Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **22:2** (2008), 53–57

AN EXAMPLE OF USING STAR COMPLEMENTS IN CLASSIFYING STRONGLY REGULAR GRAPHS*

Marko Milošević

Abstract

In this paper we show how the star complement technique can be used to reprove the result of Wilbrink and Brouwer that the strongly regular graph with parameters $(57, 14, 1, 4)$ does not exist.

1 Introduction

Let $G = (V, E)$ be a finite, undirected, simple graph. For two vertices u, v, we write $u \sim v$ if they are adjacent in G. The *neighborhood* $N(u)$ of u is the set of neighbors of u. The *closed neighborhood* $N[u]$ is the set $N(u) \cup u$.

The graph G is a *strongly regular* with parameters (n, k, λ, μ) if G is k-regular on n vertices, such that any two adjacent vertices have λ common neighbors, and any two nonadjacent vertices have μ common neighbors. Obviously, in such a graph the neighborhood of each vertex induces a λ–regular graph on k vertices.

In 1983, Wilbrink and Brouwer proved that the strongly regular graph with parameters $(57, 14, 1, 4)$ does not exist. Here, a self-contained proof of the nonexistence of this graph is given, using linear algebra and spectral graph theory, more precisely the technique of star complements. This technique was developed by Cvetković, Rowlinson and Simić in a series of papers (see, e.g., [1], [2], [3], [5], [6]).

Let ξ be an eigenvalue of G with multiplicity m. A *star set* for ξ in G is a set $X \subset V(G)$ of m vertices such that ξ is not an eigenvalue of $G - X$, the subgraph of G induced by $\overline{X} = V(G) \setminus X$. The graph $G - X$ is called a *star complement* for ξ in G. If X is a star set for an eigenvalue $\xi \notin \{-1, 0\}$ of G, then $\overline{X}$ is a location-dominating set in G, meaning that the $\overline{X}$–neighborhoods of vertices in X are distinct and nonempty [1].

The following theorem is known as the *Reconstruction Theorem*.

*This work was supported by the research grant 144026 of the Serbian Ministry of Science and Environmental Protection.

2000 *Mathematics Subject Classifications.* 05E30, 05C50.

Key words and Phrases. Strongly regular graph; Star complement; Graph eigenvalues; Graph.

Received: July 27, 2008

Communicated by Dragan Stevanović

Theorem 1 ([1]) *Let X be a set of vertices in graph G and suppose that G has adjacency matrix*

$$\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$$

where A_X is adjacency matrix of the subgraph induced by X. Then X is a star set for ξ in G if and only if ξ is not an eigenvalue of C and

$$\xi I - A_X = B^T(\xi I - C)^{-1}B. \tag{1}$$

Thus, if we know ξ, B and C, we can reconstruct the whole graph G. If we denote the columns of B by $\mathbf{b}_u$ ($u \in X$) and equate corresponding matrix entries in (1), we obtain the following result

Corollary 2 ([5]) *If X is a star set for ξ then $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \xi$, for all $u \in X$, and $\langle \mathbf{b}_u, \mathbf{b}_v \rangle \in \{-1, 0\}$ where $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \mathbf{b}_u^T \left(\xi I - C\right)^{-1} \mathbf{b}_v$. If $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = -1$, then $u \sim v$, and if $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = 0$, then $u \not\sim v$.*

One can now define the *compatibility graph* $Comp(C, \xi)$ having as vertices all $(0,1)$–vectors $\mathbf{b}$ which satisfy $\langle \mathbf{b}, \mathbf{b} \rangle = \xi$, with two vertices $\mathbf{b}'$ and $\mathbf{b}''$ adjacent if and only if $\langle \mathbf{b}', \mathbf{b}'' \rangle \in \{-1, 0\}$. Then, for each graph G that has $G - X$ as a star complement for ξ, there is a clique in $Comp(C, \xi)$ that completely determines G.

As for the strongly regular graph with parameters $(57, 14, 1, 4)$, these parameters determine the spectrum $[14, 2^{38}, -5^{18}]$, where exponents denote multiplicity. Therefore, to apply the star complements technique, one has to find an induced subgraph on 19 vertices that does not have 2 as an eigenvalue.

Let G be the strongly regular graph with parameters $(57, 14, 1, 4)$, and let u be an arbitrary vertex of G. The closed neighborhood $N[u]$ induces the windmill W_{14}. Each vertex $v \notin N[u]$ has exactly four neighbors in common with the vertex u. Any two connected vertices from $N(u)$ have u as their unique common neighbor, so they do not have any more common neighbors. Therefore we may assume, without loss of generality, that the graph H induced by $N[u] \cup \{v\}$, where v is some arbitrary, but a fixed vertex from $V(G) \setminus N[u]$, is like the one on the Figure 1.

Figure 1: Induced subgraph of G.

Graph H is a 16–vertex graph that does not have 2 as an eigenvalue. Lemma 3 from the next section enables expanding of the graph H with three additional vertices to get all of the possible star complements (which contain H as an induced subgraph) in G for the eigenvalue 2. In Section 3, it is shown that none of the possible star complements gives a rise to a desired strongly regular graph, thus proving that the graph G does not exist. These results were obtained using a computer.

2 Extending to a star complement

It this section we show that any induced subgraph that does not have the eigenvalue ξ can be extended to a star complement for the eigenvalue ξ. The lemma was proved by D. Stevanović and M. Milošević.

Lemma 3 *Let ξ be an eigenvalue of G and let H be an induced subgraph of G such that ξ is not an eigenvalue of H. Then there exists a star complement H' for ξ in G such that $H \subseteq H'$.*

Proof: Let $\mathcal{E}(\xi)$ be the eigenspace of ξ in G, and let P be the orthogonal projection of $\mathbf{R}^n$ onto $\mathcal{E}(\xi)$ with respect to the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$. Further, let $X = G - H$.

Following the proof of Theorem 7.2.3 in [1], we can show that $\langle Pe_j : j \in X \rangle = \mathcal{E}(\xi)$. Suppose, on the contrary, that $\langle Pe_j : j \in X \rangle \subset \mathcal{E}(\xi)$. Then, there is a non-zero vector $x \in \mathcal{E}(\xi) \cap \langle Pe_j : j \in X \rangle^{\perp}$. Thus, $x^T Pe_j = 0$ for all $j \in X$. Hence $(Px)^T e_j = (x^T P)e_j = 0$ for all $j \in X$. Consequently, $Px \in \langle e_j : j \in X \rangle^{\perp} = \langle e_s : s \notin X \rangle$. Since $x = Px$, we have non-zero $x \in \mathcal{E}(\xi) \cap \langle e_s : s \notin X \rangle$.

From $x = \begin{pmatrix} 0 \\ x' \end{pmatrix}$, with $x' \neq 0$, it follows that x' is an eigenvector of $G - X = H$, a contradiction.

Thus, there exists a subset $X' \subseteq X$ such that the vectors $\{Pe_j : e_j \in X'\}$ form a basis for $\mathcal{E}(\xi)$. In such case, from Theorem 7.2.9 in [1] it follows that $|X'| = \dim \mathcal{E}(\xi)$ and ξ is not an eigenvalue of $G - X' = H'$. Thus, H' is a star complement for ξ which contains H as an induced subgraph. ■

3 The nonexistence

According to Lemma 3, to classify strongly regular graphs with parameters $(57, 14, 1, 4)$, it is sufficient to extend graph H (Figure 1) with three additional vertices in all possible ways so that the second largest eigenvalue of the resulting graph is strictly lower than 2, and then examine the compatibility graphs that arise from these star complements.

So, in each of the three steps we add a new vertex that has four neighbors in common with vertex u, and we do this in all possible ways, preserving the conditions that any two adjacent vertices do not have more than one common neighbor, and that any two non-adjacent vertices do not have more than four common neighbors.

After that we get one representative of each isomorphism class, and discard those graphs that have the second largest eigenvalue greater or equal to 2.

This way, the graph H can be extended with three vertices as described to get 3720 non-isomorphic graphs. These graphs represent potential star complements in G for the eigenvalue 2. To each of them we apply the Reconstruction theorem. We do not actually need to create the whole compatibility graph since the conditions on common neighbors count must be satisfied. Therefore we only consider those $(0, 1)$–vectors that do not violate the conditions for strong regularity, i.e. we work with an induced subgraph of each compatibility graph, but we still call these subgraphs the compatibility graphs, for the ease of notation.

Sizes of the compatibility graphs vary from 4 to 265 vertices. We have used *Cliquer* [4] to determine the largest cliques in these compatibility graphs. Summary of the results is given in the following table.

The largest clique size	Number of compatibility graphs
2	6
3	2
4	13
5	32
6	18
7	173
8	358
9	403
10	131
11	220
12	502
13	400
14	58
15	123
16	303
29	19
30	49
31	910

Table 1: The largest cliques in compatibility graphs.

As we can see from Table 1, none of these compatibility graphs contains the clique of size 38 which is needed to reconstruct the graph on 57 vertices.

Thus, we conclude that

Theorem 4 *The strongly regular graph with parameters* $(57, 14, 1, 4)$ *does not exist.*

References

[1] D. Cvetković, P. Rowlinson, S. Simić, Eigenspaces of graphs, Cambridge University Press, Cambridge, 1997.

[2] D. Cvetković, P. Rowlinson, S. Simić, *Some characterizations of graphs by star complements*, Linear Algebra Appl. 301 (1999), 81–97.

[3] D. Cvetković, P. Rowlinson, S. Simić, Spectral generalizations of line graphs, Cambridge University Press, Cambridge, 2004.

[4] P. Östergård, *Cliquer*, http://users.tkk.fi/~pat/cliquer.html

[5] P. Rowlinson, *Co-cliques and star complements in extremal strongly regular graphs*, Linear Algebra and its Appl. 421 (2007), 157–162.

[6] P. Rowlinson, *Star sets in regular graphs*, J. Comb. Math. Comb. Comput. 34 (2000), 3–22.

[7] D. Stevanović, M. Milošević, *A spectral proof of the uniqueness of a strongly regular graph with parameters* $(81, 20, 1, 6)$, Eur. J. Comb., to appear

[8] H.A. Wilbrink, A.E. Brouwer, *A* $(57, 14, 1, 4)$ *strongly regular graph does not exist*, Indag. Math. 45 (1983), 117–121

Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mail: ninja643@gmail.com