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SOME NEW SEQUENCE SPACES AND ALMOST CONVERGENCE

Sameer Ahmad Gupkari

Abstract. The sequence space a_c^r have been defined and the classes $\left(a_c^r : \ell_p\right)$ and $\left(a_c^r : c\right)$ of infinite matrices have been characterized by Aydin and Başar (On the new sequence spaces which include the spaces c_0 and c, Hokkaido Math. J. 33(2) (2004), 383-398) [1], where $1 \leq p \leq \infty$. The main purpose of the present paper is to characterize the classes $\left(a_c^r : f\right)$ and $\left(a_c^r : f_0\right)$, where f and f_0 denote the spaces of almost convergent and almost convergent null sequences with real or complex terms.

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1. Introduction

Let ω be the space of all sequences, real or complex and let ℓ_∞ and c respectively be the Banach spaces of bounded and convergent sequences $x = \left(x_k\right)$ with the usual norm $\|x\| = \sup_k |x_k|$. Let $S : \ell_\infty \to \ell_\infty$ be the shift operator defined by $\left(S\, x\right)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non negative linear functional such that $L\left(Sx\right) = L\left(x\right)$ and $L(e) = 1$, $e = \left(1,1,1,...\right)$ [2]. A sequence $x \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are α [3]. We denote the set of almost convergent sequences by f and almost convergent null sequences by f_0, i.e.

$$f = \left\{ x \in \ell_\infty : \lim_m \, t_{mn}(x) = \alpha, \ uniformly \ in \ n \right\}$$

and $$f_0 = \left\{ x \in \ell_\infty : \lim_m \, t_{mn}(x) = 0, \ uniformly \ in \ n \right\}$$

Where $$t_{mn}(x) = \frac{1}{m+1} \sum_{k=0}^{m} x_{k+n}, \qquad t_{-1,n} = 0$$

and $$\alpha = f - \lim x.$$

Let λ and μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k \in \mathbb{N} = \{0,1,2,...\}$. Then we say that A defines a matrix mapping from λ in to μ, and denote it by writing $A:\lambda \to \mu$ if for every sequence $x=(x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk}\, x_k \,, \; (n \in \mathbb{N}) \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We denote by $(\lambda : \mu)$ the class of all matrices A such that $A:\lambda \to \mu$. Thus $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for every $n \in \mathbb{N}$ and every $x \in \lambda$.

For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega \; : \; Ax \in \lambda\}$$

The object of this paper is to characterize the classes $(a_c^r : f)$ and $(a_c^r : f_0)$ of infinite matrices.

The sequence space a_c^r is defined as the set of all sequences whose A^r - transform is in c [1], i.e.

$$a_c^r = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(1+r^k\right) x_k \;\; exists \right\}$$

Where A^r denotes the matrix $A^r = (a_{nk}^r)$ defined by

$$a_{nk}^r = \begin{cases} \dfrac{1+r^k}{n+1} \;\;, & (0 \le k \le n) \\ 0 \;\;, & (k > n) \end{cases}$$

We refer the reader to [1] for relevant terminology and additional references on the space a_c^r.

2. Main Results

Define the sequence $y = (y_k(r))$, which will be used as the A^r - transform of a sequence $x = (x_k)$, i.e.

$$y_k(r) = \sum_{j=0}^{k} \frac{1+r^j}{k+1} \, x_j \qquad ; \; (k \in \mathbb{N}) \tag{2.1}$$

For brevity in notation, we write

$$a(n,k,m) = \frac{1}{m+1}\sum_{j=0}^{m} a_{n+j,k}$$

and

$$\tilde{a}(n,k,m) \;=\; \Delta\left[\frac{a(n,k,m)}{1+r^k}\right](k+1) \;=\; \left[\frac{a(n,k,m)}{1+r^k} - \frac{a(n,k+1,m)}{1+r^{k+1}}\right](k+1)$$

for $n,k,m \in \mathbb{N}$.

We denote by λ^{β}, the β-dual of a sequence space λ and mean the set of the sequences $x = (x_k)$ such that $x\,y = (x_k\,y_k) \in cs$ for all $y = (y_k) \in \lambda$. Now, we may give the following lemma which is needed in proving the Theorem (2.1) below.

***Lemma* 2.1[1]:** Define the sets d_1^r and d_2^r as follows

$$d_1^r = \left\{ a = (a_k) \in \omega \;:\; \sum_k \left| \Delta\left(\frac{a_k}{1+r^k}\right)(k+1) \right| < \infty \right\}$$

and $$d_2^r = \left\{ a = (a_k) \in \omega \;:\; \left(\frac{a_k}{1+r^k}\right) \in cs \right\}$$

where $\Delta\left(\frac{a_k}{1+r^k}\right) = \frac{a_k}{1+r^k} - \frac{a_{k+1}}{1+r^{k+1}}$ for all $k \in \mathbb{N}$.

Then $\left[a_c^r\right]^{\beta} = d_1^r \cap d_2^r$.

***Theorem* 2.1**: $A \in \left(a_c^r \;:\; f \right)$ if and if

$$\sup_{m,n \in \mathbb{N}} \sum_k \left| \tilde{a}(n,k,m) \right| < \infty \tag{2.2}$$

$$\left\{\frac{a_{nk}}{1+r^k}\right\}_{k \in \mathbb{N}} \in cs \text{ for all } n \in \mathbb{N} . \tag{2.3}$$

$$\lim_{m\to\infty} \tilde{a}(n,k,m) = \alpha_k \quad \text{uniformly in n, for each } k \in \mathbb{N} \tag{2.4}$$

$$\lim_{m\to\infty} \sum_k \left|\tilde{a}(n,k,m) - \alpha_k\right| = 0 \text{ uniformly in n.} \tag{2.5}$$

***Proof*:** Suppose that the conditions (2.2), (2.3), (2.4) and (2.5) hold and $x \in a_c^r$. Then Ax exists and at this stage, we observe from (2.4) and (2.2) that

$$\sum_{j=0}^{k} \left| \alpha_j \right| \le \sup_{m,n \in \mathbb{N}} \sum_j \left| \tilde{a}(n,j,m) \right| < \infty$$

holds for every $k \in \mathbb{N}$. This gives that $(\alpha_k) \in \ell_1$. Since $x \in a_c^r$ by the hypothesis, and $a_c^r \cong c$, we have $y \in c$. Therefore, one can easily see that $(\alpha_k\, y_k) \in \ell_1$ for each $y \in c$ and also there exists $M > 0$ such that $\sup_k |y_k| < M$. Now for any $\varepsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$, there is some $m_0 \in \mathbb{N}$ by (2.4) such that

$$\left| \sum_{k=0}^{k_0} \left[\tilde{a}(n,k,m) - \alpha_k \right] y_k \right| < \frac{\varepsilon}{2}$$

for every $m \geq m_0$, uniformly in n .

Also, by (2.5), there is some $m_1 \in \mathbb{N}$, such that

$$\sum_{k=k_0+1}^{\infty} \left| \tilde{a}(n,k,m) - \alpha_k \right| < \frac{\varepsilon}{2M}$$

for every $m \geq m_1$ uniformly in n . Therefore, we have

$$\begin{aligned}
\left| \frac{1}{m+1} \sum_{i=0}^{m} (Ax)_{n+i} - \sum_k \alpha_k\, y_k \right| &= \left| \sum_k \left[\tilde{a}(n,k,m) - \alpha_k \right] y_k \right| \\
&\leq \left| \sum_{k=0}^{k_0} \left[\tilde{a}(n,k,m) - \alpha_k \right] y_k \right| + \left| \sum_{k=k_0+1}^{\infty_0} \left[\tilde{a}(n,k,m) - \alpha_k \right] y_k \right| \\
&< \frac{\varepsilon}{2} + \sum_{k=k_0+1}^{\infty} \left| \tilde{a}(n,k,m) - \alpha_k \right| \left| y_k \right| \\
&< \frac{\varepsilon}{2} + M\, \frac{\varepsilon}{2M} = \varepsilon
\end{aligned}$$

for all sufficiently large m, uniformly in n. Hence $Ax \in f$, which proves the sufficiency.

Conversely suppose that $A \in \left(a_c^r : f\right)$. Then Ax exists for every $x \in a_c^r$ and this implies that $\{a_{nk}\}_{k \in \mathbb{N}} \in \left[a_c^r\right]^{\beta}$ for each $n \in \mathbb{N}$; the necessity of (2.3) is immediate. Now $\sum_k a(n,k,m)\, x_k$ exists for each m, n and $x \in a_c^r$, the sequence $a_{mn} = \{a(n,k,m)\}_{k \in \mathbb{N}}$ define the continuous linear functionals ϕ_{mn} on a_c^r by

$$\phi_{mn}(x) = \sum_k a(n,k,m)\, x_k \ , \ \left(m\ ,\ n \in \mathbb{N}\right).$$

Since a_c^r and c are norm isomorphic ([1], Theorem 2.2), it should follow with (2.1) that

$$\left\| \phi_{mn} \right\| = \left\| \tilde{a}_{mn} \right\|$$

This just says that the functionals defined by ϕ_{mn} on a_c^r are point wise bounded. Hence, by the Banach- Steinhauss theorem, they are uniformly bounded, which yields that there exists a constant $M > 0$ such that

$$\|\phi_{mn}\| \le M \quad \text{for all } m,n \in \mathbb{N}$$

It therefore follows, using the complete identification just referred to that

$$\sum_k |\tilde{a}(n,k,m)| = \|\phi_{mn}\| \le M$$

holds for all $m,n \in \mathbb{N}$ which shows the necessity of the condition (2.2).

To prove the necessity of (2.4), consider the sequence $b^{(k)}(r) = \{b_n^{(k)}(r)\}_{n\in\mathbb{N}} \in a_c^r$ for every $k \in \mathbb{N}$, where

$$b_n^{(k)}(r) = \begin{cases} (-1)^{n-k}\dfrac{1+k}{1+r^k}, & (k \le n \le k+1) \\ 0, & (0 \le n \le k-1 \quad or \quad n > k+1) \end{cases} \quad ; \ (n,k \in \mathbb{N})$$

Since Ax exists and is in f for each $x \in a_c^r$, one can easily see that

$$Ab^{(k)}(r) = \left\{\Delta\left(\frac{a_{nk}}{1+r^k}\right)(k+1)\right\}_{n\in\mathbb{N}} \in f$$

for each $k \in \mathbb{N}$, which shows the necessity of (2.4).

Similarly by taking $x = e \in a_c^r$, we also obtain that

$$Ax = \left\{\sum_k \Delta\left(\frac{a_{nk}}{1+r^k}\right)(k+1)\right\}_{n\in\mathbb{N}} \in f$$

and this shows the necessity of (2.5) . This completes the proof.

If the space f is replaced by f_0, then Theorem (2.1) is reduced to

***Corollary* 2.1:** $A \in (a_c^r : f_0)$ if and if (2.2), (2.3) and (2.4), (2.5) also hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

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Department of Mathematics, National Institute of Technology, Srinagar-190006, India

E-mail: sameergupkari@rediffmail.com