho_3 &= \min_{\substack{k+1\leq i,j\leq n\\ i\neq j}} \left\{\left|z_j^{(0)} - z_i^{(0)}\right| - r_i^{(0)}\right\}.
\end{aligned}$$

**Lemma 3.1** *The following estimates*

$$\left|W_j^{(m)}\right| \le a_1^{(m)} r^{(m)}, \quad a_1^{(m)} = \left(1+\frac{r^{(m)}}{\rho_1^{(m)}}\right)^{k-1}\left(1+\frac{d}{\rho_2^{(m)}}\right)^{n-k} \ (j=1,\dots,k)$$

*and*

$$\left|\widetilde{W}_j^{(m)}\right| \le a_2^{(m)} d, \quad a_2^{(m)} = \left(1+\frac{r^{(m)}}{\rho_2^{(m)}}\right)^{k-1}\left(1+\frac{d}{\rho_3}\right)^{n-k} \ (j=k+1,\dots,n)$$

*hold for* $m=0,1,\dots$ .

**Proof.** First, for $j=1,\dots,k$ we estimate

$$\begin{aligned}
\left|W_j^{(m)}\right| &= \left|z_j^{(m)}-\zeta_j\right| \prod_{\substack{\lambda=1\\ \lambda\ne j}}^{k} \frac{\left|z_j^{(m)}-\zeta_\lambda\right|}{\left|z_j^{(m)}-z_\lambda^{(m)}\right|} \prod_{\lambda=k+1}^{n} \frac{\left|z_j^{(m)}-\zeta_\lambda\right|}{\left|z_j^{(m)}-z_\lambda^{(0)}\right|} \\
&\le \left|\varepsilon_j^{(m)}\right| \prod_{\substack{\lambda=1\\ \lambda\ne j}}^{k} \frac{\left|z_j^{(m)}-z_\lambda^{(m)}\right|+r_\lambda^{(m)}}{\left|z_j^{(m)}-z_\lambda^{(m)}\right|} \prod_{\lambda=k+1}^{n} \frac{\left|z_j^{(m)}-z_\lambda^{(0)}\right|+r_\lambda^{(0)}}{\left|z_j^{(m)}-z_\lambda^{(0)}\right|} \\
&\le r_j^{(m)}\left(1+\frac{r^{(m)}}{\rho_1^{(m)}}\right)^{k-1}\left(1+\frac{d}{\rho_2^{(m)}}\right)^{n-k} \le a_1^{(m)} r^{(m)}.
\end{aligned}$$

Similarly, for $j=k+1,\dots,n$ we have

$$\begin{aligned}
\left|\widetilde{W}_j^{(m)}\right| &= \left|z_j^{(0)}-\zeta_j\right| \prod_{\lambda=1}^{k} \frac{\left|z_j^{(0)}-\zeta_\lambda\right|}{\left|z_j^{(0)}-z_\lambda\right|} \prod_{\substack{\lambda=k+1\\ \lambda\ne j}}^{n} \frac{\left|z_j^{(0)}-\zeta_\lambda\right|}{\left|z_j^{(0)}-z_\lambda^{(0)}\right|} \\
&\le \left|\varepsilon_j^{(0)}\right| \prod_{\lambda=1}^{k} \frac{\left|z_j^{(0)}-z_\lambda^{(m)}\right|+r_\lambda^{(m)}}{\left|z_j^{(0)}-z_\lambda^{(m)}\right|} \prod_{\substack{\lambda=k+1\\ \lambda\ne j}}^{n} \frac{\left|z_j^{(0)}-z_\lambda^{(0)}\right|+r_\lambda^{(0)}}{\left|z_j^{(0)}-z_\lambda^{(0)}\right|} \\
&\le d\left(1+\frac{r^{(m)}}{\rho_2^{(m)}}\right)^{k}\left(1+\frac{d}{\rho_3}\right)^{n-k-1} = a_2^{(m)} d. \quad \blacksquare
\end{aligned}$$

Estimating the denominator in (2.4) yields for $i \in \boldsymbol{I}_k$

$$\begin{aligned}
\sum_{\substack{j=1\\ j\ne i}}^{k} \frac{W_j^{(m)}}{z_j^{(m)}-Z_i^{(m)}} &+ \sum_{j=k+1}^{n} \frac{\widetilde{W}_j^{(m)}}{z_j^{(0)}-Z_i^{(m)}} \\
&\subset \left\{\sum_{\substack{j=1\\ j\ne i}}^{k} \frac{W_j^{(m)}}{z_j^{(m)}-z_i^{(m)}}; \frac{(k-1)a_1^{(m)}\left(r^{(m)}\right)^2}{\left(\rho_1^{(m)}\right)^2}\right\} \\
&\quad + \left\{\sum_{j=k+1}^{n} \frac{\widetilde{W}_j^{(m)}}{z_j^{(0)}-z_i^{(m)}}; \frac{(n-k)a_2^{(m)} r^{(m)} d}{\left(\rho_2^{(m)}\right)^2}\right\} \\
&= \left\{u_i^{(m)}; r^{(m)} B^{(m)}\right\},
\end{aligned}$$

where

$$u_i^{(m)} = \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{W_j^{(m)}}{z_j^{(m)}-z_i^{(m)}} + \sum_{j=k+1}^{n} \frac{\widetilde{W}_j^{(m)}}{z_j^{(0)}-z_i^{(m)}},$$

$$B^{(m)} = (k-1)a_1^{(m)}\frac{r^{(m)}}{\left(\rho_1^{(m)}\right)^2} + (n-k)a_2^{(m)}\frac{d}{\left(\rho_2^{(m)}\right)^2}.$$

In regard to the above relations, the inclusion method (2.4) can be written in the form

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1-\left\{u_i^{(m)}; r^{(m)}B^{(m)}\right\}}, \quad (i \in \boldsymbol{I}_k).$$

The center of the disk in the denominator is bounded by

$$|1-u_i^{(m)}| > 1-\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{|W_j^{(m)}|}{\left|z_j^{(m)}-z_i^{(m)}\right|} - \sum_{j=k+1}^{n} \frac{|\widetilde{W}_j^{(m)}|}{\left|z_j^{(0)}-z_i^{(m)}\right|}$$

$$> 1-(k-1)a_1^{(m)}\frac{r^{(m)}}{\rho_1^{(m)}} - (n-k)a_2^{(m)}\frac{d}{\rho_2^{(m)}}.$$

Now, we are able to state the convergence theorem of the method (2.4).

**Theorem 3.1** *Under the initial conditions*

$$\rho^{(0)} > 4(n-1)r^{(0)} \tag{3.1}$$

*and*

$$\rho_3 > 4(n-1)d, \quad d \le r^{(0)} \tag{3.2}$$

*the method* (2.4) *is convergent.*

**Proof.** First, under the initial conditions (3.1) and (3.2) we estimate the quantities from Lemma 3.1 for $m=0$,

$$a_1^{(0)} = \left(1+\frac{r^{(0)}}{\rho_1^{(0)}}\right)^{k-1}\left(1+\frac{d}{\rho_2^{(0)}}\right)^{n-k} < \left(1+\frac{1}{4(n-1)}\right)^{n-1} < \frac{4}{3} \tag{3.3}$$

and

$$a_2^{(0)} = \left(1+\frac{r^{(0)}}{\rho_2^{(0)}}\right)^{k-1}\left(1+\frac{d}{\rho_3}\right)^{n-k} < \left(1+\frac{1}{4(n-1)}\right)^{n-1} < \frac{4}{3}. \tag{3.4}$$

Using the inequalities (3.1) – (3.4) we obtain the bounds

$$|1-u_i^{(0)}| > 1-(k-1)a_1^{(0)}\frac{r^{(0)}}{\rho_1^{(0)}} - (n-k)a_2^{(0)}\frac{d}{\rho_2^{(0)}} > \frac{3}{5} \tag{3.5}$$

and

$$r^{(0)}B^{(0)} = (k-1)a_1^{(0)}\frac{\big(r^{(0)}\big)^2}{\big(\rho_1^{(0)}\big)^2} + (n-k)a_2^{(0)}\frac{r^{(0)}d}{\big(\rho_2^{(0)}\big)^2} < \frac{1}{20}. \tag{3.6}$$

According to this we get

$$r_i^{(1)} = \operatorname{rad}\left(\frac{W_i^{(0)}}{\{1-u_i^{(0)}; r^{(0)}B^{(0)}\}}\right) < \frac{\frac{4}{3}\frac{1}{20}}{\frac{9}{25}-\frac{1}{400}} r_i^{(0)} < \frac{1}{5}r_i^{(0)}. \tag{3.7}$$

Using a geometric construction and the fact that the disks $Z_i^{(m)}$ and $Z_i^{(m+1)}$ must have at least one joint point (the zero $\zeta_i$), for $i \in \boldsymbol{I}_k$ the following relation can be derived (see [2])

$$\rho_1^{(m+1)} \geq \rho_1^{(m)} - r^{(m)} - 3r^{(m+1)}. \tag{3.8}$$

Using the initial condition (3.1) and the inequalities (3.7) and (3.8) (for $m = 0$), we find

$$\rho_1^{(1)} \geq \rho_1^{(0)} - r^{(0)} - 3r^{(1)} > 4(n-1)r^{(0)} - r^{(0)} - 3\frac{r^{(0)}}{5} > 5r^{(1)}\Big(4(n-1) - 1 - \frac{3}{5}\Big),$$

wherefrom it follows

$$\rho_1^{(1)} > 4(n-1)r^{(1)}. \tag{3.9}$$

The inequality (3.7) of the form $r^{(1)} < r^{(0)}/5$ points to the contraction of the new circular approximations $Z_1^{(1)}, \ldots, Z_k^{(1)}$.

Using the definition of $\rho_1$, the initial condition (3.1) and (3.9), we have for arbitrary pair of indices $i, j \in \boldsymbol{I}_k$ $(i \neq j)$

$$|z_i^{(1)} - z_j^{(1)}| \geq \rho_1^{(1)} > 4(n-1)r^{(1)} > 2r^{(1)} \geq r_i^{(1)} + r_j^{(1)}.$$

Therefore the disks $Z_1^{(1)}, \ldots, Z_k^{(1)}$ produced by (2.4) are pairwise disjoint.

Similarly to the relation (3.8), using a geometric construction and the fact that the disks $Z_i^{(0)}, \ldots, Z_i^{(m)}$ must have at least one common point, for $i \in \boldsymbol{I}_k$ we obtain the following inequality

$$\rho_2^{(m)} \geq \rho_2^{(0)} - 2r^{(m)}. \tag{3.10}$$

According to (3.10) (for $m = 1$) we estimate

$$\rho_2^{(1)} \geq \rho_2^{(0)} - 2r^{(1)} > \rho_2^{(0)} - \frac{2}{5}r^{(0)} > \rho_2^{(0)} - r^{(0)} \tag{3.11}$$

and

$$\rho_2^{(1)} \geq \rho_2^{(0)} - 2r^{(1)} > 4(n-1)r^{(0)} - \frac{2r^{(0)}}{5} > 5r^{(1)}\Big(4(n-1) - \frac{2}{5}\Big),$$

wherefrom it follows

$$\rho_2^{(1)} > 4(n-1)r^{(1)}. \tag{3.12}$$

Similarly as before, for arbitrary pair of indices $i \in \boldsymbol{I}_k$ and $j = k+1, \ldots, n$ we have

$$|z_i^{(1)} - z_j^{(0)}| \geq \rho_2^{(1)} > \rho_2^{(0)} - r^{(0)} > 4(n-1)r^{(0)} - r^{(0)} > r^{(1)} + d.$$

Therefore the disks $Z_1^{(1)}, \ldots, Z_k^{(1)}, Z_{k+1}^{(0)}, \ldots, Z_n^{(0)}$ are pairwise disjoint.

From the relation (3.9) and (3.12) we conclude that the initial condition (3.1) holds for $m = 1$, in other words

$$\rho^{(1)} > 4(n-1)r^{(1)}. \tag{3.13}$$

Let us now estimate the quantities from Lemma 3.1 for $m = 1$. Using the inequality (3.11) we obtain

$$a_1^{(1)} = \left(1 + \frac{r^{(1)}}{\rho_1^{(1)}}\right)^{k-1} \left(1 + \frac{d}{\rho_2^{(1)}}\right)^{n-k} < \left(1 + \frac{r^{(1)}}{\rho_1^{(1)}}\right)^{k-1} \left(1 + \frac{d}{\rho_2^{(0)} - r^{(0)}}\right)^{n-k}$$

and using the initial conditions (3.1) and (3.2) we estimate

$$a_1^{(1)} < \left(1 + \frac{1}{4(n-1)}\right)^{k-1} \left(1 + \frac{1}{4(n-1)-1}\right)^{n-k} < \frac{4}{3} \tag{3.14}$$

and

$$a_2^{(1)} = \left(1 + \frac{r^{(1)}}{\rho_2^{(1)}}\right)^{k-1} \left(1 + \frac{d}{\rho_3}\right)^{n-k} < \left(1 + \frac{1}{4(n-1)}\right)^{n-1} < \frac{4}{3}. \tag{3.15}$$

In a similar way we estimate the center and radius of the disk in the denominator of the inclusion formula (2.4) for $m = 1$. Using (3.1) and the bounds (3.14) and (3.15), we obtain

$$|1 - u_i^{(1)}| > 1 - (k-1)a_1^{(1)} \frac{r^{(1)}}{\rho_1^{(1)}} - (n-k)a_2^{(1)} \frac{d}{\rho_2^{(1)}} > \frac{3}{5} \tag{3.16}$$

and

$$r^{(1)} B^{(1)} = (k-1)a_1^{(1)} \frac{\left(r^{(1)}\right)^2}{\left(\rho_1^{(1)}\right)^2} + (n-k)a_2^{(1)} \frac{r^{(1)} d}{\left(\rho_2^{(1)}\right)^2} < \frac{1}{20}. \tag{3.17}$$

Finally we find

$$r_i^{(2)} < \frac{1}{5} r_i^{(1)}. \tag{3.18}$$

Since $r^{(2)} < r^{(1)}$ we conclude from (3.10) that the inequality (3.11) holds for $m = 2$, that is,

$$\rho_2^{(2)} \geq \rho_2^{(0)} - r^{(0)}.$$

Since the estimates (3.14) – (3.18) coincide with the bounds (3.3) – (3.7), we conclude that the inequality (3.13) holds for the index $m = 2$. Let us note that the generated disks $Z_1^{(2)}, \ldots, Z_k^{(2)}, Z_{k+1}^{(0)}, \ldots, Z_n^{(0)}$ are pairwise disjoint.

Repeating the above procedure and the argumentation for arbitrary index $m \geq 1$, we can derive all above relations for the index $m+1$. Since these relations have already proved for $m=1$, by mathematical induction it follows that, if the conditions (3.1) and (3.2) hold, they are valid for all $m \geq 1$. In particular, we have

$$\rho^{(m)} > 4(n-1)r^{(m)} \tag{3.19}$$

and

$$r^{(m+1)} < \frac{r^{(m)}}{5}. \tag{3.20}$$

In addition, we note that the inequality (3.19) means that the estimates (3.14) – (3.18) hold for each $m = 1, 2, \ldots$ . Finally, from (3.20) we conclude that the sequence $\{r^{(m)}\}$ monotonically converges to 0, in other words, the inclusion method (2.4) is convergent under the initial conditions (3.1) and (3.2). ■

**Theorem 3.2** *Let* $(Z_1, \ldots, Z_n) =: \big(Z_1^{(0)}, \ldots, Z_n^{(0)}\big)$ *be initial disks such that* $\zeta_i \in Z_i$ $(i \in \boldsymbol{I}_n)$ *and let* $\big\{Z_i^{(m)}\big\}$ *denote the sequences of the disks obtained by the iterative formula* (2.4). *Then, under the conditions* (3.1) *and* (3.2), *for each* $i \in \boldsymbol{I}_k$ $(1 \leq k \leq n)$ *and* $m = 0, 1, \ldots$ *we have*

1° $\zeta_i \in Z_i^{(m)}$;

2° *the R-order of convergence of the iterative process* (2.4) *is two if* $k < n$ *and three if* $k = n$.

**Proof.** We will prove the assertion 1° by induction. Let $\zeta_i \in Z_i^{(m)}$ for any $m$ and $i \in \boldsymbol{I}_k$. According to (2.4) we obtain

$$\zeta_i \in z_i^{(m)} - \frac{W_i^{(m)}}{1 - \left(\displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{W_j^{(m)}}{z_j^{(m)} - Z_i^{(m)}} + \sum_{j=k+1}^{n} \frac{\widetilde{W}_j^{(m)}}{z_j^{(0)} - Z_i^{(m)}}\right)} = Z_i^{(m+1)}.$$

Since $\zeta_i \in Z_i^{(0)}$ it follows by induction that $\zeta_i \in Z_i^{(m+1)}$ for each $m = 0, 1, \ldots$ .

Let us prove now the assertion 2°. From the relation (2.4) we estimate

$$r_i^{(m+1)} = \text{rad } Z_i^{(m+1)} < \frac{a_1^{(m)}\big(r^{(m)}\big)^2 B^{(m)}}{\big|1 - u_i^{(m)}\big|^2 - \big(r^{(m)}\big)^2 \big(B^{(m)}\big)^2}. \tag{3.21}$$

Since

$$B^{(m)} = (k-1)a_1^{(m)} \frac{r^{(m)}}{\big(\rho_1^{(m)}\big)^2} + (n-k)a_2^{(m)} \frac{d}{\big(\rho_2^{(m)}\big)^2} = \begin{cases} O(d), & k < n, \\ O(r^{(m)}), & k = n \end{cases}$$

we conclude from (3.21) that the $R$-order of convergence of the method (2.4) is two in the case when $k < n$ and three when $k = n$. ■

# 4 Numerical examples

The inclusion method (2.4) have been tested on a number of polynomial equations. Experimental results coincide very well with the theoretical results concerning the convergence speed of the interval method (2.4). Besides, these results show that initial approximations can be chosen under weaker conditions compared to (3.1) and (3.2). It is worth noting that only "point" approximations $z_{k+1}^{(0)}, \dots, z_n^{(0)}$ are sufficient in the implementation of the method (2.4) unlike the use of inclusion disks $Z_{k+1}^{(0)}, \dots, Z_n^{(0)}$ employed in the convergence analysis.

**Example 4.1** To find the circular inclusion approximations to the zeros of the polynomial

$$f(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300,$$

we implemented the interval methods (2.4). The exact zeros of $f$ are $\zeta_1 = -3$, $\zeta_{2,3} = \pm 1$, $\zeta_{4,5} = \pm 2i$, $\zeta_{6,7} = -2 \pm i$ and $\zeta_{8,9} = 2 \pm i$. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$, with the centers

$$\begin{aligned} z_1^{(0)} &= -3.1 + 0.1i, & z_2^{(0)} &= -1.2 - 0.1i, & z_3^{(0)} &= 1.2 + 0.1i, \\ z_4^{(0)} &= 0.1 - 2.1i, & z_5^{(0)} &= 0.1 + 1.9i, & z_6^{(0)} &= -1.9 + 1.1i, \\ z_7^{(0)} &= -1.9 - 0.9i, & z_8^{(0)} &= 2.1 + 1.1i, & z_9^{(0)} &= 1.9 - 0.9i. \end{aligned}$$

The radii of the inclusion disks produced in the first three iterative steps are given in Table 4.1, where the denotation $A(-q)$ means $A \times 10^{-q}$.

| | $r^{(1)}$ | $r^{(2)}$ | $r^{(3)}$ |
|---|---|---|---|
| $r_1$ | 1.02(−2) | 6.75(−8) | 1.45(−23) |
| $r_2$ | 2.58(−2) | 3.46(−7) | 9.26(−23) |
| $r_3$ | 2.25(−2) | 8.33(−7) | 5.35(−21) |
| $r_4$ | 7.96(−3) | 1.69(−8) | 3.02(−25) |
| $r_5$ | 8.59(−3) | 7.94(−8) | 5.14(−23) |
| $r_6$ | 1.28(−2) | 1.73(−7) | 1.12(−22) |
| $r_7$ | 1.61(−2) | 1.63(−7) | 3.31(−23) |
| $r_8$ | 8.45(−3) | 1.05(−7) | 1.70(−22) |
| $r_9$ | 1.22(−2) | 2.80(−7) | 1.29(−21) |

Table 4.1 The radii of inclusion disks

In finding inclusion disks of the first five zeros ($k = 5$), we obtain the radii in the first three iterative steps given in Table 4.2.

| | $r^{(1)}$ | $r^{(2)}$ | $r^{(3)}$ |
|---|---|---|---|
| $r_1$ | 1.02(−2) | 2.40(−6) | 1.64(−14) |
| $r_2$ | 2.58(−2) | 9.25(−6) | 1.45(−13) |
| $r_3$ | 2.25(−2) | 2.74(−5) | 2.01(−11) |
| $r_4$ | 7.96(−3) | 3.48(−7) | 4.04(−16) |
| $r_5$ | 8.59(−3) | 1.71(−6) | 2.34(−14) |

Table 4.2 The radii of inclusion disks

From Table 4.1 we can observe the cubic convergence of the interval methods, while the convergence is quadratic when only a part of inclusion disks is calculated, see Table 4.2. This is in accordance to the assertion of Theorem 3.2.

**Example 4.2** To find the circular inclusion approximations to the first four zeros of the polynomial

$$\begin{aligned} f(z) &= z^9-(4+6i)z^8-(18-3i)z^7-(45-63i)z^6+(377+324i)z^5 \\ &\quad -(158+1128i)z^4+(18-3i)z^3+(45-63i)z^2-(378+324i)z \\ &\quad +(162+1134i), \end{aligned}$$

we implemented the same interval method (2.4). The exact zeros of $f$ are $\zeta_{1,2}=\pm 1$, $z_{3,4}=\pm i$, $\zeta_5=3+3i$, $\zeta_6=4+3i$, $\zeta_7=-3-3i$, $\zeta_8=-3+3i$ and $\zeta_9=3$. The initial disks were selected to be $Z_i^{(0)}=\{z_i^{(0)};0.5\}$, with the centers

$$z_1^{(0)}=1.2-0.1i,\quad z_2^{(0)}=-1.1+0.2i,\quad z_3^{(0)}=0.1+1.2i,\quad z_4^{(0)}=0.2-0.9i.$$

The initial approximations of the remaining zeros are taken equidistantly on the circle $x^2+y^2=4$. The radii of the inclusion disks produced in the first three iterative steps are given in Table 4.3. The presented entries point to the quadratic convergence.

| | $r^{(1)}$ | $r^{(2)}$ | $r^{(3)}$ |
|---|---|---|---|
| $r_1$ | 2.44(−1) | 3.01(−2) | 2.20(−4) |
| $r_2$ | 1.30(−1) | 2.40(−3) | 1.83(−7) |
| $r_3$ | 1.78(−1) | 9.01(−3) | 7.49(−6) |
| $r_4$ | 1.67(−1) | 9.72(−3) | 1.03(−5) |

Table 4.3 The radii of inclusion disks

**Example 4.3** To find the circular inclusion approximations to the zeros of the polynomial

$$\begin{aligned} f(z) &= z^{20}+z^{19}+12z^{17}+124z^{16}+268z^{15}-432z^{14}+2784z^{13} \\ &\quad +1302z^{12}+34710z^{11}-91824z^{10}+324696z^9-3275380z^8 \\ &\quad +620972z^7-9722256z^6-2270592z^5-1056847z^4 \\ &\quad -28303951z^3+313942512z^2-25704900z+308458800, \end{aligned}$$

we implemented the interval method (2.4). The exact zeros of $f$ are: $\zeta_{1,2} = 1 \pm 2i$, $z_{3,4} = -1 \pm 2i$, $\zeta_{5,6} = \pm 2$, $\zeta_{7,8} = \pm i$, $\zeta_{9,10} = 3 \pm 2i$, $\zeta_{11,12} = -3 \pm 2i$, $\zeta_{13,14} = 2 \pm 3i$, $\zeta_{15,16} = -2 \pm 3i$, $\zeta_{17,18} = \pm 3i$, $\zeta_{19} = 3$ and $\zeta_{20} = -4$. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$, with the centers

$$
\begin{aligned}
z_1^{(0)} &= 1.1 + 2.2i, & z_2^{(0)} &= 1.2 - 2.1i, & z_3^{(0)} &= -1.2 + 2.1i,\\
z_4^{(0)} &= -1.2 - 2.1i, & z_5^{(0)} &= 2.2 + 0.1i, & z_6^{(0)} &= -2.1 + 0.1i,\\
z_7^{(0)} &= -0.2 + 0.9i, & z_8^{(0)} &= 0.2 - 1.1i, & z_9^{(0)} &= 3.1 + 1.9i,\\
z_{10}^{(0)} &= 3.2 - 1.9i, & z_{11}^{(0)} &= -3.2 + 1.9i, & z_{12}^{(0)} &= -3.2 - 1.9i,\\
z_{13}^{(0)} &= 2.2 + 2.9i, & z_{14}^{(0)} &= 2.2 - 2.9i, & z_{15}^{(0)} &= -2.2 + 2.9i,\\
z_{16}^{(0)} &= -2.2 - 2.9i, & z_{17}^{(0)} &= 0.2 + 3.1i, & z_{18}^{(0)} &= -0.2 - 2.9i,\\
z_{19}^{(0)} &= 3.2 - 0.1i, & z_{20}^{(0)} &= -4.2 - 0.1i.
\end{aligned}
$$

The radii of the inclusion disks produced in the first three iterative steps, are given in Table 4.4.

| | $r^{(1)}$ | $r^{(2)}$ | $r^{(3)}$ |
|---|---|---|---|
| $r_1$ | 5.18(−2) | 2.53(−5) | 1.02(−15) |
| $r_2$ | 5.66(−2) | 6.45(−5) | 1.69(−14) |
| $r_3$ | 5.35(−2) | 3.55(−5) | 1.29(−15) |
| $r_4$ | 5.02(−2) | 2.25(−5) | 1.07(−15) |
| $r_5$ | 7.21(−2) | 7.49(−5) | 9.51(−15) |
| $r_6$ | 2.14(−2) | 2.06(−6) | 2.59(−18) |
| $r_7$ | 6.51(−2) | 5.72(−5) | 2.23(−15) |
| $r_8$ | 7.62(−2) | 1.16(−4) | 3.71(−14) |
| $r_9$ | 1.41(−2) | 1.19(−6) | 4.52(−19) |
| $r_{10}$ | 1.93(−2) | 1.98(−6) | 6.59(−19) |
| $r_{11}$ | 1.20(−2) | 2.53(−6) | 1.40(−18) |
| $r_{12}$ | 1.97(−2) | 3.10(−6) | 9.06(−18) |
| $r_{13}$ | 2.86(−2) | 8.12(−6) | 9.66(−17) |
| $r_{14}$ | 3.40(−2) | 8.57(−6) | 1.21(−16) |
| $r_{15}$ | 3.25(−2) | 7.23(−6) | 3.12(−17) |
| $r_{16}$ | 3.26(−2) | 9.77(−6) | 5.63(−17) |
| $r_{17}$ | 3.67(−2) | 8.94(−6) | 9.37(−17) |
| $r_{18}$ | 5.34(−2) | 4.72(−5) | 6.65(−15) |
| $r_{19}$ | 2.32(−2) | 3.86(−6) | 2.52(−17) |
| $r_{20}$ | 1.27(−2) | 1.42(−7) | 1.24(−21) |

Table 4.4 The radii of inclusion disks

In finding inclusion disks of the first seven zeros ($k = 7$), we obtain the radii in the first three iterative steps given in Table 4.5.

| | $r^{(1)}$ | $r^{(2)}$ | $r^{(3)}$ |
|---|---|---|---|
| $r_1$ | 5.18(−2) | 2.71(−4) | 2.44(−9) |
| $r_2$ | 5.66(−2) | 6.71(−4) | 5.97(−8) |
| $r_3$ | 5.35(−2) | 4.91(−4) | 1.83(−8) |
| $r_4$ | 5.02(−2) | 2.24(−4) | 1.67(−9) |
| $r_5$ | 7.21(−2) | 1.13(−3) | 2.81(−8) |
| $r_6$ | 2.14(−2) | 2.46(−5) | 1.31(−11) |
| $r_7$ | 6.51(−2) | 4.79(−4) | 5.99(−9) |

Table 4.5 The radii of inclusion disks

As in Example 4.1, from Table 4.4 and 4.5 we can observe the cubic and quadratic convergence of the interval methods (2.4), respectively.

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