Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **23:2** (2009), 1–11

VISUALISATION - INTERPRETATION UNDERSTANDING: TRAVELLING ON THE ROYAL ROAD TO MATHEMATICAL ABSTRACTION IN OLD BOOTS: DESCRIPTIVE GEOMETRY

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Abstract

Sketching and computer visualisation are standard communication media in Technology and Natural Science as well as in Mathematics. Of any visualisation we demand easy interpretability by the "educated" viewer not only can "read" the meaning of the figure but also gains some understanding of the visualised problem. Such a "visual communication" needs schooling and training. Descriptive Geo-metry provides some simple but effective rules and techniques for such a visual communication. Using properties classical geometric mappings, e.g. normal projections or cyclography, can give insight to problems, which sometimes are rather hard to tackle purely by mathematical calculation. Sometimes we receive even an easy proof of the problem, a proof "by looking at the figure", such that one is encouraged to speak of a "geometric royal road" to the problem. Some special examples of such problems shall illustrate this statement. Most of the shown examples are not new, but they are not at all very well known! What should be shown is that Descriptive Geometry is much more than just an engineering graphics tool for visualizing 3D-objects. It strongly supports mathematicians, too. To emphasize that Descriptive Geometry is an intellectual tool besides for visualization the figures in this work are freehand drawn sketches instead of perhaps more beautiful computer generated drawings.

1 Preface

In most countries we face decreasing geometrical contents in mathematical syllabuses in spite of the existence of advanced dynamical graphics software and the increasing need of skills of interpreting computer generated images and spatial and multi-parametric representation. One of reasons for this development is that new mathematical disciplines are branching up, while classical geometry is meant to be "ready developed" and "easy to find in references". But geometry as a tool for

2000 *Mathematics Subject Classifications.* 51Nxx, 51M04
Key words and Phrases. : descriptive geometry, geometric mappings, normal projection, cyclography, generalised stereographic mapping, quadrics of revolution, pipe connections.

problem proving has to be thought and trained, otherwise it will not be available as a tool and its elegance will never be felt. This statement implies the existence of "trainers" and of course their education, too! In the following some special examples of mathematical problems shall show the special geometric way of treatment called "synthetic geometric reasoning". In these cases the geometric approach is clearly easier than the (usual) analytic approach by calculation.

2 Equiareal-Faced Tetrahedra

Theorem 1. *A tetrahedron ABCD with faces having equal area has congruent faces.*

This statement has a long history and it is hard to say, when it was discovered and who published it for the first time. For references and generalisations see [4]; (Related to this topic is also [7]). An analytic proof by using Heron's formula for the area of a triangle was recently given by N. Alexandrova*. The author came in contact with the problem in the late 1960ies as a student in Vienna, via H. Vogler. We now will give a geometric

Proof: We keep the distance d of the two (skew) edge lines AB and CD for the moment fixed, as well as e.g. vertex A and line CD. Now, moving segment $[CD]$ along line CD will keep the area $a(ACD)$ of triangle ACD fixed because of Cavalieri's principle. Let P resp. Q be the feet of the common bi-normal n of AB and CD on AB resp. CD. Let further φ_A mean the angle between n and face ACD and φ_B be the angle between n and face BCD according to Fig.1. This figure shows a top and side view projection of $ABCD$ with rays parallel to n resp. parallel to edge CD. The distances of A resp. B to plane PCD are labelled with h_A resp. h_B and $d := dist(P, Q)$.

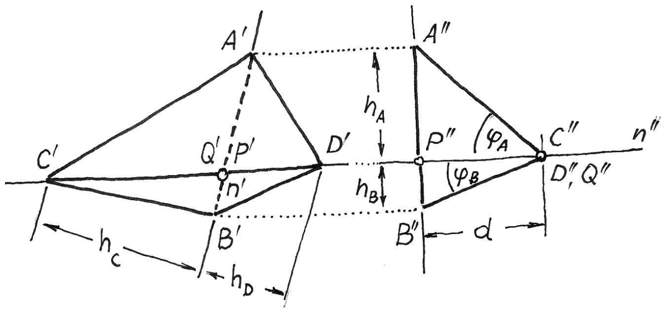

Fig. 1: Ground view and side view of a tetrahedron $ABCD$ (Ground view parallel to bi-normal n of edges AB and CD, side view parallel to edge CD.)

*The problem was posed in one of the National Mathematical Olympic Competition in Russia; the mentioned solution was communicated in 2007 by Mrs. N. Alexandrova (Novosibirsk) to the author, thus stimulating him to write this article.

For the area a of face (ACD) we find

$$a(ACD) = a(PCD)/\cos\varphi_A = \overline{CD}\frac{d}{2}/\cos\varphi_A$$

and for $a(BCD)$ we have a similar expression. Therefore, from $a(ACD) = a(BCD)$, follows

$$a(ACD) = (\overline{CD}\frac{d}{2})/\cos\varphi_A = a(BCD) = (\overline{CD}\frac{d}{2})/\cos\varphi_B.$$

Because of $\tan\varphi_i = d/h_i$ this has $\cos\varphi_A = \cos\varphi_B \wedge h_A = h_B$ as a consequence. Therewith, in a normal projection parallel to the bi-normal $n = PQ$ the vertices A and B occur at equal distances $h_A = h_B$ from the image of CD, c.f. Fig. 1. Of course $h_A = h_B \neq 0$, and A and B cannot be on the same side of plane PCD, otherwise $ABCD$ would be degenerate.

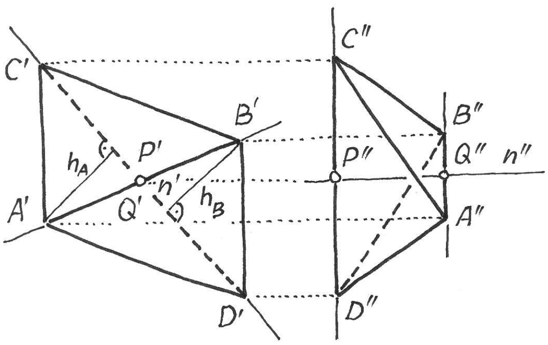

Fig. 2: Top view of tetrahedron $ABCD$ parallel to common normal n of edge lines AB and CD and side view with an image plane π_2 parallel to AC and BD.

Now we demand $a(CAB) = a(DAB)$ and we conclude that also C and D occur at equal distances from the image of AB. Thus the top view image of the tetrahedron necessarily must be a parallelogram, see Fig. 2, (top view at left)! Therefore the edges $[AC]$ and $[BD]$ have equal lengths, as their images are congruent and they have the same slope with respect to the top view image plane π_1 (which means that their images are shortened by the same factor). The same argument is valid for the edges $[AD]$ and $[BC]$.

As a result in between we can state

Theorem 2. *A tetrahedron, where two faces have equal area and the remaining two faces also have equal area, has two pairs of (skew) edges of equal length. The third pair may consist of edges of different lengths.*

Demanding all faces having equal area we can project the tetrahedron $ABCD$ normally e.g. onto a plane parallel to AD and CB (see Fig. 2, right). Also this image should be a parallelogram. The only possibility to receive it from the former parallelogram (Fig. 2, top view) is that the parallelogram is indeed a rectangle. Then the diagonals $[AB]$ and $[CD]$ have equal lengths, too, and, as they are shown in true length in Fig. 2, (they are parallel to the image plane), also this third pair consists of (skew) edges of equal length. Therewith the (in general three different) side lenghts of the face-triangles of $ABCD$ must be the same for all four triangles and thus they are congruent. Therewith Theorem 1 is proved. □

Remark: Some people call an equifaced tetrahedron also a "sphenisk", c.f. [2].

3 Conformity of a Stereographic Projection

Theorem 3. *A central projection $\sigma : E^3 \longrightarrow \pi$ of a euclidean 3-space E^3 onto a (projec-tive extended) plane π, restricted to a sphere Σ, is a conformal mapping, if and only if the projection centre N is a point of Σ and π is parallel to the tangent plane of Σ in N.*

Remark: This theorem has a long history, too, and it nicely connects different parts of mathematics. For example, the real and two-dimensional plane π is seen as a model for the (one-dimensional) complex line; it is known by the name "Gaussian plane". The sphere Σ models this projective extended complex line and with this interpretation Σ is named "Riemannian sphere". Usually one finds analytic proofs of theorem 3, see e.g. [1], this idea of a geometric proof can be found e.g. in [3]. (One can also find references, where the Gaussian plane and the Riemannian sphere are called "complex plane" resp. "complex sphere", which is at least misleading and conceptually wrong.)

Proof: Let us take the "equator plane" of Σ as the image plane π of σ. At first we show that the centre N of σ must necessarily be the "North pole" of Σ. For an arbitrary centre point $N \in E^3$ conformity is at least true for the point S ("South pole"). For a point P on the equator and two tangents a, b in P conformity is true, only if the centre N belongs to the tangent in the North pole to the "meridian" through P, see Fig. 3.

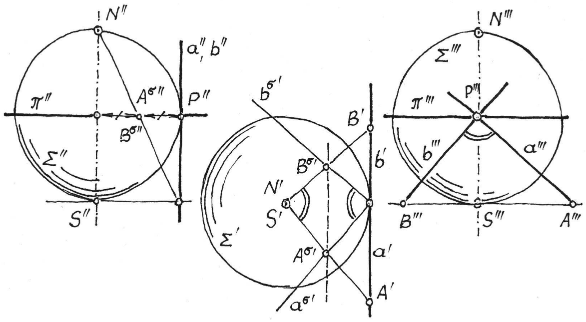

Fig. 3: Front- and top- and side view of the stereographic projection of two tangents a, b at an equator point P.

We intersect a and b with the tangent plane τ_S in S receiving intersection points A, B, whereby we have equal angle measures $\angle aPb = \angle ASB$. Now we project the tangents and angles onto π; for conformity to the images A^σ, B^σ, must belong to the symmetry line between S^σ and $P = P^\sigma$. As conformity should yield for any equator point, the only possible position for centre N is the North pole itself. So we have already $Z = N \in \Sigma$ as a necessary condition.

Now we consider an arbitrarily given point $P \in \Sigma$, as in Fig. 4. From the front view it is obvious that a central projection with North pole as the centre N delivers two congruent angles $\angle A^\sigma S^\sigma B^\sigma \cong \angle A^\sigma S^\sigma B^\sigma$, too.

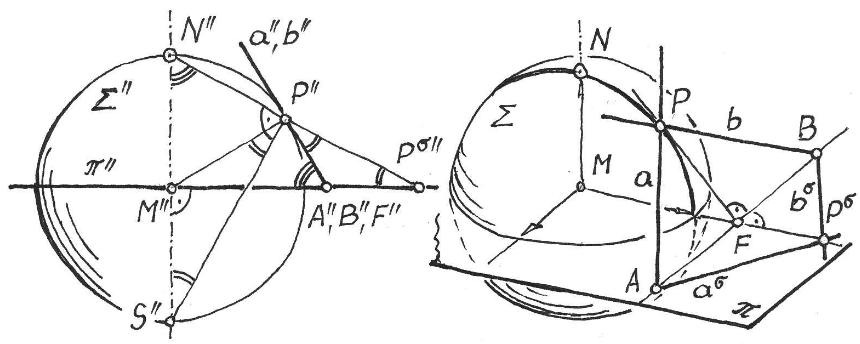

Fig. 4: Front- and axonometric view of the stereographic projection of two tangents a, b at an arbitrary point P.

For general points $P \in \Sigma$ we use similar projections as before, see Fig. 4: Let the tangents $a, b \subset \tau_P$ intersect the equator plane π in points A and B. The axonometric view Fig. 4 (at right) shows that the triangles $\triangle ABP$ and $\triangle ABP^\sigma$ have the same base AB and the same altitude foot point F on AB. Therefore they are congruent only if they have altitudes FP, FP^σ of the equal lengths. This we read off from the front view Fig. 4 (at left): The right-angled triangles $\triangle NPS$ and $\triangle NPM^\sigma$ are similar as well as the isosceles triangles $\triangle SMP$ and $\triangle PFP^\sigma$.

To prove that σ is "circle true" in the sense of Möbius, which means that circles on Σ are mapped onto circles or straight lines in π, we again use the front view projection and show by an elementary geometric statement that π intersects the projecting cone Γ of a circle $c \subset \Sigma$ $(N \in c)$ again by a circle c^σ. Supposed this be true, then there would exist a sphere Φ containing both circles c and c^σ.

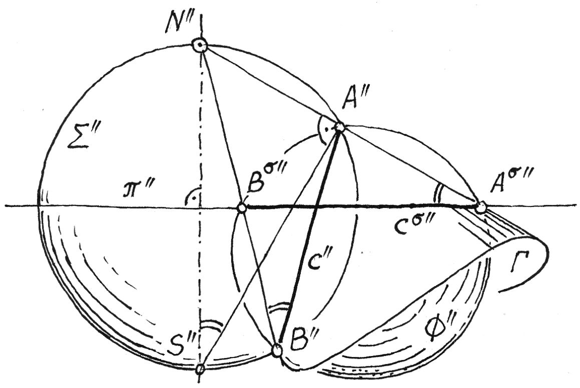

Fig. 5: Front view of the stereographic projection of a circle c of Σ onto the equator plane π.

In the front view image Fig.5 the lowest and highest point A and B of c as well as their σ-images must belong to a circle, namely the contour of the sphere Φ we are looking for. Evidently the triangles $\triangle A''S''N''$, $\triangle M''A^{\sigma''}N''$ are similar and thus $\angle N''S''A''$, $= \angle N''A^{\sigma''}M''$, because they are 'orthogonal angles'. Furthermore we find that $\angle N''S''A''$, $= \angle N''B''M''$, because they are angles over the same segment $[A'', N'']$ of the contour circle of Σ. Therefore the angles at B" and $A^{\sigma''}$ over the segment $[A'', B^{\sigma''}]$ are equal, too, and thus the points must have a circum circle, which could act as the contour of the sphere Φ.

There is still a gap in the proof: We now have the contour of a sphere Φ containing points $A^{\sigma''}, B^{\sigma''}$. But why is the image of c^σ circular?

Here the argument is not an elementary geometric one but one of elementary algebraic geometry: Sphere Φ contains the circle c as one planar 2^{nd} order component of the intersection$\Phi \cap \Gamma$, which in total is algebraic of order 4. Therefore the remaining component must be a curve of 2^{nd} order, too. So it must be a *planar* intersection component of Γ and a sphere Φ and thus it must be circle. □

4 C^1-Curves Consisting of Circular Arcs

In the planar case the historical "three-centred curves" belong to this topic. Here we shall study C^1-curves consisting of two circular arcs and interpolating two given (skew or intersecting) oriented line elements. Such circular arc curves are basic for the design of pipes consisting of cylindrical and torus parts.

Let us first start with the planar case, i.e. with two line elements $(A, a), (B, b) \subset \pi$ This gives the possibility to introduce the classical cyclography (see e.g. [5]) as a tool to solve that problem, before we use a higher dimensional generalisation of that mapping for the more general problem of skew line elements.

Remark: Cyclography is the well known mapping $^c : \{cycles \in \pi\} \to \{points \in \Pi\}$, whereby "cycles" mean oriented circles or points of a Euclidean plane π. Thereby the image of a positively oriented circle is a point P in the upper half space of Π with respect to π and its z-coordinate equals the radius of that cycle. Zero cycles are points in π. As a consequence, the cyclographic images of all cycles touching a given oriented line element $(A, a) \subset \pi$ form a line $\tilde{a}^c \subset \Pi$ orthogonal to a and with the slope angle 45 with respect to π. Lines with this property are named "isotropic" lines. The c-images of two touching cycles belong to such a line. All cycles touching a given cycle p are mapped to points of the cone of revolution through p with vertex P. Thus cyclography is a mean to transfer the planar problem concerning oriented circles in a problem of points and lines in the 3-space Π. Therewith the original problem now reads as follows: Find isotropic lines $\tilde{x}^c$ in Π, which meet the isotropic lines $\tilde{a}^c, \tilde{b}^c$ to the given line elements $(A, a), (B, b) \subset \pi$, see Fig. 6.

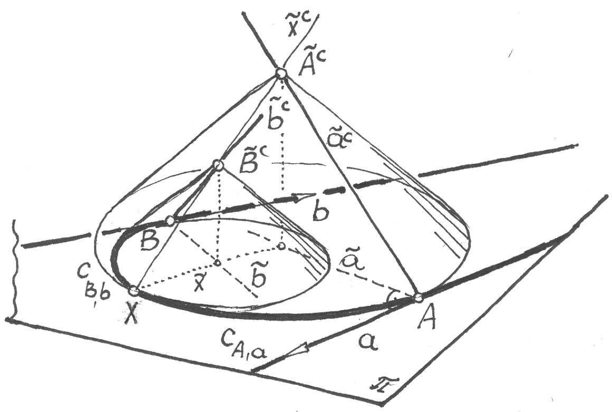

Fig. 6: Touching circular arcs through 2 given (oriented) line elements (A, a), (B, b); transformation by cyclography into a problem in 3-space.

The set of those isotropic lines is a ruled surface Φ with a cone of rotation as its director cone. As also the lines $\tilde{a}^c, \tilde{b}^c$ are isotropic, this ruled surface Φ must be a regulus on a hyperboloid Φ of rotation. It intersects π in a circle k through A and B and k is therefore the set of all possible points X, where cycles c_{Aa}, c_{Bb} through the given oriented line elements touch each other, see Fig.7. To construct this circle $k = \{X\}$ we consider the degenerate cases, where one of the circular arcs becomes a straight line, say a. Then the other arc through (B, b) must belong to a circle touching both, a and b, in B and T_0. Analogously there is a circle through (A, a) touching b in T_1 and the points A, B, T_0, T_1 are concyclic with the point k, Fig. 7.

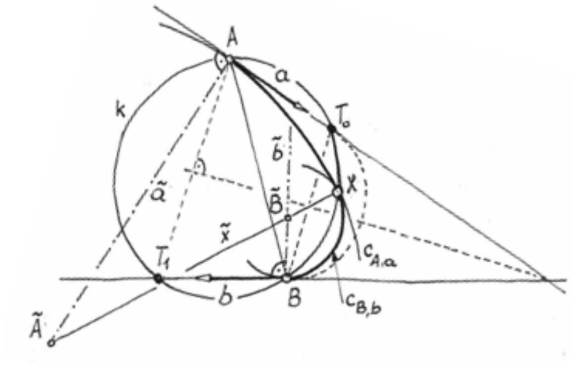

Fig. 7: Touching circular arcs through 2 given (oriented) line elements $(A, a), (B, b)$; construction of circle k of possible points X of contact.

Now we admit that the two (oriented) line elements (A, a) and (B, b) are skew in Π , see Fig. 8.

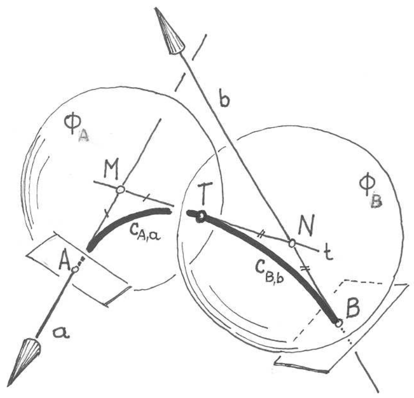

Fig. 8: Touching circular arcs through 2 given (oriented) skew line elements $(A, a), (B, b)$; construction of common line element (T, MN).

Two touching cycles c_{Aa}, c_{Bb} must have a common tangent t, which hits the lines a and b. Let these intersection points be $M \in a, N \in b$. On t the point of contact of c_{Aa}, c_{Bb} shall be labelled with T. Obviously the tangent segments $[MA]$ and $[MT]$ have equal lengths and so do the segments $[NB]$ and $[NT]$. So M and N are the centers of spheres Φ_A and Φ_B, which touch each other in T.

Now again we apply a generalized cyclography c, mapping each (oriented) sphere $\Psi \subset \Pi$ to a point $P := \Psi^\sigma$ in a 4-space Π^4. Analogously to the former cyclographic mapping the (signed) radius of Ψ gives the 4^{th} coordinate of P, while the other 3 coordinates are those of the centre of Ψ, c.f. [5]. So again, from the parabolic pencils of spheres $\{\Phi_A\}$ resp. $\{\Phi_B\}$ we receive two lines a^c resp. b^c in Π^4 with a

45^o-slope with respect to Π. As these lines span only a three-space Π^3, we have exactly the same problem as before: Find 45^o-slope lines x^c (they must belong to that three-space Π^3), which intersect a^σ as resp. b^σ. As we already know from the planar case, the result is a regulus of isotropic lines on a hyperboloid $\Phi \subset \Pi^3 \subset \Pi^4$. The normal projection of this regulus onto the given 3-space Π is the set of all common tangents t of the two circular arcs we look for. It is therefore an affine transform Φ of Φ^c.

We are now interested in the set of common points $T \in t$. This set must be the trace curve k of the hyperboloid $\Phi \subset \Pi^3 \subset \Pi^4$ in the space Π spanned by the given two line elements (A, a) and (B, b). Therewith k is a planar curve, which as in the planar case passes trough the given points A and B. We note that Φ is a quadric of revolution with an axis orthogonal to Π; therefore the trace curve k again must be a circle!

The plane of this circle k is spanned by A and B and a third point T. Such a point can be found to an arbitrarily chosen point M on a as follows, see Fig. 9:

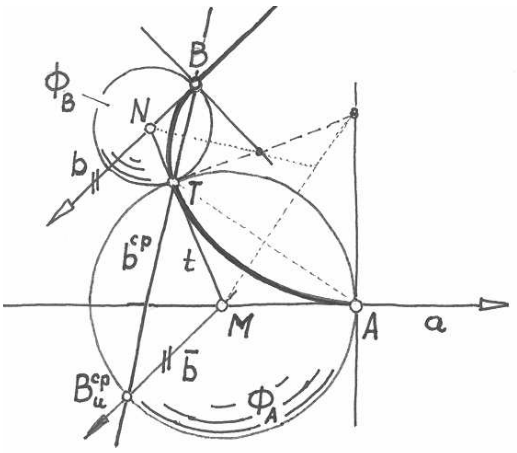

Fig. 9: Construction of a line element (T, t) to an arbitrarily chosen sphere ΦA using a central procection of Π^4 onto the 3-space Π spanned by the given line elements $(A, a), (B, b)$.

Choose a sphere Φ_A with centre $M \in a$ and map it to its cyclographic point M^c. Then the corresponding sphere Φ_B with centre $N \in a$ and radius $\overline{NB}$ will have to be mapped to a point N^c on b^c and the cone $\Gamma^c := M^c \vee \Phi_A$.

For the problem to intersect b^c with Γ^c we again use descriptive geometric ideas: We use the vertex M^c of Γ^c as the centre of a central projection $p : \Pi^4 \to \Pi^3$. Because of the 45^o-slope of b^c the vanishing point B_u^{cp} of b^c must be a point of Φ_A. We can get it by intersecting Φ_A with the (oriented!) parallel line $\bar{b}$ to b through M. The line connecting the trace point B of b^c with B_u^{cp} is already the central projection image of b^{cp} and, as the central projection image of Γ^c coincides with Φ_A (-Γ^c is "projecting" under the central projection with vertex M^c as centre -), we only need to intersect Φ_A with the line b^{cp} to receive the point T, where the two spheres Φ_A, Φ_B touch each other. That will say

$$T^c = b^c \cap \Gamma^c = T^{cp} = b^{cp} \cap \Phi_A = b^c \cap \Phi_A = T$$

and (T, MN) is the common line element of two circular arcs $c_{A,a}, c_{B,b}$, and all possible points of type T belong to the circumcircle k of triangle ABT.

Remark 1: H. Stachel and W. Fuhs [6] used another nice and very simple geometric idea to solve this "circular arc problem": If two circles touch each other, their axes must intersect and these circles belong to one sphere Σ. A normal plane at any point of one of the circles must contain the circle's axis and therewith pass through the centre O of Σ. To construct O and the sphere Σ one only has to intersect the normal planes to a at A and to b at B and the symmetry plane of $[A, B]$. The radius of Σ then is $\overline{OA}$. To find a point T on this sphere Σ we proceed as follows: Choose M on a, subscribe Σ the cone Γ_M with vertex M and intersect it with b. Of the two (in algebraic sense) possible intersection points only one fulfils the conditions caused by orientation of a and b. The rest of the construction then is obvious. This elegant procedure has the only little disadvantage that is does not work in the planar case, while the cyclographic approach described for the 3D-case also comprises the planar case, as it is depicted in Fig. 9.

Remark 2: For a calculation, which leads to a computer supported pipe design we only need to follow the geometric constructions described above:

Let $\underline{A}$ and $\underline{B}$ be the vectors describing the points A and B, and let $\underline{a}$, $\underline{b}$ be the normed direction vectors of a and b as well as $\underline{B} - \underline{A} = d$ Then

$$\underline{M} = \underline{A} + u\underline{a},\ \underline{N} = \underline{B} + v\underline{b},\ \underline{T} = (1-u)\underline{M} + u\underline{N}, \tag{1}$$

and from condition

$$||N - M|| = u + v \tag{2}$$

follows

$$v(u) = \frac{u(\underline{d} \cdot \underline{a})}{u(1 + \underline{a} \cdot \underline{b}) + \underline{d} \cdot \underline{b}}. \tag{3}$$

So we find indeed a projectivity from line a to line b generating the regulus Φ of common tangents MN, as expected. Further steps, e.g. to find the points T, are obvious, too.

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