Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **22:2** (2008), 23–42

APPROXIMATING COMMON FIXED POINTS OF FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE NON-SELF MAPPINGS

G. S. Saluja

Abstract

Let K be a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. Let $T_1, T_2, \dots, T_N : K \longrightarrow E$ be N asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \le i \le N$ and $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \dots, N$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (6), where $\{u_n^i\}$ are bounded sequences in K with $\sum_{n=1}^{\infty} u_n^i < \infty$. (i) If the dual E^* of E has the *Kadec-Klee* property, then $\{x_n\}$ converges weakly to a common fixed point $x^* \in F$; (ii) if $\{T_1, T_2, \dots, T_N\}$ satisfies condition (B), then $\{x_n\}$ converges strongly to a common fixed point $x^* \in F$.

1 Introduction and preliminaries

Let K be a nonempty closed convex subset of a Banach space E. A self mapping $T : K \longrightarrow K$ is called asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$; $u_n \to 0$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\|T^n x - T^n y\| \le (1 + u_n) \|x - y\|, \quad \forall n \ge 1. \tag{1}$$

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for

2000 *Mathematics Subject Classifications.* 47H10.
Key words and Phrases. Asymptotically nonexpansive nonself mapping, common fixed point, multi step iterative sequences with errors for nonself mappings, real uniformly convex Banach space.
Received: December 17, 2007
Communicated by Vladimir Rakočević

all $x, y \in K$,

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall n \geq 1. \tag{2}$$

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [12] as an important generalization of the class of nonexpansive maps (i.e., mappings $T: K \longrightarrow K$ such that $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in K$) who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authers (see e.g., [24], [2, 3], [5], [21], [17], [6], [23], [1], [10, 11], [7], [13, 14, 15, 16]) using the Mann iteration method (see e.g., [27]) or the Ishikawa iteration method (see e.g., [20]).

In 1978, Bose [19] proved that if K is a bounded closed convex nonempty subset of a uniformly convex Banach space E satisfying Opial's [29] condition and $T: K \longrightarrow K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in K$, i.e., $\lim_{n\to\infty} \left\|T^n x - T^{n+1} x\right\| = 0$. Passty [6] and also Xu [8] proved that the requirement that E satisfies Opial's condition can be replaced by the condition that E has a Frechet differentiable norm. Furthermore, Tan and Xu [13, 14] later proved that the asymptotic regularity of T can be weakened to the weakly asymptotic regularity of T at x, i.e., $\omega - \lim_{n\to\infty}(T^n x - T^{n+1} x) = 0$.

In [10, 11], Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space H.

In 1994, Rhoades [1] extended the Schu's result to uniformly convex Banach space using a modified Ishikawa iteration method.

In all the above results, the operator T remains a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space. If, however, the domain of T, $D(T)$ is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

In 2003, Chidume et al [4] studied the iterative scheme defined by

$$x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1} x_n), \quad n \geq 1, \tag{3}$$

in the framework of uniformly convex Banach space, where K is a closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retract. $T\colon K \to E$ is an asymptotically nonexpansive nonself map with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0, 1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive nonself maps.

In 2005, Shahzad [18] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n TP[(1 - \beta_n)x_n + \beta_n Tx_n]), \tag{4}$$

where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. He proved weak and strong convergence theorems for nonself nonexpansive mappings in Banach spaces. Recently, Su and Qin [28] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K,$$

$$\begin{aligned} z_n &= P(\alpha''_n T(PT)^{n-1}x_n + (1 - \alpha''_n)x_n), \\ y_n &= P(\alpha'_n T(PT)^{n-1}z_n + (1 - \alpha'_n)x_n), \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}y_n + (1 - \alpha_n)x_n), \end{aligned} \tag{5}$$

where $\{\alpha_n\}$, $\{\alpha'_n\}$ and $\{\alpha''_n\}$ are real sequences in $(0, 1)$ and K is a nonempty closed convex nonexpansive retract of a uniformly convex Banach space E with P as a nonexpansive retraction. They proved weak and strong convergence theorems for asymptotically nonexpansive nonself mappings in uniformly convex Banach space. Motivated by Su and Qin [28] and some others, the purpose of this paper is to construct a multi step iterative scheme with errors for approximating common fixed point of a finite family of asymptotically nonexpansive nonself mappings (when such a fixed point exists) and to prove weak and strong convergence theorems for such maps.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. In this paper, the following iteration scheme is studied:

$$\begin{aligned} x_n^1 &= P(\alpha_n^1 T_1(PT_1)^{n-1}x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) \\ x_n^2 &= P(\alpha_n^2 T_2(PT_2)^{n-1}x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2) \end{aligned}$$

$$
\begin{aligned}
\ldots\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (6)\\
\ldots\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_{n+1} = x_n^N &= P(\alpha_n^N T_N (PT_N)^{n-1} x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N)
\end{aligned}
$$

with $x_1 \in K$, $n \geq 1$, where $\{\alpha_n^1\}, \{\alpha_n^2\}, \ldots, \{\alpha_n^N\}$, $\{\beta_n^1\}, \{\beta_n^2\}, \ldots, \{\beta_n^N\}$, $\{\gamma_n^1\}, \{\gamma_n^2\}, \ldots, \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \ldots, N$, and $\{u_n^1\}, \{u_n^2\}, \ldots, \{u_n^N\}$ are bounded sequences in K.

Our theorems improve and generalize some previous results. Our weak convergence result applies not only to L^p-spaces with $1 < p < \infty$ but also to other spaces which do not satisfy Opial's condition or have a Fréchet differentiable norm. More precisely, we prove weak convergence of the above defined iteration scheme with errors (6) in a real uniformly convex Banach space whose dual has the *Kadec-Klee* property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial's property; however their dual does have the *Kadec-Klee* property (see, e.g., [9, 25]).

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P\colon E \to E$ such that $Px = x$ for all $x \in K$. A map $P\colon E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A set K is optimal if each point outside K can be moved to be closure to all points of K. It is well known (see, e.g., [26]) that

(i) if E is a separable, strictly convex, smooth, reflexive Banach space, and if $K \subset E$ is an optimal set with interior, then K is a nonexpansive retract of E;

(ii) a subset of ℓ^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

A Banach space E is said to have the *Kadec-Klee* property if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$.

Recall that the following:

(i) A mapping $T\colon K \to K$ with $F(T) \neq \phi$ is said to satisfy condition (A) [7] on K if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$, $\|x - Tx\| \geq f(d(x,F))$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

(ii) A family $\{T_1, T_2, \ldots, T_N\}$ of N self mappings on K with $F = \cap_{i=1}^{N} F(T_i) \neq \phi$ is said to satisfy condition (B) on K if there exist f and d as in (i) such that $\max_{1\leq i\leq N}\{\|x - T_i x\|\} \geq f(d(x,F))$ for all $x \in K$.

Note that condition (B) reduces to condition (A), when $T_i = T$ for all $i = 1, 2, \ldots, N$.

In order to prove our main results, we will make use of the following lemmas:

Lemma 1.1 (Tan and Xu [15]): Let $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \quad \forall n \in N.$$

If $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. Then

(i) $\lim_{n\longrightarrow\infty} a_n$ exists.

(ii) If $\liminf_{n\longrightarrow\infty} a_n = 0$, then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 1.2 (see [8]): Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \leq R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 1.3 (demiclosed principle for nonselfmap [4]): Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E. Let $T\colon K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed with respect to zero.

Lemma 1.4 (see [9]): Let E be a real reflexive Banach space such that its dual E^* has the *Kadec-Klee* property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in w_w(x_n)$; here $w_w(x_n)$ denotes the weak w-limit set of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.

2 Main results

Definition 2.1 (see [4]): Let E be a real normed linear space, K a nonempty subset of E. Let $P\colon E \longrightarrow K$ be the nonexpansive retraction of E onto K. A map

$T\colon K \longrightarrow E$ is said to be asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0,\infty)$; $u_n \to 0$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq (1+u_n)\left\|x-y\right\|, \quad \forall n \geq 1. \tag{7}$$

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq L\left\|x-y\right\|, \quad \forall n \geq 1. \tag{8}$$

Lemma 2.2: Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2, \ldots, T_N\colon K \longrightarrow K$ be N asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leq i \leq N$ and $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \ldots, N$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (6), where $\{u_n^i\}$ are bounded sequences in K with $\sum_{n=1}^{\infty} u_n^i < \infty$. Then

(**i**) $\|x_{n+1} - x^*\| = \left\|x_n^N - x^*\right\| \leq (1 + b_n^{N-1})\left\|x_n - x^*\right\| + d_n^{N-1}$, for all $n \geq 1$, $x^* \in F$ and for some sequences $\{b_n^i\}$ and $\{d_n^i\}$ for all $i = 1, 2, \ldots, N$ of numbers such that $\sum_{n=1}^{\infty} b_n^i < \infty$ and $\sum_{n=1}^{\infty} d_n^i < \infty$.

(**ii**) There exists a constant $M > 0$ such that $\|x_{n+m} - x^*\| \leq M\left\|x_n - x^*\right\|$ for all $n, m \geq 1$ and $x^* \in F$.

Proof: (i) Let $x^* \in F$, then from (6) we have

$$\begin{aligned}
\left\|x_n^1 - x^*\right\| &= \left\|P(\alpha_n^1 T_1(PT_1)^{n-1}x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) - Px^*\right\| \\
&\leq \alpha_n^1\left\|T_1(PT_1)^{n-1}x_n - x^*\right\| + \beta_n^1\left\|x_n - x^*\right\| + \gamma_n^1\left\|u_n^1 - x^*\right\| \\
&\leq \alpha_n^1(1+r_n^1)\left\|x_n - x^*\right\| + \beta_n^1\left\|x_n - x^*\right\| + \gamma_n^1\left\|u_n^1 - x^*\right\| \\
&\leq (1-\beta_n^1)(1+r_n^1)\left\|x_n - x^*\right\| + \beta_n^1(1+r_n^1)\left\|x_n - x^*\right\| \\
&\quad + \gamma_n^1\left\|u_n^1 - x^*\right\| \\
&\leq (1+r_n^1)\left\|x_n - x^*\right\| + \gamma_n^1\left\|u_n^1 - x^*\right\| \\
&\leq (1+r_n^1)\left\|x_n - x^*\right\| + d_n^0
\end{aligned}$$

where $d_n^0 = \gamma_n^1\left\|u_n^1 - x^*\right\|$. Since $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, then $\sum_{n=1}^{\infty} d_n^0 < \infty$. Next, we note that

$$\begin{aligned}
\left\|x_n^2 - x^*\right\| &= \left\|P(\alpha_n^2 T_2(PT_2)^{n-1}x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2) - Px^*\right\| \\
&\leq \alpha_n^2\left\|T_2(PT_2)^{n-1}x_n^1 - x^*\right\| + \beta_n^2\left\|x_n - x^*\right\| + \gamma_n^2\left\|u_n^2 - x^*\right\| \\
&\leq \alpha_n^2(1+r_n^2)\left\|x_n^1 - x^*\right\| + \beta_n^2\left\|x_n - x^*\right\| + \gamma_n^2\left\|u_n^2 - x^*\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n^2(1+r_n^2)[(1+r_n^1)\|x_n-x^*\|+d_n^0]+\beta_n^2\|x_n-x^*\| \\
&\quad +\gamma_n^2\left\|u_n^2-x^*\right\| \\
&\leq [(1+r_n^1)(1+r_n^2)\alpha_n^2+\beta_n^2]\|x_n-x^*\|+\alpha_n^2(1+r_n^2)d_n^0 \\
&\quad +\gamma_n^2\left\|u_n^2-x^*\right\| \\
&\leq (\alpha_n^2+\beta_n^2)(1+r_n^1)(1+r_n^2)\|x_n-x^*\|+\alpha_n^2(1+r_n^2)d_n^0 \\
&\quad +\gamma_n^2\left\|u_n^2-x^*\right\| \\
&\leq (1+r_n^1+r_n^2+r_n^1r_n^2)\|x_n-x^*\|+\alpha_n^2(1+r_n^2)d_n^0+\gamma_n^2\left\|u_n^2-x^*\right\| \\
&\leq (1+b_n^1)\|x_n-x^*\|+d_n^1
\end{aligned}
$$

where $d_n^1=\alpha_n^2(1+r_n^2)d_n^0+\gamma_n^2\left\|u_n^2-x^*\right\|$ and $b_n^1=(1+r_n^1+r_n^2+r_n^1r_n^2)$. Since $\sum_{n=1}^{\infty}d_n^0<\infty$, $\sum_{n=1}^{\infty}\gamma_n^2<\infty$, $\sum_{n=1}^{\infty}r_n^i<\infty$ for $i=1,2$, and so $\sum_{n=1}^{\infty}d_n^1<\infty$ and $\sum_{n=1}^{\infty}b_n^1<\infty$. Similarly, we have

$$
\begin{aligned}
&\left\|x_n^3-x^*\right\|=\left\|P(\alpha_n^3T_3(PT_3)^{n-1}x_n^2+\beta_n^3x_n+\gamma_n^3u_n^3)-Px^*\right\| \\
&\leq \alpha_n^3\left\|T_3(PT_3)^{n-1}x_n^2-x^*\right\|+\beta_n^3\|x_n-x^*\|+\gamma_n^3\left\|u_n^3-x^*\right\| \\
&\leq \alpha_n^3(1+r_n^3)\left\|x_n^2-x^*\right\|+\beta_n^3\|x_n-x^*\|+\gamma_n^3\left\|u_n^3-x^*\right\| \\
&\leq \alpha_n^3(1+r_n^3)[(1+b_n^1)\|x_n-x^*\|+d_n^1]+\beta_n^3\|x_n-x^*\| \\
&\quad +\gamma_n^3\left\|u_n^3-x^*\right\| \\
&\leq [\alpha_n^3(1+r_n^3)(1+b_n^1)+\beta_n^3]\|x_n-x^*\|+\alpha_n^3(1+r_n^3)d_n^1 \\
&\quad +\gamma_n^3\left\|u_n^3-x^*\right\| \\
&\leq (\alpha_n^3+\beta_n^3)(1+b_n^1)(1+r_n^3)\|x_n-x^*\|+\alpha_n^3(1+r_n^3)d_n^1 \\
&\quad +\gamma_n^3\left\|u_n^3-x^*\right\| \\
&\leq (1+b_n^1)(1+r_n^3)\|x_n-x^*\|+d_n^2 \\
&\leq (1+b_n^2)\|x_n-x^*\|+d_n^2
\end{aligned}
$$

where $b_n^2=b_n^1+r_n^3+b_n^1r_n^3$ and $d_n^2=\alpha_n^3(1+r_n^3)d_n^1+\gamma_n^3\left\|u_n^3-x^*\right\|$. Since $\sum_{n=1}^{\infty}b_n^1<\infty$, $\sum_{n=1}^{\infty}r_n^3<\infty$, $\sum_{n=1}^{\infty}d_n^1<\infty$ and $\sum_{n=1}^{\infty}\gamma_n^3<\infty$, so $\sum_{n=1}^{\infty}b_n^2<\infty$ and $\sum_{n=1}^{\infty}d_n^2<\infty$.
By continuing the above process, there exists a nondecreasing sequences $\{d_n^{l-1}\}$ and $\{b_n^{l-1}\}$ such that $\sum_{n=1}^{\infty}d_n^{l-1}<\infty$ and $\sum_{n=1}^{\infty}b_n^{l-1}<\infty$ and

$$
\left\|x_n^i-x^*\right\|\leq(1+b_n^{i-1})\|x_n-x^*\|+d_n^{i-1},\quad \forall n\geq 1,\quad \forall i=1,2,\ldots,N.
$$

Thus

$$
\|x_{n+1}-x^*\|=\left\|x_n^N-x^*\right\|\leq(1+b_n^{N-1})\|x_n-x^*\|+d_n^{N-1},\quad \forall n\in N.
$$

This completes the proof of (i).

(ii) Since $1+x \le e^x$ for all $x>0$. Then from (i) it can be obtained that

$$
\begin{aligned}
& \|x_{n+m}-x^*\| \le (1+b^{N-1}_{n+m-1})\,\|x_{n+m-1}-x^*\| + d^{N-1}_{n+m-1} \\
\le\ & e^{b^{N-1}_{n+m-1}}\,\|x_{n+m-1}-x^*\| + d^{N-1}_{n+m-1} \\
=\ & e^{(b^{N-1}_{n+m-1}+b^{N-1}_{n+m-2})}\,\|x_{n+m-2}-x^*\| + e^{b^{N-1}_{n+m-1}} d^{N-1}_{n+m-2} + d^{N-1}_{n+m-1} \\
=\ & e^{(b^{N-1}_{n+m-1}+b^{N-1}_{n+m-2})}\,\|x_{n+m-2}-x^*\| + e^{b^{N-1}_{n+m-1}} (d^{N-1}_{n+m-1} + d^{N-1}_{n+m-2}) \\
=\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
=\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
=\ & e^{\sum_{k=n}^{n+m-1} b^{N-1}_k}\,\|x_n-x^*\| + e^{\sum_{k=n}^{n+m-1} b^{N-1}_k} . \sum_{k=n}^{n+m-1} d^{N-1}_k \\
=\ & M.\,\|x_n-x^*\| + M. \sum_{k=n}^{n+m-1} d^{N-1}_k,\ \ where\ \ M = e^{\sum_{k=n}^{\infty} b^{N-1}_k}
\end{aligned}
$$

This completes the proof of (ii).

Lemma 2.3: Let E be a normed linear space and K be a nonempty closed and convex subset which is also a nonexpansive retract of E. Let $T_1, T_2, \ldots, T_N \colon K \to K$ be N uniformly L-Lipschitzian mappings. Let $\{x_n\}$ be the sequence defined by (6) with sequences $\{u^i_n\}$ in K for all $i = 1, 2, \ldots, N$ and $\{\alpha^i_n\}$, $\{\beta^i_n\}$ and $\{\gamma^i_n\}$ are sequences in $[0,1]$ satisfying $\alpha^i_n + \beta^i_n + \gamma^i_n = 1$ for all $i = 1, 2, \ldots, N$. Set $c^i_n = \left\|x_n - T_i(PT_i)^{n-1}x_n\right\|$ for all $i = 1, 2, \ldots, N$. If $\lim_{n\to\infty} \left\|x_n - T_i(PT_i)^{n-1}x_n\right\| = 0$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$.

Proof: Since T_i is uniformly L-Lipschitzian for all $i = 1, 2, \ldots, N$, we have

$$
\begin{aligned}
\|x_{n+1} - T_i x_{n+1}\| &\le \|x_{n+1} - T_i(PT_i)^n x_{n+1}\| + \|T_i(PT_i)^n x_{n+1} - T_i x_{n+1}\| \\
&\le c^i_{n+1} + L\left\|T_i(PT_i)^{n-1}x_{n+1} - x_{n+1}\right\| \\
&\le c^i_{n+1} + L\{\|x_{n+1} - x_n\| + \left\|x_n - T_i(PT_i)^{n-1}x_n\right\| \\
&\quad + \left\|T_i(PT_i)^{n-1}x_n - T_i(PT_i)^{n-1}x_{n+1}\right\|\} \\
&\le c^i_{n+1} + L\{\|x_{n+1} - x_n\| + c^i_n + L\,\|x_{n+1} - x_n\|\} \\
&\le c^i_{n+1} + L(L+1)\,\|x_{n+1} - x_n\| + Lc^i_n \to 0 \ \ as \ \ n \to \infty
\end{aligned}
$$

This completes the proof.

Remark 2.4: If we put $P = I$ (identity map) in Lemma 2.2, then it generalizes the corresponding Lemma of Schu [11] for one mapping. Further, if $F = \cap_{i=1}^N F(T_i) \ne \phi$ and $\lim_{n\to\infty} \left\|x_n - T_i(PT_i)^{n-1}x_n\right\| = 0$ for all $i = 1, 2, \ldots, N$, then we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Theorem 2.5: Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2, \ldots,$

$T_N\colon K \longrightarrow K$ be N uniformly continuous asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leq i \leq N$ and $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence defined by (6) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$, for some $\varepsilon \in (0,1)$. Then $\|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$.

Proof: Let $x^* \in F = \cap_{i=1}^{N} F(T_i)$. Then by Lemma 2.1 (i) and Lemma 1.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$. If $a = 0$, then by the continuity of each T_i the conclusion follows. Now suppose that $a > 0$. Firstly, we are now to show that $\lim_{n\to\infty} \left\|T_N(PT_N)^{n-1}x_n - x_n\right\| = 0$. Since $\{x_n\}$ and $\{u_n^i\}$ are bounded for all $i = 1, 2, \ldots, N$, there exists $R > 0$ such that $x_n - x^* + \gamma_n^i(u_n^i - x_n), T_i(PT_i)^{n-1}x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n) \in B_R(0)$ for all $n \geq 1$ and for all $i = 1, 2, \ldots, N$. Using Lemma 1.2, we have

$$\begin{aligned}
\left\|x_n^N - x^*\right\|^2 &= \left\|P(\alpha_n^N T_N(PT_N)^{n-1}x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N) - Px^*\right\|^2 \\
&= \left\|\alpha_n^N T_N(PT_N)^{n-1}x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N - x^*\right\|^2 \\
&= \left\|\alpha_n^N (T_N(PT_N)^{n-1}x_n^{N-1} - x^* + \gamma_n^N(u_n^N - x_n)) \right. \\
&\quad \left. +(1-\alpha_n^N)(x_n - x^* + \gamma_n^N(u_n^N - x_n))\right\|^2 \\
&\leq \alpha_n^N \left\|T_N(PT_N)^{n-1}x_n^{N-1} - x^* + \gamma_n^N(u_n^N - x_n)\right\|^2 + (1-\alpha_n^N) \\
&\quad \left\|x_n - x^* + \gamma_n^N(u_n^N - x_n)\right\|^2 \\
&\quad -W_2(\alpha_n^N)g(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\|) \\
&\leq \alpha_n^N(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x^*\right\| + \gamma_n^N \left\|u_n^N - x_n\right\|)^2 + (1-\alpha_n^N) \\
&\quad (\|x_n - x^*\| + \gamma_n^N \left\|u_n^N - x_n\right\|)^2 \\
&\quad -W_2(\alpha_n^N)g(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\|) \\
&\leq \alpha_n^N[(1+b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \left\|u_n^N - x_n\right\|]^2 + (1-\alpha_n^N) \\
&\quad [(1+b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \left\|u_n^N - x_n\right\|]^2 \\
&\quad -W_2(\alpha_n^N)g(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\|) \\
&\leq [(1+b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \left\|u_n^N - x_n\right\|]^2 \\
&\quad -W_2(\alpha_n^N)g(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\|) \\
&\leq [\|x_n - x^*\| + \lambda_n^{N-2}]^2 - W_2(\alpha_n^N)g(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\|) \qquad (9)
\end{aligned}$$

where$\lambda_n^{N-2} = d_n^{N-2} + \gamma_n^N \left\|u_n^N - x_n\right\|$. Observe that $\varepsilon^3 \leq W_2(\alpha_n^N)$ now (9)implies that $\varepsilon^3 g(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-2}$, where $\rho_n^{N-2} = 2\lambda_n^{N-2} + (\lambda_n^{N-2})^2$. Since $\sum_{n=1}^{\infty} d_n^{N-2} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{N-2} < \infty$, we get $\sum_{n=1}^{\infty} \rho_n^{N-2} < \infty$. This implies that $\lim_{n\to\infty} g(\left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\|) = 0$. Since g is strictly increasing and continuous at 0, it follows that

$$\lim_{n\to\infty} \left\|T_N(PT_N)^{n-1}x_n^{N-1} - x_n\right\| = 0.$$

Since $T_N, \forall N$ is asymptotically nonexpansive, note that

$$\begin{aligned} \|x_n - x^*\| &\leq \left\|x_n - T_N(PT_N)^{n-1}x_n^{N-1}\right\| + \left\|T_N(PT_N)^{n-1}x_n^{N-1} - x^*\right\| \\ &= \left\|x_n - T_N(PT_N)^{n-1}x_n^{N-1}\right\| + (1 + r_n^N)\left\|x_n^{N-1} - x^*\right\| \end{aligned}$$

for all $n \geq 1$. Thus $a = \lim_{n\to\infty} \|x_n - x^*\| \leq \liminf_{n\to\infty} \left\|x_n^{N-1} - x^*\right\| \leq \limsup_{n\to\infty} \left\|x_n^{N-1} - x^*\right\| \leq a$ and therefore $\lim_{n\to\infty} \left\|x_n^{N-1} - x^*\right\| = a$. Using the same argument in the proof above, we have

$$\begin{aligned} & \left\|x_n^{N-1} - x^*\right\|^2 \\ \leq\ & \alpha_n^{N-1}\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\right\|^2 + (1 - \alpha_n^{N-1}) \\ & \left\|x_n - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\right\|^2 \\ & -W_2(\alpha_n^{N-1})g(\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x_n\right\|) \\ \leq\ & \alpha_n^{N-1}[(1 + b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\left\|u_n^{N-1} - x_n\right\|]^2 + (1 - \alpha_n^{N-1}) \\ & [(1 + b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\left\|u_n^{N-1} - x_n\right\|]^2 \\ & -W_2(\alpha_n^{N-1})g(\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x_n\right\|) \\ \leq\ & [(1 + b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\left\|u_n^{N-1} - x_n\right\|]^2 \\ & -W_2(\alpha_n^{N-1})g(\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x_n\right\|) \\ \leq\ & [\|x_n - x^*\| + \lambda_n^{N-3}]^2 \\ & -W_2(\alpha_n^{N-1})g(\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x_n\right\|) \qquad (10) \end{aligned}$$

where $\lambda_n^{N-3} = d_n^{N-3} + \gamma_n^{N-1}\left\|u_n^{N-1} - x_n\right\|$.

This implies that

$\varepsilon^3 g(\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x_n\right\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-3}$,

where $\rho_n^{N-3} = 2\lambda_n^{N-3} + (\lambda_n^{N-3})^2$

and therefore $\lim_{n\to\infty}\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x_n\right\| = 0$.

Thus, we have

$$\begin{aligned} & \left\|x_n - T_N(PT_N)^{n-1}x_n\right\| \\ \leq\ & \left\|x_n - T_N(PT_N)^{n-1}x_n^{N-1}\right\| + \left\|T_N(PT_N)^{n-1}x_n^{N-1} - T_N(PT_N)^{n-1}x_n\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\|x_n - T_N(PT_N)^{n-1}x_n^{N-1}\right\| + (1+r_n^N)\left\|x_n^{N-1} - x_n\right\| \\
&\leq \left\|x_n - T_N(PT_N)^{n-1}x_n^{N-1}\right\| \\
&\quad +(1+r_n^N)\left\|\alpha_n^{N-1}T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} + \beta_n^{N-1}x_n + \gamma_n^{N-1}u_n^{N-1} - x_n\right\| \\
&\leq \left\|x_n - T_N(PT_N)^{n-1}x_n^{N-1}\right\| \\
&\quad +(1+r_n^N)[\alpha_n^{N-1}\left\|T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2} - x_n\right\| + \gamma_n^{N-1}\left\|u_n^{N-1} - x_n\right\|]
\end{aligned}
$$

since

$$\lim_{n\to\infty}\left\|x_n - T_N(PT_N)^{n-1}x_n^{N-1}\right\| = 0,\ \ \lim_{n\to\infty}\left\|x_n - T_{N-1}(PT_{N-1})^{n-1}x_n^{N-2}\right\| = 0$$

and $\sum_{n=1}^{\infty}\gamma_n^{N-1} < \infty$, $\sum_{n=1}^{\infty}r_n^N < \infty$, it follows that

$$\lim_{n\to\infty}\left\|x_n - T_N(PT_N)^{n-1}x_n\right\| = 0.$$

Similarly, by using the same argument as in the proof above we have

$\lim_{n\to\infty}\left\|x_n - T_{N-2}(PT_{N-2})^{n-1}x_n^{N-3}\right\| = \lim_{n\to\infty}\left\|x_n - T_{N-3}(PT_{N-3})^{n-1}x_n^{N-4}\right\|$
$=, \ldots, = \lim_{n\to\infty}\left\|x_n - T_2(PT_2)^{n-1}x_n^1\right\| = 0.$

This implies that

$\lim_{n\to\infty}\left\|x_n - T_{N-1}(PT_{N-1})^{n-1}x_n\right\| = \lim_{n\to\infty}\left\|x_n - T_{N-2}(PT_{N-2})^{n-1}x_n\right\|$
$=, \ldots, = \lim_{n\to\infty}\left\|x_n - T_3(PT_3)^{n-1}x_n\right\| = 0.$

It remains to show that

$$\lim_{n\to\infty}\left\|x_n - T_1(PT_1)^{n-1}x_n\right\| = 0, \qquad \lim_{n\to\infty}\left\|x_n - T_2(PT_2)^{n-1}x_n\right\| = 0.$$

Note that

$$
\begin{aligned}
&\left\|x_n^1 - x^*\right\|^2 \\
\leq\ &\alpha_n^1(\left\|T_1(PT_1)^{n-1}x_n - x^*\right\| + \gamma_n^1\left\|u_n^1 - x^*\right\|)^2 + (1-\alpha_n^1) \\
&(\|x_n - x^*\| + \gamma_n^1\left\|u_n^1 - x^*\right\|)^2 - W_2(\alpha_n^1)g(\left\|T_1(PT_1)^{n-1}x_n - x_n\right\|) \\
\\
\leq\ &\alpha_n^1[(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\left\|u_n^1 - x^*\right\|]2 + (1-\alpha_n^1) \\
&[(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\left\|u_n^1 - x^*\right\|]^2 - W_2(\alpha_n^1)g(\left\|T_1(PT_1)^{n-1}x_n - x_n\right\|) \\
\\
\leq\ &[(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\left\|u_n^1 - x^*\right\|]^2 \\
&-W_2(\alpha_n^1)g(\|T_1(PT_1)^n x_{n-1} - x_n\|) \\
\\
\leq\ &[\|x_n - x^*\| + \gamma_n^1\left\|u_n^1 - x^*\right\|]^2 - W_2(\alpha_n^1)g(\left\|T_1(PT_1)^{n-1}x_n - x_n\right\|)
\end{aligned}
$$

Thus, we have $\varepsilon^3 g(\left\|T_1(PT_1)^{n-1}x_n - x_n\right\|) \leq [\|x_n - x^*\| + \gamma_n^1 \left\|u_n^1 - x^*\right\|]^2 - \left\|x_n^1 - x^*\right\|^2$ and therefore $\lim_{n\to\infty} \left\|T_1(PT_1)^{n-1}x_n - x_n\right\| = 0$.

Since

$$\begin{aligned}
& \left\|x_n - T_2(PT_2)^{n-1}x_n\right\| \\
\leq\ & \left\|x_n - T_2(PT_2)^{n-1}x_n^1\right\| + \left\|T_2(PT_2)^{n-1}x_n^1 - T_2(PT_2)^{n-1}x_n\right\| \\
\leq\ & \left\|x_n - T_2(PT_2)^{n-1}x_n^1\right\| + (1 + r_n^2)\left\|x_n^1 - x_n\right\| \\
\leq\ & \left\|x_n - T_2(PT_2)^{n-1}x_n^1\right\| \\
& + (1 + r_n^2)\left\|\alpha_n^1 T_1(PT_1)^{n-1}x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - x_n\right\| \\
\leq\ & \left\|x_n - T_2(PT_2)^{n-1}x_n^1\right\| \\
& + (1 + r_n^2)[\alpha_n^1 \left\|T_1(PT_1)^{n-1}x_n - x_n\right\| + \gamma_n^1 \left\|u_n^1 - x_n\right\|]
\end{aligned}$$

it implies that $\lim_{n\to\infty} \left\|T_2(PT_2)^{n-1}x_n - x_n\right\| = 0$. Therefore

$$\lim_{n\to\infty} \left\|T_i(PT_i)^{n-1}x_n - x_n\right\| = 0$$

for all $i = 1, 2, \ldots, N$.

On the other hand, by Remark 2.3, it is clear that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Therefore, by Lemma 2.2, we can conclude that $\|x_n - T_i x_n\| = 0$ as $n \to \infty$. This completes the proof.

Theorem 2.6: Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2, \ldots,$
$T_N\colon K \longrightarrow K$ be N asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leq i \leq N$ and $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence defined by (6) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$, for some $\varepsilon \in (0,1)$. Then for all $u, v \in F$, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$$

exists for all $t \in [0,1]$.

Proof: By Lemma 2.2(i) and by Lemma 1.1, we have $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. This implies that $\{x_n\}$ is bounded. Observe that there exists $R > 0$ such that $\{x_n\} \subset C = B_R(0) \cap K$, where $B_R(0) = \{x \in E : \|x\| \leq R\}$. Then C is a nonempty closed convex bounded subset of E. Let $a_n(t) = \|tx_n + (1-t)u - v\|$. Then $\lim_{n\to\infty} a_n(0) = \|u - v\|$ and from Lemma 2.2(i) and 1.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - v\|$ exists. Without loss of generality, we may assume that $\lim_{n\to\infty} \|x_n - u\| = r > 0$ and $t \in (0,1)$.

For any $n \geq 1$ and for all $i = 1, 2, \ldots, N$, we define $A_n^i\colon C \to C$ by

$$A_n^1 = P(\alpha_n^1 T_1(PT_1)^{n-1} + \beta_n^1 I + \gamma_n^1 u_n^1)$$

$$\begin{aligned}
A_n^2 &= P(\alpha_n^2 T_2 (PT_2)^{n-1} A_n^1 + \beta_n^2 I + \gamma_n^2 u_n^2) \\
..... &= .... \\
..... &= .... \\
A_n^N &= P(\alpha_n^N T_N (PT_N)^{n-1} A_n^{N-1} + \beta_n^N I + \gamma_n^N u_n^N)
\end{aligned}$$

Thus for all $x, y \in K$, we have

$$\begin{aligned}
& \left\| A_n^i x - A_n^i y \right\| \\
= & \left\| P(\alpha_n^i T_i (PT_i)^{n-1} A_n^{i-1} x + \beta_n^i x + \gamma_n^i u_n^i) \right. \\
& \left. - P(\alpha_n^i T_i (PT_i)^{n-1} A_n^{i-1} y + \beta_n^i y + \gamma_n^i u_n^i) \right\| \\
\leq & \left\| (\alpha_n^i T_i (PT_i)^{n-1} A_n^{i-1} x + \beta_n^i x + \gamma_n^i u_n^i) - (\alpha_n^i T_i (PT_i)^{n-1} A_n^{i-1} y + \beta_n^i y + \gamma_n^i u_n^i) \right\| \\
\leq & \alpha_n^i \left\| T_i (PT_i)^{n-1} A_n^{i-1} x \right. \\
& \left. - T_i (PT_i)^{n-1} A_n^{i-1} y \right\| + \beta_n^i \left\| x - y \right\| \\
\leq & \alpha_n^i (1 + r_n^i) \left\| A_n^{i-1} x - A_n^{i-1} y \right\| + \beta_n^i \left\| x - y \right\| \\
\leq & (1 + r_n^i) \left\| A_n^{i-1} x - A_n^{i-1} y \right\| + \left\| x - y \right\| \\
\leq & k_n^i \left\| A_n^{i-1} x - A_n^{i-1} y \right\| + \left\| x - y \right\|
\end{aligned}$$

where $k_n^i = (1 + r_n^i)$ for all $i = 2, 3, \ldots, N$. Since $\sum_{n=1}^{\infty} r_n^i < \infty$ for all $i = 1, 2, \ldots, N$, then $\prod_{i=1}^{N} k_n^i < \infty$, and

$$\begin{aligned}
\left\| A_n^1 x - A_n^1 y \right\| &\leq \alpha_n^1 (1 + r_n^1) \left\| x - y \right\| + \beta_n^1 \left\| x - y \right\| \\
&\leq [\alpha_n^1 (1 + r_n^1) + \beta_n^1] \left\| x - y \right\| \\
&\leq [\alpha_n^1 + \beta_n^1 + \alpha_n^1 r_n^1] \left\| x - y \right\| \\
&\leq (1 + r_n^1) \left\| x - y \right\| = k_n^1 \left\| x - y \right\| .
\end{aligned}$$

This implies that

$$\begin{aligned}
\left\| A_n^N x - A_n^N y \right\| &\leq \alpha_n^N \left\| T_N (PT_N)^{n-1} A_n^{N-1} x - T_N (PT_N)^{n-1} A_n^{N-1} y \right\| + \beta_n^N \left\| x - y \right\| \\
&\leq \alpha_n^N (1 + r_n^N) \left\| A_n^{N-1} x - A_n^{N-1} y \right\| + \beta_n^N \left\| x - y \right\| \\
&\leq (1 + r_n^N) \left\| A_n^{N-1} x - A_n^{N-1} y \right\| + \left\| x - y \right\| \\
&\leq k_n^N \left\| A_n^{N-1} x - A_n^{N-1} y \right\| + \left\| x - y \right\| , \ \textit{where} \ k_n^N = (1 + r_n^N) \\
&\leq k_n^N [k_n^{N-1} \left\| A_n^{N-2} x - A_n^{N-2} y \right\| + \left\| x - y \right\|] + \left\| x - y \right\| \\
&\leq k_n^N k_n^{N-1} \left\| A_n^{N-2} x - A_n^{N-2} y \right\| + (1 + k_n^N) \left\| x - y \right\| \\
&\leq .... \\
&\leq ....
\end{aligned}$$

$$\begin{aligned}
&\leq k_n^N k_n^{N-1} \ldots k_n^2 \left\| A_n^1 x - A_n^1 y \right\| + [1 + k_n^N + k_n^N k_n^{N-1} + \\
&\quad \ldots + k_n^N k_n^{N-1} \ldots k_n^2] \left\| x - y \right\| \\
&\leq k_n^N k_n^{N-1} \ldots k_n^2 k_n^1 \left\| x - y \right\| + [1 + k_n^N + k_n^N k_n^{N-1} + \\
&\quad \ldots + k_n^N k_n^{N-1} \ldots k_n^2] \left\| x - y \right\| \\
&\leq (\prod_{i=1}^{N} k_n^i) \left\| x - y \right\| + [1 + (\prod_{i=N}^{N} k_n^i) + (\prod_{i=N-1}^{N} k_n^i) + \\
&\quad \ldots + (\prod_{i=2}^{N} k_n^i)] \left\| x - y \right\| \\
&\leq [\prod_{i=1}^{N} k_n^i + \prod_{i=2}^{N} k_n^i + \ldots + \prod_{i=N}^{N} k_n^i + 1] \left\| x - y \right\| \\
&\leq [\delta_n^1 + \delta_n^2 + \ldots + \delta_n^N + 1] \left\| x - y \right\|, \ \textit{where} \ \delta_n^j = \prod_{i=j}^{N} k_n^i \\
&\leq (\sum_{p=0}^{N} \delta_n^p) \left\| x - y \right\| \\
&\leq \left\| x - y \right\|
\end{aligned}$$

that is

$$\left\| A_n^N x - A_n^N y \right\| \leq \left\| x - y \right\|.$$

Set $S_{n,m} = A_{n+m-1}^N A_{n+m-2}^N \ldots A_n^N$, $m \geq 1$, and

$b_{n,m} = \|S_{n,m}(tx_n + (1-t)u) - (tS_{n,m}x_n + (1-t)S_{n,m}u)\|$.

It is easy to see that $A_n^N x_n = x_{n+1}$, $S_{n,m}x_n = x_{n+m}$ and $\|S_{n,m}x - S_{n,m}y\| \leq \|x - y\|$.

We show first that, for any $x^* \in F$, $\|S_{n,m}x^* - x^*\| \to 0$ uniformly for all $m \geq 1$ as $n \to \infty$. Indeed, for any $x^* \in F$, we have

$$\begin{aligned}
\left\| A_n^i x^* - x^* \right\| &\leq \alpha_n^i \left\| T_i(PT_i)^{n-1} A_n^{i-1} x^* - x^* \right\| + \gamma_n^i \left\| u_n^i - x^* \right\| \\
&\leq \alpha_n^i (1 + r_n^i) \left\| A_n^{i-1} x^* - x^* \right\| + \gamma_n^i \left\| u_n^i - x^* \right\| \\
&\leq (1 + r_n^i) \left\| A_n^{i-1} x^* - x^* \right\| + \left\| u_n^i - x^* \right\| \\
&\leq k_n^i \left\| A_n^{i-1} x^* - x^* \right\| + \left\| u_n^i - x^* \right\|, \ \textit{where} \ k_n^i = (1 + r_n^i)
\end{aligned}$$

for all $i = 2, 3, \ldots, N$, and $k_n^i = (1 + r_n^i)$ for all $i = 1, 2, \ldots, N$ and

$$\begin{aligned}
\left\| A_n^1 x^* - x^* \right\| &\leq \gamma_n^1 \left\| u_n^1 - x^* \right\| \\
&\leq \left\| u_n^1 - x^* \right\|.
\end{aligned}$$

Therefore

$$
\begin{aligned}
\left\|A_n^N x^* - x^*\right\| &\leq \alpha_n^N \left\|T_N(PT_N)^{n-1}A_n^{N-1}x^* - x^*\right\| + \gamma_n^N \left\|u_n^N - x^*\right\| \\
&\leq \alpha_n^N(1+r_n^N)\left\|A_n^{N-1}x^* - x^*\right\| + \gamma_n^N \left\|u_n^N - x^*\right\| \\
&\leq (1+r_n^N)\left\|A_n^{N-1}x^* - x^*\right\| + \left\|u_n^N - x^*\right\| \\
&\leq k_n^N \left\|A_n^{N-1}x^* - x^*\right\| + \left\|u_n^N - x^*\right\| \\
&\leq \delta_n^1 \left\|u_n^1 - x^*\right\| + \delta_n^2 \left\|u_n^2 - x^*\right\| + \ldots \\
&\quad + \delta_n^N \left\|u_n^{N-1} - x^*\right\| + \left\|u_n^N - x^*\right\| \\
&\leq M \sum_{p=0}^{N} \delta_n^p
\end{aligned}
$$

for all $n \geq 1$, where $M = \max\{\sup_{n\geq 1}\{\left\|u_n^1 - x^*\right\|\}, \ldots, \sup_{n\geq 1}\{\left\|u_n^N - x^*\right\|\}\}$ and $\delta_n^j = \prod_{i=j}^{N} k_n^i$ such that $\sum_{n=1}^{\infty} \delta_n^j < \infty$. Hence

$$
\begin{aligned}
\|S_{n,m}x^* - x^*\| &= \left\|A_{n+m-1}^N A_{n+m-2}^N \ldots A_n^N x^* - x^*\right\| \\
&\leq \left\|A_{n+m-1}^N A_{n+m-2}^N \ldots A_n^N x^* - A_{n+m-1}^N A_{n+m-2}^N \ldots A_{n+1}^N x^*\right\| \\
&\quad + \left\|A_{n+m-1}^N A_{n+m-2}^N \ldots A_{n+1}^N x^* - A_{n+m-1}^N A_{n+m-2}^N \ldots A_{n+2}^N x^*\right\| \\
&\quad + \ldots + \left\|A_{n+m-1}^N x^* - x^*\right\| \\
&\leq \left\|A_n^N x^* - x^*\right\| + \left\|A_{n+1}^N x^* - x^*\right\| + \ldots + \left\|A_{n+m-1}^N x^* - x^*\right\| \\
&\leq M \sum_{p=0}^{N} \delta_n^p + M \sum_{p=0}^{N} \delta_{n+1}^p + \ldots + M \sum_{p=0}^{N} \delta_{n+m-1}^p \\
&\leq M \sum_{l=n}^{n+m-1} (\sum_{p=0}^{N} \delta_l^p) = \xi_n^{x^*}
\end{aligned}
$$

since $\sum_{l=1}^{\infty} \delta_l^p < \infty$, we have $\xi_n^{x^*} \to 0$ as $n \to \infty$ and hence $\|S_{n,m}x^* - x^*\| \to 0$ as $n \to \infty$. Observe that

$$
\begin{aligned}
a_{n+m}(t) &= \|tS_{n,m}x_n + (1-t)u - v\| \\
&\leq \|tS_{n,m}x_n + (1-t)u - S_{n,m}(tx_n + (1-t)u)\| \\
&\quad + \|S_{n,m}(tx_n + (1-t)u) - v\| \\
&\leq \|tS_{n,m}x_n + (1-t)S_{n,m}u - S_{n,m}(tx_n + (1-t)u) + (1-t)(u - S_{n,m}u)\| \\
&\quad + \|S_{n,m}(tx_n + (1-t)u) - v\| \\
&\leq b_{n,m} + (1-t)\|u - S_{n,m}u\| + \|S_{n,m}(tx_n + (1-t)u) - v\| \\
&\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)u) - S_{n,m}v\| + \|S_{n,m}v - v\| \\
&\quad + (1-t)\|u - S_{n,m}u\| \\
&\leq b_{n,m} + \|tx_n + (1-t)u - v\| + \|S_{n,m}v - v\| + (1-t)\|u - S_{n,m}u\| \\
&\leq b_{n,m} + a_n(t) + \xi_n^v + (1-t)\xi_n^u \qquad (*)
\end{aligned}
$$

By using [[9], Theorem 2.3], we have

$$\begin{aligned} b_{n,m} &\le \varphi^{-1}(\|x_n-u\|-\|S_{n,m}x_n-S_{n,m}u\|) \\ &\le \varphi^{-1}(\|x_n-u\|-\|x_{n+m}-u+u-S_{n,m}u\|) \\ &\le \varphi^{-1}(\|x_n-u\|-(\|x_{n+m}-u\|-\|S_{n,m}u-u\|)) \end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0 as $n\to\infty$ for all $m\ge 1$. Thus, fixing n and letting $m\to\infty$ in $(*)$, we have

$$\limsup_{m\to\infty} a_{n+m}(t) \le \varphi^{-1}(\|x_n-u\|-(\lim_{m\to\infty}\|x_m-u\|-\xi_n^u))+a_n(t)+\xi_n^v+(1-t)\xi_n^u$$

and again letting $n\to\infty$

$$\limsup_{n\to\infty} a_n(t) \le \varphi^{-1}(0)+\liminf_{n\to\infty} a_n(t)+0+0=\liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty}\|tx_n+(1-t)u-v\|$$

exists for all $t\in[0,1]$. This completes the proof.

Theorem 2.7: Let E be a real uniformly convex Banach space such that its dual E^* has the *Kadec-Klee* property and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1,T_2,\dots,T_N\colon K\longrightarrow K$ be N asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^\infty r_n^i<\infty$, for all $1\le i\le N$ and $F=\cap_{i=1}^N F(T_i)\ne\phi$. Let $\{x_n\}$ be the sequence defined by (6) with $\sum_{n=1}^\infty \gamma_n^i<\infty$ and $\{\alpha_n^i\}\subseteq[\varepsilon,1-\varepsilon]$ for all $i=1,2,\dots,N$, for some $\varepsilon\in(0,1)$. Then $\{x_n\}$ converges weakly to some common fixed point $x^*\in F$.

Proof: By Lemma 2.2 (i) and 1.1, we have $\lim_{n\to\infty}\|x_n-x^*\|$ exists for all $x^*\in F$. This implies that $\{x_n\}$ is bounded. Since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to some $x^*\in K$. By Theorem 2.5, we have $\lim_{j\to\infty}\left\|x_{n_j}-T_ix_{n_j}\right\|=0$ for all $i=1,2,\dots,N$. Now Lemma 1.3 guarantees that $I-T_i$ is demiclosed at zero for all $i=1,2,\dots,N$. This implies that $T_ix^*=x^*$ for all $i=1,2,\dots,N$, hence this means that $x^*\in F$. It remains to show that $\{x_n\}$ converges weakly to x^*. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some y^*. Then $y^*\in K$ and so $x^*,y^*\in\omega_w(x_n)\cap F$. By Theorem 2.6, the limit

$$\lim_{n\to\infty}\|tx_n+(1-t)x^*-y^*\|$$

exists for all $t\in[0,1]$. By Lemma 1.4, we have $x^*=y^*$. As a result, $\omega_w(x_n)\cap F$ is a singleton, and so $\{x_n\}$ converges weakly to some common fixed point $x^*\in F$.

This completes the proof.

Theorem 2.8: Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2, \ldots,$
$T_N\colon K \longrightarrow K$ be N asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leq i \leq N$ and $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Suppose $\{T_1, T_2, \ldots, T_N\}$ satisfies condition (B). Let $\{x_n\}$ be the sequence defined by (6) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$, for some $\varepsilon \in (0,1)$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.

Proof: From Lemma 2.2(i) and 1.1 we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F = \cap_{i=1}^{N} F(T_i)$. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$ for some $a \geq 0$. Without loss of generality, if $a = 0$, then there is nothing to prove. Assume that $a > 0$, as proved in Lemma 2.2(i), we have

$$\|x_{n+1} - x^*\| \leq (1 + b_n^{N-1}) \|x_n - x^*\| + d_n^{N-1}, \qquad \forall n \in N$$

where $\{b_n^i\}_{n=1}^{\infty}$ and $\{d_n^i\}_{n=1}^{\infty}$ for all $i = 1, 2, \ldots, N$ are nonnegative real sequences such that $\sum_{n=1}^{\infty} b_n^i < \infty$ and $\sum_{n=1}^{\infty} d_n^i < \infty$ for all $i = 1, 2, \ldots, N$. This gives that

$$d(x_{n+1}, F) \leq (1 + b_n^{N-1}) d(x_n, F) + d_n^{N-1}, \qquad \forall n \in N.$$

Applying Lemma 1.1 to the above inequality, we obtained that $\lim_{n\to\infty} d(x_n, F)$ exists. Also by Theorem 2.5, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$. Since $\{T_1, T_2, \ldots, T_N\}$ satisfies condition (B), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, given any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \frac{\varepsilon}{3}$ for all $n \geq n_0$. So we can find $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$. For all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq \|x_{n_0} - p^*\| + \|x_{n_0} - p^*\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_1 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_1$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a natural number $n_2 \geq n_1$ such that for all $n \geq n_2$ we have $d(x_n, F) < \frac{\varepsilon_1}{5}$ and in particular, we have $d(x_{n_2}, F) < \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in F$ such that $\|x_{n_2} - w^*\| < \frac{\varepsilon_1}{4}$. For any $i \in I$ and

$n \geq n_2$, we have

$$\begin{aligned} \|T_i q^* - q^*\| &\leq \|T_i q^* - w^*\| + \|w^* - q^*\| \\ &\leq 2\,\|q^* - w^*\| \\ &\leq 2(\|q^* - x_{n_2}\| + \|x_{n_2} - w^*\|) \\ &< 2(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}) \\ &< \varepsilon_1. \end{aligned}$$

This implies that $T_i q^* = q^*$. Hence $q^* \in F(T_i)$ for all $i \in I$ and so $q^* \in F = \cap_{i=1}^N F(T_i)$. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$. This completes the proof.

Remark 2.9: Theorem 2.5 to 2.8 extend the corresponding results of Su and Qin [28] to the case of multistep iterative sequences with errors for a finite family of asymptotically nonexpansive nonself mappings and also they extend many known results.

Remark 2.10: Our results also extend the corresponding results of Plubtieng and Ungchittrakool [22] to more general class of nonexpansive nonself mappings.

Remark 2.11: Our results also extend the corresponding results of Shahzad [18] to the case of multistep iterative sequences with errors for a finite family of more general class of nonexpansive mappings.

References

[1] B.E. Rhoades, *Fixed point iteration for certain nonlinear mappings*, J. Math. Anal. Appl. 183(1994), 118-120.

[2] C.E. Chidume, *On the approximation of fixed points of nonexpansive mappings*, Houston J. Math. 7(1981), 345-355.

[3] C.E. Chidume, *Nonexpansive mappings, generalizations and iterative algorithms.* In: Agarwal R.P., O'Reagan D.eds. Nonlinear Analysis and Application. To V. Lakshmikantam on his 80th Birthday (Research Monograph), Dordrecht: Kluwer Academic Publishers, pp. 383-430.

[4] C.E. Chidume; E.U. Ofoedu and H. Zegeye, *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, J. of Mathematical Analysis and Applications, 280(2003), 364-374.

[5] C.E. Chidume; N. Shahzad and H. Zegeye, *Strong convergence theorems for nonexpansive mappings in arbitrary Banach spaces*, Nonlinear Anal. Submitted.

[6] G.B. Passty, *Construction of fixed points for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 84(1982), 212-216.

[7] H.F. Senter and W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. 44(1974), 375-380.

[8] H.K. Xu, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Analysis, 16(1991), 1139-1146.

[9] J. Garcia Falset; W. Kaczor; T. Kuczumow and S. Reich, *Weak convergence theorems for asymptotically nonexpansive mappings and semigroups*, Nonlinear Analysis. Theory, Methods and Applications 43(2001), no.3, 377-401.

[10] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 158(1991), 407-413.

[11] J. Schu, *Weak and strong convergence theorems to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. 43(1991), 153-159.

[12] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35(1972), 171-174.

[13] K.K. Tan and H.K. Xu, *A nonlinear ergodic theorem for asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. 45(1992), 25-36.

[14] K.K. Tan and H.K. Xu, *The nonlinear ergodic theorem for asymptotically nonexpansive mapping in Banach spaces*, Proc. Amer. Math. Soc. 114(1992), 399-404.

[15] K.K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178(1993), 301-308.

[16] K.K. Tan and H.K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 122(1994), 733-739.

[17] M.O. Osilike and S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Comput. Modelling 32(2000), 1181-1191.

[18] N. Shahzad, *Approximating fixed points of non-self nonexpansive mappings in Banach spaces*, Nonlinear Analysis 61(2005), no.6, 1031-1039.

[19] S.C. Bose, *Weak convergence to a fixed point of an asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 68(1978), 305-308.

[20] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. 44(1974), 147-150.

[21] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. 59(1976), 65-71.

[22] S. Plubtieng and K. Ungchittrakool, *Weak and strong convergence of finite family with errors of nonexpansive non-self mappings*, Fixed Point Theory and Applications, Volume 2006, Article ID 81493, pages 1-12.

[23] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 67(1979), 274-276.

[24] S.S. Chang; Y.J. Cho and H. Zhou, *Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings*, J. Korean Math. Soc. 38(6) (2001), 1245-1260.

[25] W. Kaczor, *Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups*, J. Mathematical Analysis and Applications 272(2002), no.2, 565-574.

[26] W.J. Davis and P. Enflo, *Contractive projections on l_p spaces*, Analysis at Urbana, Vol.I (Urbana, IL, 1986-1987), Londan Math. Soc. Lecture Note Ser., vol. 137, Cambridge University Press, Cambridge, 1989, pp. 151-161.

[27] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4(1953), 506-510.

[28] Yongfu Su and Xiaolong Qin, *Approximating fixed points of non-self asymptotically nonexpansive mappings in Banach spaces*, J. Applied Mathematics and Stochastic Analysis, Volume 2006, Article ID 21961, pages 1-13.

[29] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73(1967), 591-597.

Department of Mathematics & I.T., Govt. College of Science, Raipur (C.G.), India

E-mail: `saluja_1963@rediffmail.com`