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ON $\mathcal{DS}^*$-SETS AND DECOMPOSITIONS OF CONTINUOUS FUNCTIONS

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Abstract

In this paper, the notions of $\mathcal{DS}^*$-sets and $\mathcal{DS}^*$-continuous functions are introduced and their properties and their relationships with some other types of sets are investigated. Moreover, some new decompositions of continuous functions are obtained by using $\mathcal{DS}^*$-continuous functions, $\mathcal{DS}$-continuous functions and $\mathcal{D}$-continuous functions.

1 Introduction

In a recent paper, Ekici and Jafari [12] have studied $\mathcal{DS}$-sets and $\mathcal{D}$-sets and obtained some decompositions of continuous functions via $\mathcal{DS}$-continuous functions and $\mathcal{D}$-continuous functions. In this paper, we introduce a new class of sets called $\mathcal{DS}^*$-sets. Properties of this class are investigated. Furthermore, the notion of $\mathcal{DS}^*$-continuous functions is introduced via $\mathcal{DS}^*$-sets to establish some new decompositions of continuous functions. On the other hand, by using $\mathcal{DS}$-sets and $\mathcal{D}$-sets, other new decompositions of continuous functions are obtained.

In this paper (X, τ) and (Y, σ) represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. A subset A of a space X is called regular open (resp regular closed) [22] if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A is called δ-open [24] if for each $x \in A$, there exists a regular open set U such that $x \in U \subset A$. A is called δ-closed if its complement is δ-open. A point $x \in X$ is called a δ-cluster point of A if $A \cap int(cl(U)) \neq \varnothing$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta\text{-}cl(A)$. The union of all regular open sets, each contained in A called the δ-interior of A and is denoted by $\delta\text{-}int(A)$. A subset A of a space (X, τ) is called semiopen [15] (resp. semi-regular [7], α-open [19], preopen [16] or locally dense [6], b-open [4] or γ-open [13] or sp-open [8], β-open [1] or

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semi-preopen [3], δ-semiopen [20], δ-preopen [21]) if $A \subset cl(int(A))$ (resp. semiopen and semiclosed, $A \subset int(cl(int(A)))$, $A \subset int(cl(A))$, $A \subset int(cl(A)) \cup cl(int(A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(\delta\text{-}int(A))$, $A \subset int(\delta\text{-}cl(A))$). The complement of a δ-semiopen (resp. semiopen) set is called a δ-semiclosed (resp. semiclosed) set. The union (resp. intersection) of all δ-preopen (resp. δ-semiclosed) sets, each contained in (resp. containing) a set A in a topological space X is called the δ-preinterior (resp. δ-semiclosure) of A and it is denoted by $\delta\text{-}pint(A)$ (resp. $\delta\text{-}scl(A)$).

Definition 1 *A subset A of a space (X, τ) is called*
(1) a $\mathcal{D}$-set [12] if $A = U \cap V$, where U is open and V is δ-closed,
(2) a $\mathcal{DS}$-set [12] if $A = U \cap V$, where U is open and V is δ-semiclosed,
(3) a $\mathcal{B}$-set [23] if $A \in \mathcal{B}(X) = \{U \cap V : U \in \tau,\ int(cl(V)) \subset V\}$,
(4) an $\mathcal{AB}$-set [9] if $A \in \mathcal{AB}(X) = \{U \cap V : U \in \tau$ and V is semi-regular$\}$.

The family of all $\mathcal{DS}$-sets (resp. $\mathcal{D}$-sets) of a topological space X will be denoted by $\mathcal{DS}(X)$ (resp. $\mathcal{D}(X)$). A topological space X is called a locally indiscrete [10] if every open subset of X is closed and called submaximal [5] if every dense subset of X is open.

Definition 2 *A function $f : X \to Y$ is called*
(1) β-continuous [1] if $f^{-1}(A)$ is β-open for each $A \in \sigma$.
(2) α-continuous [17] if $f^{-1}(A)$ is α-open for each $A \in \sigma$.
(3) γ-continuous [13] if $f^{-1}(A)$ is γ-open for each $A \in \sigma$.
(4) quasi-continuous [14] if $f^{-1}(A)$ is semiopen for each $A \in \sigma$.
(5) precontinuous [16] if $f^{-1}(A)$ is preopen for each $A \in \sigma$.
(6) δ-almost continuous [21] if $f^{-1}(A)$ is δ-preopen for each $A \in \sigma$.
(7) δ-semicontinuous [11] if $f^{-1}(A)$ is δ-semiopen for each $A \in \sigma$.
(8) super-continuous [18] if $f^{-1}(A)$ is δ-open for each $A \in \sigma$.

2 $\mathcal{DS}^*$-sets in topological spaces

Definition 3 *A subset A of a topological space X is called a $\mathcal{DS}^*$-set if $A = U \cap V$, where U is open and V is δ-semiclosed and $int(\delta\text{-}cl(V)) = cl(\delta\text{-}int(V))$.*

The family of all $\mathcal{DS}^*$-sets of a topological space X will be denoted by $\mathcal{DS}^*(X)$.

Remark 4 *The following diagram holds for a subset of a space X:*

$$\begin{array}{c} \mathcal{B}\text{-set} \\ \uparrow \\ \mathcal{DS}\text{-set} \\ \uparrow \\ \mathcal{DS}^*\text{-set} \end{array}$$

The following examples show that these implications are not reversible.

Example 5 *Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing,\ \{a\},\ \{b\},\ \{a, b\},\ X\}$. The set $\{a, c, d\}$ is a $\mathcal{DS}$-set but it is not a $\mathcal{DS}^*$-set.*

Example 6 *([12]) Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing,\ \{a\},\ \{c\},\ \{a, b\},\ \{a, c\},\ \{a, b, c\},\ \{a, c, d\},\ X\}$. The set $\{b, c, d\}$ is a $\mathcal{B}$-set but it is not a $\mathcal{DS}$-set.*

Remark 7 *Every open set is a $\mathcal{DS}^*$-set. The converse is not true.*

Example 8 *Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing,\ \{a\},\ \{b\},\ \{a, b\},\ X\}$. The set $\{a, c\}$ is a $\mathcal{DS}^*$-set but it is not open.*

Theorem 9 *The following are equivalent for a subset A of a space X:*
(1) A is open,
(2) A is α-open and a $\mathcal{DS}^*$-set,
(3) A is semiopen and a $\mathcal{DS}^*$-set,
(4) A is preopen and a $\mathcal{DS}^*$-set,
(5) A is γ-open and a $\mathcal{DS}^*$-set.
(6) A is β-open and a $\mathcal{DS}^*$-set.

Proof. $(1) \Rightarrow (2)$: It follows from the fact that every open set is α-open and a $\mathcal{DS}^*$-set.
$(2) \Rightarrow (3) \Rightarrow (5)$: Obvious.
$(2) \Rightarrow (4) \Rightarrow (5)$: Obvious.
$(5) \Rightarrow (6)$: Obvious.
$(6) \Rightarrow (1)$: Let A be β-open and a $\mathcal{DS}^*$-set. Since A is β-open, $A \subset cl(int(cl(A)))$. Since A is a $\mathcal{DS}^*$-set, then $A = U \cap V$, where U is open and V is δ-semiclosed and $int(\delta\text{-}cl(V)) = cl(\delta\text{-}int(V))$. Also, by δ-semiclosedness of V, we have $\delta\text{-}int(V) = \delta\text{-}int(\delta\text{-}cl(V))$. Furthermore, we obtain

$$\begin{aligned} A = A \cap U \subset cl(int(cl(A))) \cap U &= cl(int(cl(U \cap V))) \cap U \\ &\subset cl(int(cl(U))) \cap cl(int(cl(V))) \cap U \\ &= cl(int(cl(V))) \cap U \\ &\subset cl(int(\delta\text{-}cl(V))) \cap U \\ &= cl(\delta\text{-}int(V)) \cap U \\ &= int(\delta\text{-}cl(V)) \cap U \\ &= \delta\text{-}int(V) \cap U. \end{aligned}$$

Thus, $A = \delta\text{-}int(V) \cap U$ and hence A is open. ■

Theorem 10 *([12]) For a space X, the following are equivalent:*
(1) X is indiscrete,
(2) the $\mathcal{DS}$-sets in X are only the trivial ones,
(3) the $\mathcal{D}$-sets in X are only the trivial ones.

Theorem 11 *The following are equivalent for a space X:*
(1) X is indiscrete,
(2) the $\mathcal{DS}^*$-sets in X are the trivial ones.

Proof. (1) $\Rightarrow$ (2) : Let A be a $\mathcal{DS}^*$-set in X. Then there exist an open set U and a δ-semiclosed set V such that $A = U \cap V$ and $\delta\text{-}int(\delta\text{-}cl(V)) = \delta\text{-}cl(\delta\text{-}int(V))$. Let $A \neq \varnothing$. We have $U \neq \varnothing$. By (1), $U = X$ and $A = V$. Hence, $X = \delta\text{-}scl(A) \subset A$ and so $A = X$.
(2) $\Rightarrow$ (1) : Since every $\mathcal{DS}^*$-set is $\mathcal{DS}$-set, it follows from Theorem 10. ■

Theorem 12 *Let X be a topological space and $A \subset X$. If $A \in \mathcal{DS}(X)$, then $\delta\text{-}pint(A) = int(A)$.*

Proof. Let $A \in \mathcal{DS}(X)$. Then, $A = U \cap V$, where U is open and V is δ-semiclosed. Since V is δ-semiclosed, then we have $\delta\text{-}int(V) = \delta\text{-}int(\delta\text{-}cl(V))$. Moreover, we obtain

$$\begin{aligned} \delta\text{-}pint(A) = A \cap \delta\text{-}int(\delta\text{-}cl(A)) &\subset U \cap \delta\text{-}int(\delta\text{-}cl(V)) \\ &= U \cap \delta\text{-}int(V) \\ &\subset U \cap int(V) \\ &= int(A). \end{aligned}$$

Thus, $\delta\text{-}pint(A) = int(A)$. ■

Theorem 13 *The following are equivalent for a subset A of a space X:*
(1) A is open,
(2) A is δ-preopen and a $\mathcal{D}$-set,
(3) A is δ-preopen and a $\mathcal{DS}$-set.

Proof. (1) $\Rightarrow$ (2) : Since every open set is δ-preopen and a $\mathcal{D}$-set, it is completed.
(2) $\Rightarrow$ (3) : Obvious.
(3) $\Rightarrow$ (1) : Let A be δ-preopen and a $\mathcal{DS}$-set. By Theorem 12, $\delta\text{-}pint(A) = int(A)$. Also, since A is δ-preopen, $A = \delta\text{-}pint(A) = int(A)$. Thus, A is open. ■

Theorem 14 *Let X be a topological space and $A \subset X$. If $A \in \mathcal{DS}^*(X)$, then $A = U \cap \delta\text{-}scl(A)$ for some open set U.*

Proof. Let $A \in \mathcal{DS}^*(X)$. This implies that $A = U \cap V$, where U is open and V is δ-semiclosed and $int(\delta\text{-}cl(V)) = cl(\delta\text{-}int(V))$. Since $A \subset V$, $\delta\text{-}scl(A) \subset \delta\text{-}scl(V) = V$. Moreover, $U \cap \delta\text{-}scl(A) \subset U \cap V = A \subset U \cap \delta\text{-}scl(A)$ and hence $A = U \cap \delta\text{-}scl(A)$. ■

Theorem 15 *([12]) The following are equivalent for a subset P of a space X:*
(1) P is an $\mathcal{AB}$-set,
(2) P is semiopen and a $\mathcal{DS}$-set,
(3) P is b-open and a $\mathcal{DS}$-set,
(4) P is β-open and a $\mathcal{DS}$-set.

Theorem 16 *Let X be a topological space and $A \subset X$. If A is β-open and a $\mathcal{DS}^*$-set, then it is an $\mathcal{AB}$-set.*

Proof. Let A be β-open and a $\mathcal{DS}^*$-set. Since A is a $\mathcal{DS}$-set, by Theorem 15, A is an $\mathcal{AB}$-set. ■

Definition 17 *Let X be a topological space and $A \subset X$. Then A is called a δ^*-set if δ-$int(A)$ is δ-closed.*

Theorem 18 *Let X be a topological space and $A \subset X$. If A is a δ^*-set and δ-semiopen, then it is δ-open.*

Proof. Let A be a δ^*-set and δ-semiopen. Then $A \subset cl(\delta\text{-}int(A)) = \delta\text{-}int(A)$. Thus, A is δ-open. ■

Theorem 19 *Let X be a topological space and $A \subset X$. Then A is open if A is a δ-semiopen $\mathcal{DS}^*$-set and A is preopen or a δ^*-set.*

Proof. Let A be a δ-semiopen $\mathcal{DS}^*$-set. Suppose that A is preopen or a δ^*-set. If A is a preopen $\mathcal{DS}^*$-set, then it is a preopen $\mathcal{B}$-set. So, by Proposition 9 [23], A is open. Also, if A is a δ^*-set and δ-semiopen, by Theorem 18, A is open. Thus, the proof is completed. ■

Theorem 20 *The following are equivalent for a space X :*
(1) X is a locally indiscrete space,
(2) every $\mathcal{DS}^*$-set is clopen,
(3) every $\mathcal{DS}^*$-set is closed.

Proof. (1) $\Rightarrow$ (2) : Let A be a $\mathcal{DS}^*$-set. Then there exist an open set U and a δ-semiclosed set V such that $A = U \cap V$ and $int(\delta\text{-}cl(V)) = cl(\delta\text{-}int(V))$. Since U is clopen, then A is semiclosed. By [2], since X is a locally indiscrete space, then A is clopen.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let $A \subset X$ be an open set. Since A is a $\mathcal{DS}^*$-set, then A is closed. Hence, X is a locally indiscrete space. ■

Theorem 21 *([12]) For a space X the following are equivalent:*
(1) X is submaximal,
(2) every dense subset of X is a $\mathcal{D}$-set,
(3) every dense subset of X is a $\mathcal{DS}$-set.

Theorem 22 *Let X be a topological space. Then X is submaximal if and only if every dense subset of X is a $\mathcal{DS}^*$-set.*

Proof. ($\Rightarrow$) : Let A be a dense subset of X. Since X submaximal, then A is open and so A is a $\mathcal{DS}^*$-set.

($\Leftarrow$) : Since every dense subset is a $\mathcal{DS}^*$-set and every $\mathcal{DS}^*$-set is a $\mathcal{DS}$-set, then by Theorem 21, X is submaximal. ■

3 Some new decompositions of continuity

Definition 23 *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called*
(1) $\mathcal{DS}^*$*-continuous if* $f^{-1}(V) \in \mathcal{DS}^*(X)$ *for each* $V \in \sigma$.
(2) δ^**-continuous if* $f^{-1}(V)$ *is a* δ^**-set for each* $V \in \sigma$.

Definition 24 *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called*
(1) $\mathcal{D}$*-continuous [12] if* $f^{-1}(V) \in \mathcal{D}(X)$ *for each* $V \in \sigma$.
(2) $\mathcal{DS}$*-continuous [12] if* $f^{-1}(V) \in \mathcal{DS}(X)$ *for each* $V \in \sigma$.
(3) $\mathcal{AB}$*-continuous [9] if* $f^{-1}(V) \in \mathcal{AB}(X)$ *for each* $V \in \sigma$.
(4) $\mathcal{B}$*-continuous [23] if* $f^{-1}(V) \in \mathcal{B}(X)$ *for each* $V \in \sigma$.

Remark 25 *The following diagram holds for a function* $f : X \to Y$*:*

$$\begin{array}{c} \mathcal{B}\text{-continuous} \\ \Uparrow \\ \mathcal{DS}\text{-continuous} \\ \Uparrow \\ \mathcal{DS}^*\text{-continuous} \end{array}$$

None of these implications is reversible as shown in the following examples.

Example 26 *Let* $X = Y = \{a, b, c, d\}$ *and* $\tau = \{\varnothing,\ X,\ \{a\},\ \{b\},\ \{a, b\}\}$, $\sigma = \{\varnothing,\ X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. *Then the function* $f : (X, \tau) \to (Y, \sigma)$, *defined as:* $f(a) = c$, $f(b) = b$, $f(c) = c$, $f(d) = d$, *is* $\mathcal{DS}$*-continuous but it is not* $\mathcal{DS}^*$*-continuous.*

Example 27 *([12]) Let* $X = Y = \{a, b, c, d\}$ *and* $\tau = \sigma = \{\varnothing,\ X,\ \{a\},\ \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. *Then the function* $f : (X, \tau) \to (Y, \sigma)$, *defined as:* $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d) = d$, *is* $\mathcal{B}$*-continuous but it is not* $\mathcal{DS}$*-continuous.*

Remark 28 *Every continuous function is* $\mathcal{DS}^*$*-continuous but not conversely.*

Example 29 *Let* $X = Y = \{a, b, c, d\}$ *and* $\tau = \{\varnothing,\ X,\ \{a\},\ \{b\},\ \{a, b\}\}$, $\sigma = \{\varnothing,\ X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. *Then the function* $f : (X, \tau) \to (Y, \sigma)$, *defined as:* $f(a) = c$, $f(b) = b$, $f(c) = c$, $f(d) = b$, *is* $\mathcal{DS}^*$*-continuous but it is not continuous.*

Theorem 30 *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function. If* f *is* β*-continuous and* $\mathcal{DS}^*$*-continuous, then it is* $\mathcal{AB}$*-continuous.*

Proof. It follows from Theorem 16. ■

Theorem 31 *The following are equivalent for a function $f : X \to Y$:*
(1) f is continuous,
(2) f is α-continuous and $\mathcal{DS}^*$-continuous,
(3) f is quasi-continuous and $\mathcal{DS}^*$-continuous,
(4) f is precontinuous and $\mathcal{DS}^*$-continuous,
(5) f is γ-continuous and $\mathcal{DS}^*$-continuous,
(6) f is β-continuous and $\mathcal{DS}^*$-continuous.

Proof. It is an immediate consequence of Theorem 9. ■

Theorem 32 *The following are equivalent for a function $f : X \to Y$:*
(1) f is continuous,
(2) f is δ-almost continuous and $\mathcal{D}$-continuous,
(3) f is δ-almost continuous and $\mathcal{DS}$-continuous.

Proof. It follows from Theorem 13. ■

Theorem 33 *Let $f : X \to Y$ be a function. Then f is continuous if f is δ-semicontinuous, $\mathcal{DS}^*$-continuous and precontinuous or δ^*-continuous.*

Proof. It is an immediate consequence of Theorem 19. ■

Theorem 34 *Let $f : X \to Y$ be a function. Then f is super-continuous if f is δ^*-continuous and δ-semicontinuous.*

Proof. It is an immediate consequence of Theorem 18. ■

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