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MIXED AND COMPONENTWISE CONDITION NUMBERS FOR WEIGHTED MOORE-PENROSE INVERSE AND WEIGHTED LEAST SQUARES PROBLEMS

Zhao Li* **and Jie Sun**

Abstract

Condition numbers play an important role in numerical analysis. Classical condition numbers are normwise: they measure both input perturbations and output errors with norms. To take into account the relative scaling of data components or a possible sparseness, componentwise condition numbers have been increasingly considered. In this paper, we give explicit expressions for the mixed and componentwise condition numbers for the weighted Moore-Penrose inverse of a matrix A, as well as for the solution and residue of a weighted linear least squares problem $\|W^{\frac{1}{2}}(Ax-b)\|_2 = \min_{v\in\mathbb{R}^n} \|W^{\frac{1}{2}}(Av-b)\|_2$, where the matrix A with full column rank.

1 Introduction

1.1 General consideration.

A general theory of condition numbers are first given by Rice in [16], the relative normwise condition number of a_0 is given by

$$\mathrm{cond}(a_0) := \lim_{\varepsilon\to 0} \sup_{\|\Delta a\|\le\varepsilon} \Big(\frac{\|\phi(a_0+\Delta a)-\phi(a_0)\|}{\|\phi(a_0)\|} / \frac{\|\Delta a\|}{\|a_0\|} \Big) = \frac{\|\phi'(a_0)\|\|a_0\|}{\|\phi(a_0)\|},$$

where $\phi'(a_0)$ is Fréchet derivative of ϕ at a_0. A drawback of condition numbers of this type is that they ignore the structure of both input and output data with respect to scaling and/or sparsity. When the data is badly scaled or contains many zeros, measuring a perturbation in terms of its norm, we are left in the dark concerning the relative size of the perturbation on its small (or zero) entries.

To tackle this drawback, another approach in perturbation theory, known as *componentwise analysis,* has been increasingly considered. To be precise, two kinds

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of condition numbers were studied: first, those measuring the errors in the output with norms and the input perturbation componentwisely, and second, those measuring both the error in the output and the perturbation in the input componentwisely. The resulting condition numbers are called *mixed and componentwise,* respectively, by Gohberg and Koltracht [7]. We will use this terminology throughout this paper.

In this paper we exhibit explicit expressions for mixed and componentwise condition numbers for both weighted Moore-Penrose inverse and weighted linear least squares problems. We also exhibit upper bounds for these condition numbers which are easier to compute for large matrices.

1.2 A brief description of some previous work.

Probably the first mixed perturbation analysis was done by Skeel [18]. He performed a mixed perturbation analysis for nonsingular linear systems of equations and a mixed error analysis for Gaussian elimination. Skeel's condition number is of mixed type. It is defined using componentwise perturbation on the input data and infinity norm in the solution. In [17], Rohn introduced a new relative condition number measuring both perturbation in the input data and error in the output componentwisely.

They were Gohberg and Koltracht [7] who named Skeel's condition numbers. They also gave explicit expressions for both mixed and componentwise condition numbers of linear equations.

Perturbation theory for rectangular matrices and linear least squares problems existed for quite a while for normwise case(cf. [19, 22]) and has been further studied in [6, 9, 15]. For the mixed and componentwise settings for linear least squares problems, the existing results consisted of bounds for both condition numbers(or first order perturbations bounds) and unrestricted perturbation bounds. There were no explicit expressions for mixed and componentwise condition numbers until Cucker, Diao and Wei's work [4].

1.3 Main definition and results.

For any points $a,\ b \in \mathbb{R}^n$, Let

$$\frac{a}{b} = (c_1, \ldots, c_n) = \begin{cases} a_i/b_i, & if\ b_i \neq 0, \\ 0, & if\ a_i = b_i = 0, \\ \infty & otherwise. \end{cases}$$

Then the following form of "distance" function can be defined by

$$d(a, b) = \left\| \frac{a-b}{b} \right\|_\infty = \max_{i=1,\ldots,n} \left\{ \frac{|a_i - b_i|}{|b_i|} \right\},$$

which measures the componentwise perturbation when data a varies into b.
Note that if $d(a,b) < \infty$,

$$d(a,b) = \min_{i=1,\dots,n} \{\nu \geq 0 \mid |a_i - b_i| \leq \nu |b_i|\}.$$

In the rest of this paper we will only consider pairs (a,b) for which $d(a,b) < \infty$. We can extend the function d to matrices naturally by introducing a function vec. For a matrix $A \in \mathbb{R}^{m\times n}$, $\mathsf{vec}(A) \in \mathbb{R}^{mn}$ is defined by $\mathsf{vec}(A) = [a_1^T, \dots, a_n^T]^T$, where $A = [a_1, \dots, a_n]$ with $a_i \in \mathbb{R}^m, i = 1, \cdots, n$. Then,we define

$$d(A,B) = d(\mathsf{vec}(A), \mathsf{vec}(B)).$$

Note that vec is a homeomorphism between $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{mn}$. In addition, it transforms in the sense that, for all $A \in \mathbb{R}^{m\times n}$,

$$\|\mathsf{vec}(A)\|_2 = \|A\|_F \qquad \|\mathsf{vec}(A)\|_\infty = \|A\|_{max}, \tag{1}$$

where $\|\cdot\|_F$ is the Frobenius norm given by $\|A\|_F = \mathsf{trace}(A^TA)$, A^T denotes transpose matrix of A, and $\|\cdot\|_{max}$ is the max norm given by $\|A\|_{\max} = \max\limits_{i,j} |A_{ij}|$.

For $\varepsilon > 0$ we denote $B^0(a,\varepsilon) = \{x \mid d(x,a) \leq \varepsilon\}$. For a function $F : \mathbb{R}^p \to \mathbb{R}^q$ we denote by $\mathsf{Dom}(f)$ its domain of definition.

Definition 1. *Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous mapping defined on an open set $\mathsf{Dom}(F) \subset \mathbb{R}^p$ such that $0 \notin \mathsf{Dom}(F)$ such that $F(a) \neq 0$.*

(i) *The* mixed condition number *of F at a is defined by*

$$m(F,a) = \lim_{\varepsilon\to 0} \sup_{\substack{x\in B^0(a,\varepsilon)\\ x\neq a}} \frac{\|F(x) - F(a)\|_\infty}{\|F(a)\|_\infty} \frac{1}{d(x,a)}$$

(ii) *Suppose $F(a) = (f_1(a), \dots, f_q(a))$is such that $f_j(a) \neq 0$ for $j = 1, \dots, q$. Then the* componentwise condition number *of F at a is*

$$c(F,a) = \lim_{\varepsilon\to 0} \sup_{\substack{x\in B^0(a,\varepsilon)\\ x\neq a}} \frac{d(F(x), F(a)}{d(x,a)}.$$

In this paper we consider these condition numbers for the *weighted Moore Penrose inverse* of $A \in \mathbb{R}^{m\times n}$ with respect to a symmetric positive definite matrix $W \in \mathbb{R}^{m\times m}$. We recall that this the unique $n \times m$ matrix $A_W^\dagger$ satisfying the four matrix equation [2, 21]

$$AA_W^\dagger A = A, \quad A_W^\dagger A A_W^\dagger = A_W^\dagger, \quad (WAA_W^\dagger)^T = WAA_W^\dagger, \quad (A_W^\dagger A)^T = A_W^\dagger A.$$

Identifying $\mathbb{R}^{m\times n}$ with $\mathbb{R}^{mn}$ via vec and using (1), Definition 1 yields, respectively,

$$m(A_W^\dagger, A) := \lim_{\varepsilon\to 0} \sup_{\|\Delta A/A\|_{max}\leq\varepsilon} \frac{\left\|(A+\Delta A)_W^\dagger - A_W^\dagger\right\|_{max}}{\left\|A_W^\dagger\right\|_{max}} \frac{1}{\|\Delta A/A\|_{max}}$$

and

$$c(A_W^\dagger, A) := \lim_{\varepsilon \to 0} \sup_{\|\Delta A/A\|_{max} \le \varepsilon} \frac{1}{\|\Delta A/A\|_{max}} \left\| \frac{(A+\Delta A)_W^\dagger - A_W^\dagger}{A_W^\dagger} \right\|_{max}.$$

Here $\frac{B}{A}$ is an entrywise division defined by $\frac{B}{A} := \mathsf{vec}^{-1}(\mathsf{vec}(B) - \mathsf{vec}(A))$. Note also that in the definition of $c(A_W^\dagger, A)$ we are assuming that $A_W^\dagger$ has no zero components.

Theorem 1 gives explicit expressions for these condition numbers. Corollary 1 then gives easier-to-compute upper bounds.

In a similar way, one defines, given a full column rank matrix A a vector b and a symmetric positive definite matrix W, the condition numbers $m_{wls}(A, b)$ and $c_{wls}(A, b)$ for the solution x of the weighted least squares problem

$$\|W^{\frac{1}{2}}(Ax - b)\|_2 = \min_{v \in \mathbb{R}^n} \|W^{\frac{1}{2}}(Av - b)\|_2.$$

$W^{\frac{1}{2}}$ is the symmetric positive definite solution to the equation $Z^2 = W$. The condition numbers $m_{res}(A, b)$ and $c_{res}(A, b)$ for the residue $r = W(b - Ax)$ are also considered. The main results in Section 4, Theorem 2 and 4, give explicit expressions for them. Easier-to-compute upper bounds are also shown in this section. Theorem 3 also gives sharp bounds for unrestricted(i. e., not necessarily small) perturbations.

2 Preliminaries

2.1 Kronecker products.

If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the Kronecker product $A \otimes B \in \mathbb{R}^{mp\times nq}$ is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

The following results can be found in [8],

$$|A \otimes B| = |A| \otimes |B|, \tag{2}$$

$$\mathsf{vec}(AXB) = (B^T \otimes A)\mathsf{vec}(X), \tag{3}$$

where $|A| = (|A_{ij}|)$, A_{ij} is the (i, j)th entry of A.

It is also proven in [8] that there exists a matrix $\Pi \in \mathbb{R}^{mn\times mn}$ such that, for all $A \in \mathbb{R}^{m\times n}$,

$$\Pi(\mathsf{vec}(A)) = \mathsf{vec}(A^T). \tag{4}$$

The matrix Π is called the *vec-permutation matrix* and can be represented explicitly by

$$\Pi = \sum_{i=1}^{n}\sum_{j=1}^{m} E_{ij}(m\times n)\otimes E_{ji}(n\times m).$$

Here $E_{ij}(m\times n) = e_i^{(m)}(e_j^{(n)})^T \in \mathbb{R}^{m\times n}$ denotes the (i,j)th elementary matrix and $e_i^{(m)}$ is the vector $[0,\ldots,0,1,0,\ldots,0]^T \in \mathbb{R}^m$, the 1 in the ith component(see [8]).

For matrix Π, it is proven in [15] that for any vector $y \in \mathbb{R}^p$ and matrix $Y \in \mathbb{R}^{p\times q}$,

$$(y^T \otimes Y)\Pi = Y \otimes y^T. \tag{5}$$

2.2 Condition number and differentiability.

The following lemma gives expressions for the mixed and componentwise condition numbers for differentiable functions. In its statement, and in all that follows, if $a \in \mathbb{R}^p$, we denote by Dg the $p\times p$ diagonal matrix with $a_1,\ldots,a_p$ in the diagonal. Recall, for a matrix $A \in \mathbb{R}^{m\times n}$, that its operator norm with respect norm to $\|\cdot\|_\infty$ satisfies

$$\|A\|_\infty = \max_{i\le m}\sum_{j=1}^{n}|A_{ij}|.$$

Lemma 1 ([7]). *Let $F:\mathbb{R}^p \to \mathbb{R}^q$ be as Definition 1 and $a \in \mathsf{Dom}(F)$ be such that F is Fréchet differentiable at a. Then,*

(a) *If $F(a) \neq 0$, then $m(F,a) = \frac{\|DF(a)\mathsf{Dg}(a)\|_\infty}{\|F(a)\|_\infty}$.*

(b) *If $(F(a))_i \neq 0$ for $i = 1,\ldots,q$, then $c(F,a) = \left\|\mathsf{Dg}(F(a))^{-1}DF(a)\mathsf{Dg}(a)\right\|_\infty$.*

Lemma 2. *With the notation above, we have*

(a) *If $F(a) \neq 0$, then $m(F,a) = \frac{\||DF(a)||a|\|_\infty}{\|F(a)\|_\infty}$.*

(b) *If $(F(a))_i \neq 0$ for $i = 1,\ldots,q$, then $c(F,a) = \left\|\frac{|DF(a)||a|}{|F(a)|}\right\|_\infty$.*

Proof. It readily follows from the fact that for any matrix $A \in \mathbb{R}^{p\times q}$ and diagonal matrix $\mathsf{Dg}(d) \in \mathbb{R}^{q\times q}$ we have

$$\|A\mathsf{Dg}(d)\|_\infty = \||A\mathsf{Dg}(d)|\|_\infty = \||A||\mathsf{Dg}(d)|\|_\infty = \||A||\mathsf{Dg}(d)|\boldsymbol{e}_q|\|_\infty = \||A||d|\|_\infty.$$

Here $\boldsymbol{e}_q = (1,1,\ldots,1)^T \in \mathbb{R}^q$. $\square$

Remark 1.

- *Lemma 2 reduces the computation of condition numbers, mostly, to finding explicit expressions for $|DF(a)|$ or, more precisely, matrix expressions for the derivative $DF(a)$. This will therefore be a major concern in the rest of this paper.*
- *To simplify notation, in the rest of this paper we assume that every time we deal with componentwise condition numbers, the computed objection has no zero components.*

3 Weighted Moore-Penrose inverse

Let

$$V = \{g \in \mathbb{R}^{mn} \mid g = \mathsf{vec}(G), \text{with } G \in \mathbb{R}^{m\times n}, \mathsf{rank}(G) = n\}$$

then the set $\{G \in \mathbb{R}^{m\times n} \mid \mathsf{rank}(G) = n\}$ is open in $\mathbb{R}^{m\times n}$ since vec is a homeomorphism between $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{mn}$ and its complement is the union of the sets $\mathsf{det}(G_s) = 0$, where G_s runs over all $n \times n$. submatrices of G.

Now we define the mapping $\phi : V \to \mathbb{R}^{mn}$ by $\phi(\mathsf{vec}(G)) = \mathsf{vec}(G_W^\dagger)$. By definition we have

$$m(A_W^\dagger, A) = m(\phi, \mathsf{vec}(A)), \quad \text{and} \quad c(A_W^\dagger, A) = c(\phi, \mathsf{vec}(A)).$$

Our goal is to get an explicit expression for the derivative $D\phi$. Lemma 4 below exhibits such an expression. To prove Lemma 4, we need the following well-known result in Lemma 3.

Lemma 3. *Let $A \in \mathbb{R}^{m\times n}$ and suppose $\{A_k\}$ is a sequence of $m \times n$ matrices satisfying $\lim_{k\to\infty} A_k = A$ necessary and sufficient condition for $\lim_{k\to\infty} (A_k)_W^\dagger = A_W^\dagger$ is*

$$\mathsf{rank}(\mathsf{A_k}) = \mathsf{rank}(\mathsf{A})$$

for sufficiently large k.

Proof. In the hypotheses of the lemma, $\lim_{k\to\infty} (A_k)^\dagger = A^\dagger$ equals $\mathsf{rank}(\mathsf{A_k}) = \mathsf{rank}(\mathsf{A})$([2, 20, 21].) Note that $A_W^\dagger = (W^{\frac{1}{2}}A)^\dagger W^{\frac{1}{2}}$, the conclusion follows. □

Lemma 4. *The mapping ϕ is continuous and Fréchet differentiable at a for all $\boldsymbol{a} \in V$. Moreover, it has the matrix expression $\mathrm{D}\phi(a) = M(A)$ where*

$$M(A) = [-(A_W^\dagger)^T \otimes A_W^\dagger + W(I_m - AA_W^\dagger) \otimes (A^TWA)^{-1}\Pi].$$

Here I_m denotes the $m \times m$ identity matrix.

Proof. The continuity of ϕ on V follows immediately from Lemma 3. The following equation is well known (see [16,Page 150, eqn. (3.35)]):

$$(A + \Delta A)^\dagger - A^\dagger = -A^\dagger \Delta A A^\dagger + (A^TA)^{-1}(\Delta A)^T(I_m - AA^\dagger) + \mathcal{O}(\|\Delta A\|^2).$$

Omitting the second-order terms, combining it with the equation

$$A_W^\dagger = (W^{\frac{1}{2}}A)^\dagger W^{\frac{1}{2}},$$

we get

$$(A + \Delta A)_W^\dagger - A_W^\dagger = -A_W^\dagger \Delta A A_W^\dagger + (A^TWA)^{-1}\Delta A^T W(I_m - AA_W^\dagger). \tag{6}$$

Then using the vec function, and denoting $a = \mathsf{vec}(\Delta A)$, we have

$$\begin{aligned}
&\phi(a + \delta a) - \phi(a) \\
&\quad \approx \mathsf{vec}(-A_W^\dagger \Delta A A_W^\dagger + (A^T W A)^{-1} \Delta A^T W (I_m - A A_W^\dagger)) \\
&\quad = -((A_W^\dagger)^T \otimes A_W^\dagger)\mathsf{vec}(\Delta A) + (W(I_m - A A_W^\dagger) \otimes (A^T W A)^{-1})\mathsf{vec}(\Delta A^T) \\
&\quad = [-((A_W^\dagger)^T \otimes A_W^\dagger) + (W(I_m - A A_W^\dagger) \otimes (A^T W A)^{-1})\Pi]\delta a,
\end{aligned}$$

where the second line follows from (3) and Π is the vec-permutation matrix defined by (4).
So the Fréchet derivative of ϕ at a is given by $\mathrm{D}\phi(a) = M(A)$. □

The main result in this section is the following theorem. It provides explicit expressions for the condition number we defined for the weighted Moore-Penrose inverse .

Theorem 1. *Let* $A \in \mathbb{R}^{n \times n}$ *be such that* $\mathsf{rank}(A) = n$ *and* W *be symmetric positive definite. Then*

(a) $m(A_W^\dagger, A) = \frac{\|M(A)\mathsf{vec}(|A|)\|_\infty}{\left\|\mathsf{vec} A_W^\dagger\right\|_\infty}$,

(b) $c(A_W^\dagger, A) = \left\| \frac{|M(A)|\mathsf{vec}(|A|)}{\mathsf{vec}(A_W^\dagger)} \right\|_\infty$.

Proof. By Lemma 2 and 4,

$$m(A_W^\dagger, A) = m(\phi; a) = \frac{\||\mathrm{D}\phi(a)||a|\|_\infty}{\|\phi(a)\|_\infty} = \frac{\||M(A)|\mathsf{vec}(|A|)\|_\infty}{\left\|\mathsf{vec}(A_W^\dagger)\right\|_\infty}$$

and

$$c(A_W^\dagger, A) = c(\phi; a) = \left\| \frac{|M(A)||a|}{|\phi(a)|} \right\|_\infty = \left\| \frac{|M(A)|\mathsf{vec}(|A|)}{\mathsf{vec}(|A_W^\dagger|)} \right\|_\infty .$$

□

Theorem 1 gives explicit expressions for the condition number $m(A_W^\dagger, A)$ and $c(A_W^\dagger, A)$. While these expressions are sharp they may not be easy to compute by their dependance on the (large) matrix Π and the need to compute Kronecker products. The next corollary gives easier to compute upper bounds for these condition numbers.

Lemma 5. *For any matrices* M, N, P, Q, R *and* S *with dimensions making the following well defined*

$$[M \otimes N + (P \otimes Q)\Pi]\mathsf{vec}(R),$$

$$\frac{[M \otimes N + (P \otimes Q)\Pi]\mathsf{vec}(R)}{S},$$

$$NRM^T \text{ and } QR^T P^T,$$

we have

$$\| |[M \otimes N + (P \otimes Q)\Pi]| \mathsf{vec}(|R|) \|_\infty \leq \left\| \mathsf{vec}(|N||R||M|^T + |Q||R|^T|P|^T) \right\|_\infty$$

and

$$\left\| \frac{|[M \otimes N + (P \otimes Q)\Pi]| \mathsf{vec}(|R|)}{|S|} \right\|_\infty \leq \left\| \frac{\mathsf{vec}(|N||R||M|^T + |Q||R|^T|P|^T)}{|S|} \right\|_\infty .$$

Proof. From equation (2) and equation (3), it is easy to get that

$$\begin{aligned} |[M \otimes N + (P \otimes Q)\Pi]| \mathsf{vec}(|R|) &\leq [|M| \otimes |N| + (|P| \otimes |Q|)\Pi] \mathsf{vec}(|R|) \\ &= (|M| \otimes |N|)\mathsf{vec}(|R|) + (|P| \otimes |Q|)\mathsf{vec}(|R|^T). \end{aligned}$$

Taking norms (and dividing by $|S|$ before doing so for the second inequality in the statement) proves the lemma. □

Corollary 1. *In the hypothesis of Theorem 1 we have*

(a) $m(A_W^\dagger, A) \leq \frac{\left\| |A_W^\dagger||A||A_W^\dagger| + |(A^T W A)^{-1}||A^T||W(I - AA_W^\dagger)| \right\|_{max}}{\left\| A_W^\dagger \right\|_{max}}$,

(b) $c(A_W^\dagger, A) \leq \left\| \frac{|A_W^\dagger||A||A_W^\dagger| + |(A^T W A)^{-1}||A^T||W(I - AA_W^\dagger)|}{A_W^\dagger} \right\|_{max}$.

Proof. Using Theorem 1 and Lemma 5 with $M = -{A_W^\dagger}^T$, $N = A_W^\dagger$, $P = W(I_m - AA_W^\dagger)$, $Q = (A^T W A)^{-1}$, $R = A$ and $S = \mathsf{vec}(A_W^\dagger)$, we obtain

$$\begin{aligned} m(A_W^\dagger, A) &\leq \frac{\left\| \mathsf{vec}(|A_W^\dagger||A||A_W^\dagger| + |(A^T W A)^{-1}||A^T||W(I - AA_W^\dagger)|) \right\|_\infty}{\left\| \mathsf{vec}(A_W^\dagger) \right\|_\infty} \\ &= \frac{\left\| |A_W^\dagger||A||A_W^\dagger| + |(A^T W A)^{-1}||A^T||W(I - AA_W^\dagger)| \right\|_{max}}{\left\| A_W^\dagger \right\|_{max}} \end{aligned}$$

and

$$\begin{aligned} c(A_W^\dagger, A) &\leq \left\| \frac{\mathsf{vec}(|A_W^\dagger||A||A_W^\dagger| + |(A^T W A)^{-1}||A^T||W(I - AA_W^\dagger)|)}{\mathsf{vec}(A_W^\dagger)} \right\|_\infty \\ &= \left\| \frac{|A_W^\dagger||A||A_W^\dagger| + |(A^T W A)^{-1}||A^T||W(I - AA_W^\dagger)|}{A_W^\dagger} \right\|_{max} . \end{aligned}$$

□

4 Weighted least squares problems

We consider weighted least squares problems [10, 11, 14]

$$\|W^{\frac{1}{2}}(Ax-b)\|_2 = \min_{v\in\mathbb{R}^n} \|W^{\frac{1}{2}}(Av-b)\|_2. \tag{7}$$

where $A \in \mathbb{R}^{m\times n}$, $\mathsf{rank}(A) = n$, and $b \in \mathbb{R}^m$. As we already know that there exists a unique minimizer x for (7). This minimizer be as below,

$$x = A_W^{\dagger} b = (W^{\frac{1}{2}}A)^{\dagger} W^{\frac{1}{2}} b.$$

Let $\Delta b \in \mathbb{R}^m$, and $\Delta A \in \mathbb{R}^{m\times n}$ such that $\mathsf{rank}(A+\Delta A) = n$, Consider the problem

$$\min_{\omega\in\mathbb{R}^n} \|W^{\frac{1}{2}}((A+\Delta A)\omega - (b+\Delta b))\|_2. \tag{8}$$

Then there is a unique minimizer y and letting $\Delta x := y - x$, we have

$$\Delta x = (A+\Delta A)_W^{\dagger}(b+\Delta b) - x \tag{9}$$

The mixed and componentwise condition numbers for WLS are defined as follows:

$$m_{wls}(A,b) := \lim_{\varepsilon\to 0} \sup_{\substack{|\Delta A|\le\varepsilon|A| \\ |\Delta b|\le\varepsilon|b|}} \frac{\|\Delta x\|_\infty}{\varepsilon\,\|x\|_\infty},$$
$$c_{wls}(A,b) := \lim_{\varepsilon\to 0} \sup_{\substack{|\Delta A|\le\varepsilon|A| \\ |\Delta b|\le\varepsilon|b|}} \frac{1}{\varepsilon}\left\|\frac{\Delta x}{x}\right\|_\infty.$$

Just an in the previous section, to comfortably make use of Lemma 1, we define the mapping$\psi : V \times \mathbb{R}^m \to \mathbb{R}^n$ by

$$\psi(g,f) := (\mathsf{vec}^{-1} g)_W^{\dagger} f.$$

Note that $m_{wls} = m(\psi; a, b)$ and $c_{wls} = c(\psi; a, b)$.

Lemma 6. *The set $V \times \mathbb{R}^m$ is open and ψ is a continuous mapping on $V \times \mathbb{R}^m$. In addition, for all $(a,b) \in V \times \mathbb{R}^m$, ψ is Fréchet differentiable at (a,b) and has the matrix expression $\mathrm{D}\psi(a,b) = [H(A,b), A_W^{\dagger}]$, where*

$$H(A,b) = -(x^T \otimes A_W^{\dagger}) + (A^T W A)^{-1} \otimes r^T,$$
$$r = W(b - Ax).$$

Proof. The first statement is trivial. We next proceed with the claimed equality. To do so, note that

$$\begin{aligned}\Delta x &\approx -A_W^{\dagger}\Delta A x + (A^T W A)^{-1}(\Delta A)^T r + A_W^{\dagger}\Delta b \\ &= [-(x^T \otimes A_W^{\dagger}) + (r^T \otimes (A^T W A)^{-1})\Pi, A_W^{\dagger}] \begin{bmatrix}\delta a \\ \delta b\end{bmatrix} \\ &= [-(x^T \otimes A_W^{\dagger}) + (A^T W A)^{-1} \otimes r, A_W^{\dagger}] \begin{bmatrix}\delta a \\ \delta b\end{bmatrix}\end{aligned}$$

where the first line is from substituting equation (6) into equation (9), the second line follows from (3) and (4) and the fact that vec applied to a vector yields the vector itself, and the last line follows from (5).
We can rewrite

$$\psi(a+\Delta a, b+\Delta b) - \psi(a,b) \approx [-(x^T \otimes A_W^\dagger) + (A^TWA)^{-1} \otimes r, A_W^\dagger] \begin{bmatrix} \delta a \\ \delta b \end{bmatrix}$$

Then the Fréchet derivative of ψ at (A,b) is

$$\mathrm{D}\psi(a,b) = [-(x^T \otimes A_W^\dagger) + (A^TWA)^{-1} \otimes r, A_W^\dagger].$$

□

We can now give expressions for the mixed and componentwise condition number of WLS. Recall, for the first we assume $x \neq 0$ and for the second $x_i \neq 0$ for $i = 1, \ldots, n$.

Theorem 2. *Let $A \in \mathbb{R}^{m\times n}$, $\mathsf{rank}(A) = n$, and $b \in \mathbb{R}^m$. We have*

$$m_{wls}(A,b) = \frac{\left\| |H(A,b)|\mathsf{vec}(|A|) + |A_W^\dagger||b| \right\|_\infty}{\|x\|_\infty},$$

$$c_{wls}(A,b) = \left\| \frac{|H(A,b)|\mathsf{vec}(|A|) + |A_W^\dagger||b|}{x} \right\|_\infty.$$

Furthermore, if $r = 0$ (i.e.,for consistent case), we have

$$m_{wls}(A,b) = \frac{\left\| |A_W^\dagger||A||x| + |A_W^\dagger||b| \right\|_\infty}{\|x\|_\infty},$$

$$c_{wls}(A,b) = \left\| \frac{|A_W^\dagger||A||x| + |A_W^\dagger||b|}{x} \right\|_\infty.$$

Proof. By Lemmas 2 and 6

$$m_{wls}(A,b) = \frac{\left\| |\mathrm{D}(a,b)| \begin{bmatrix} |a| \\ |b| \end{bmatrix} \right\|_\infty}{\|x\|_\infty} = \frac{\left\| [|H(A,b)|, |A_W^\dagger|] \begin{bmatrix} |a| \\ |b| \end{bmatrix} \right\|_\infty}{\|x\|_\infty}$$
$$= \frac{\left\| |H(A,b)|\mathsf{vec}(|A|) + |A_W^\dagger||b| \right\|_\infty}{\|x\|_\infty}$$

and

$$c_{wls}(A,b) = \left\| \frac{|\mathrm{D}\psi(a,b)| \begin{bmatrix} |a| \\ |b| \end{bmatrix}}{x} \right\|_\infty = \left\| \frac{|H(A,b)|\mathsf{vec}(|A|) + |A_W^\dagger||b|}{x} \right\|_\infty.$$

For consistent case replace r by 0 in $H(A,b)$ to obtain

$$m_{wls}(A,b) = \frac{\left\| |A_W^{\dagger}||A||x| + |A_W^{\dagger}||b| \right\|_{\infty}}{\|x\|_{\infty}}$$

and

$$c_{wls}(A,b) = \left\| \frac{|A_W^{\dagger}||A||x| + |A_W^{\dagger}||b|}{x} \right\|_{\infty}.$$

□

Corollary 2. *We have the following bounds:*

$$m_{wls}(A,b) \leq m_{wls}^{upper} := \frac{\left\| |A_W^{\dagger}||A||x| + |(A^TWA)^{-1}||A^T||r| + |A_W^{\dagger}||b| \right\|_{\infty}}{\|x\|_{\infty}}$$
$$c_{wls}(A,b) \leq c_{wls}^{upper} := \left\| \frac{|A_W^{\dagger}||A||x| + |(A^TWA)^{-1}||A^T||r| + |A_W^{\dagger}||b|}{|x|} \right\|_{\infty}$$

Proof. From Theorem 2, equality (5), and Lemma 5, we have

$$\begin{aligned} m_{wls}(A,b) &= \frac{\left\| |[-(x^T \otimes A_W^{\dagger}) + (r^T \otimes (A^TWA)^{-1}\Pi]|\mathsf{vec}(|A|) + |A_W^{\dagger}||b| \right\|_{\infty}}{\|x\|_{\infty}} \\ &\leq \frac{\left\| |A_W^{\dagger}||A||x| + |(A^TWA)^{-1}||A^T||r| + |A_W^{\dagger}||b| \right\|_{\infty}}{\|x\|_{\infty}} \end{aligned}$$

and

$$\begin{aligned} c_{wls} &= \left\| \frac{|[(x^T \otimes A_W^{\dagger}) - (r^T \otimes (A^TWA)^{-1})\Pi]|\mathsf{vec}(|A|) + |A_W^{\dagger}||b|}{x} \right\|_{\infty} \\ &\leq \left\| \frac{|A_W^{\dagger}||A||x| + |(A^TWA)^{-1}||A^T||r| + |A_W^{\dagger}||b|}{x} \right\|_{\infty}. \end{aligned}$$

□

Condition numbers bound the *worst-case* sensitivity of an input data only to small perturbations. If ε is the size of the perturbation, a term $\mathcal{O}(\varepsilon^2)$ is neglected and therefore, the bound only holds for ε small enough. One says that condition numbers are *first order* bounds for these sensitivities. The following result exhibits such unrestricted perturbation bounds for WLS.

Theorem 3. *Let A, $\Delta A \in \mathbb{R}^{m\times n}$ satisfy $\mathsf{rank}(A) = \mathsf{rank}(A+\Delta A) = n$. Let $\Delta b \in \mathbb{R}^m$ and x, y the solutions of (7) and (8) respectively. If for some $E \in \mathbb{R}^{m\times n}$*

and $f \in \mathbb{R}^m$ we have $|\Delta A| \le \varepsilon E$ and $|\Delta b| \le \varepsilon f$, then

$$\frac{\|y-x\|_\infty}{\|x\|_\infty} \le \varepsilon \frac{\left\| |[(x^T \otimes A_W^\dagger) - (A^T W A)^{-1} \otimes r^T]| \mathsf{vec}(E) + |A_W^\dagger||f| \right\|_\infty}{\|x\|_\infty}$$
$$\frac{\|s-r\|_\infty}{\|r\|_\infty} \le \varepsilon \frac{\left\| |[x^T \otimes P + {A_W^\dagger}^T \otimes r^T]| \mathsf{vec}(E) + |P|f \right\|_\infty}{\|r\|_\infty}$$

where $s = W(b + \Delta b - (A+\Delta A)y)$ and $P = W(I_m - AA_W^\dagger)$. Both inequality are sharp.

To prove Theorem 3 we need some preparation.

Lemma 7. *Let A, ΔA, b, Δ b, x, y, E, and f be as in the hypothesis of Theorem 3. Then*

$$\begin{aligned} y - x &= [-(x^T \otimes A_W^\dagger) + (A^T W A)^{-1} \otimes r^T, A_W^\dagger] \begin{bmatrix} \delta a \\ \Delta b \end{bmatrix}, \\ s - r &= [-(x^T \otimes P) - {A_W^\dagger}^T \otimes r^T, P] \begin{bmatrix} \delta a \\ \Delta b \end{bmatrix}, \end{aligned} \tag{10}$$

where $s = W(b + \Delta b - (A + \Delta A)y), P = W(I_m - AA_W^\dagger)$.

Proof. By using equation (6) we can get that

$$\begin{aligned} y - x &= -A_W^\dagger \Delta A x + (A^T W A)^{-1} \Delta A^T r + A_W^\dagger \Delta b, \\ s - r &= -P \Delta A x - {A_W^\dagger}^T \Delta A^T r + P \Delta b. \end{aligned}$$

□

Then

$$\begin{aligned} y - x &= \mathsf{vec}(-A_W^\dagger \Delta A x + (A^T W A)^{-1} \Delta A^T r) + A_W^\dagger \Delta b \\ &= [-(x^T \otimes A_W^\dagger) + (A^T W A)^{-1} \otimes r^T, A_W^\dagger] \begin{bmatrix} \delta a \\ \Delta b \end{bmatrix} \end{aligned}$$

and similarly for the second equation.

For the proof of the following lemma, we use an idea from [7].

Lemma 8. *Let $A \in \mathbb{R}^{m \times n}$, $D = \mathsf{Dg}(d) \in \mathbb{R}^{n \times n}$, and $v \in \mathbb{R}^n$ satisfying $|v_i| \le \varepsilon |d_i|$ for $i = 1, \ldots, n$. Then*

$$\left\| AD \frac{v}{d} \right\|_\infty \le \varepsilon \|AD\|_\infty,$$

and v can be chosen to make the upper bound attainable.

Proof. The first inequality is easy to prove. For the second statement, let $i \leq m$ such that

$$\|AD\|_\infty = \sum_{k=1}^{n} |A_{ik} d_k|.$$

Define v by $v_k = \varepsilon \mathsf{sgn}(A_{ik} d_k) d_k$, where sgn is the sign function,i.e.,$x = 1$ if $x \geq 0$ and $\mathsf{sgn}(x) = -1$ otherwise. Then

$$\frac{v}{d} = \varepsilon \begin{bmatrix} \mathsf{sgn}(A_{i1} d_1) \\ \vdots \\ \mathsf{sgn}(A_{in} d_n) \end{bmatrix} \qquad and \qquad \left\| \frac{v}{d} \right\|_\infty = \varepsilon$$

and the ith row of $AD\frac{v}{d}$ is given by

$$(AD\frac{v}{d})_i = \varepsilon[A_{i1} d_1 \mathsf{sgn}(A_{i1} d_1) + \cdots + A_{in} d_n \mathsf{sgn}(A_{in} d_n)] = \varepsilon[|A_{i1} d_1| + \ldots + |A_{in} d_n|].$$

Therefore

$$\left\| AD\frac{v}{d} \right\|_\infty \leq \varepsilon \|AD\|_\infty .$$

□

Proof of Theorem 3. By Lemma 7, writing $e = \mathsf{vec}(E)$,

$$\begin{aligned} y - x &= [-(x^T \otimes A_W^\dagger) + (A^T W A)^{-1} \otimes r^T, A_W^\dagger] \begin{bmatrix} \delta a \\ \Delta b \end{bmatrix} \\ &= [-(x^T \otimes A_W^\dagger) + (A^T W A)^{-1} \otimes r^T, A_W^\dagger] \begin{bmatrix} \mathsf{Dg}(e) & 0 \\ 0 & \mathsf{Dg}(f) \end{bmatrix} \begin{bmatrix} \frac{\delta a}{e} \\ \frac{\Delta b}{f} \end{bmatrix}. \end{aligned}$$

Taking norms and using Lemma 8, we obtain

$$\begin{aligned} \|y - x\|_\infty &\leq \varepsilon \left\| [-(x^T \otimes A_W^\dagger) + (A^T W A)^{-1} \otimes r^T, A_W^\dagger] \begin{bmatrix} \mathsf{Dg}(e) & 0 \\ 0 & \mathsf{Dg}(f) \end{bmatrix} \right\|_\infty \\ &\leq \varepsilon \left\| |[(x^T \otimes A_W^\dagger) - (A^T W A)^{-1} \otimes r^T]| \mathsf{vec}(E) + |A_W^\dagger||f| \right\|_\infty \end{aligned}$$

the second line as in Lemma 2. This proves first inequality of (10). Sharpness follows from Lemma 8.

The second inequality can be proved similarly.

In what follows we assume that $b \notin \mathcal{R}(A)$, where $\mathcal{R}$ denotes the range of A. That is , the residual vector $r \neq 0$. For componentwise results, we also assume that $r_i \neq 0$ for $i = 1, \ldots, m$. Define the mixed and componentwise condition numbers for r as

$$\begin{aligned} m_{res}(A, b) &:= \lim_{\varepsilon \to 0} \sup_{\substack{|\Delta A| \leq \varepsilon |A| \\ |\Delta b| \leq \varepsilon |b|}} \frac{\|\Delta r\|_\infty}{\varepsilon \|r\|_\infty}, \\ c_{res}(A, b) &:= \lim_{\varepsilon \to 0} \sup_{\substack{|\Delta A| \leq \varepsilon |A| \\ |\Delta b| \leq \varepsilon |b|}} \frac{1}{\varepsilon} \left\| \frac{\Delta r}{r} \right\|_\infty . \end{aligned}$$

Define the function $\Phi : V \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$\Phi(g, f) := W(I_m - (\mathsf{vec}^{-1} g)(\mathsf{vec}^{-1} g)_W^{\dagger})f.$$

Then $m_{res}(A, b) = m(\Phi; a, b)$ and $c_{res}(A, b) = c(\Phi; a, b)$.

Lemma 9. *The function Φ is continuous. Moreover, for all $(a, b) \in V \times \mathbb{R}^m$, it is Fréchet differentiable at (a, b) and has the matrix expression $\mathrm{D}\Phi(a, b) = [Q(A, b), P]$ where*

$$Q(A, b) = -(x^T \otimes P) + {A_W^{\dagger}}^T \otimes r \quad \text{and} \quad P = W(I_m - AA_W^{\dagger}).$$

Proof. Let $(\delta a, \Delta b)$ are a perturbation of (a, b) and $\Delta A = \mathsf{vec}^{-1}(\delta a)$. It is easy to see that, omitting second (and higher)-order terms, the perturbed residual vector $r + \Delta r$ satisfies

$$r + \Delta r \approx r + P\Delta b - P\Delta A x - {A_W^{\dagger}}^T \Delta A^T r,$$

using (3),

$$\Delta r \approx [-(x^T \otimes P) + {A_W^{\dagger}}^T \otimes r^T, P] \begin{bmatrix} \delta a \\ \delta b \end{bmatrix}.$$

So

$$\mathrm{D}\Phi(a, b) = [-(x^T \otimes P) + {A_W^{\dagger}}^T \otimes r^T, P].$$

□

Theorem 4. *With the notation above, the mixed and componentwise condition numbers for r satisfy*

$$m_{res}(A, b) = \frac{\| |Q(A, b)| \mathsf{vec}(|A|) + |P||b| \|_\infty}{\|r\|_\infty}$$

$$c_{res}(A, b) = \left\| \frac{|Q(A, b)| \mathsf{vec}(|A|) + |P||b|}{r} \right\|_\infty.$$

Proof. It follows from Lemmas 2 and 9 that

$$\begin{aligned} m_{res}(A, b) = m(\Phi; a, b) &= \frac{\left\| |\mathrm{D}\Phi(a, b) \begin{bmatrix} |a| \\ |b| \end{bmatrix} | \right\|_\infty}{\|\Phi(a, b)\|_\infty} \\ &= \frac{\| |Q(A, b)| \mathsf{vec}(|A|) + |P||b| \|_\infty}{\|r\|_\infty} \end{aligned}$$

and

$$\begin{aligned} c_{res}(A, b) = c(\Phi; a, b) &= \left\| \frac{|\mathrm{D}\Phi(a, b) \begin{bmatrix} |a| \\ |b| \end{bmatrix} |}{\Phi(a, b)} \right\|_\infty \\ &= \left\| \frac{|Q(A, b)| \mathsf{vec}(|A|) + |P||b|}{r} \right\|_\infty. \end{aligned}$$

□

The next corollary gives easier to compute upper bounds for the residual vector mixed and componentwise condition numbers.

Corollary 3. *Let* $A \in \mathbb{R}^{m\times n}$ *satisfy* $\mathsf{rank}(\mathsf{A}) = n$,

$$m_{res}(A,b) = \frac{\||Q(A,b)|\mathsf{vec}(|A|) + |P||b|\|_\infty}{\|r\|_\infty}$$
$$c_{res}(A,b) = \left\|\frac{|Q(A,b)|\mathsf{vec}(|A|) + |P||b|}{r}\right\|_\infty .$$

Proof. Using Theorem 4, equality (5) and Lemma 5, we obtain

$$\begin{aligned} m_{res}(A,b) &= \frac{\||Q(A,b)|\mathsf{vec}(|A|) + |P||b|\|_\infty}{\|r\|_\infty} \\ &\leq \frac{\left\||P||A||x| + |{A_W^\dagger}^T||A^T||r| + |P||b|\right\|_\infty}{\|r\|_\infty} \end{aligned}$$

and

$$\begin{aligned} c_{res}(A,b) &= \left\|\frac{|Q(A,b)|\mathsf{vec}(|A|) + |P||b|}{r}\right\|_\infty \\ &\leq \left\|\frac{|P||A||x| + |{A_W^\dagger}^T||A^T||r| + |P||b|}{|r|}\right\|_\infty . \end{aligned}$$

□

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Address:

Zhao Li
Institute of Mathematics, School of Mathematical Sciences, Fudan University, Shanghai, 200433, P. R. China
E-mail: 062018032@fudan.edu.cn

Jie Sun
Department of Mathematics and Physics, Shanghai Institute of Technology, Shanghai, 200235, P. R. of China
E-mail: sunjie@sit.edu.cn