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NOTES ON CONSTANT MEAN CURVATURE SURFACES AND THEIR GRAPHICAL PRESENTATION

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Abstract

In this paper graphical presentation some of the constant mean curvature surfaces (CMC surfaces) is given. This work is an extension of the results [3]. Interesting shapes and complicate structures of CMC surfaces obtained using Mathematica computer program are given.

1 Introduction

Let M be a 2-dimensional manifold and $f : M \longrightarrow R^3$ an immersion with at least C^2 differentiability. The Euclidean metrics on R^3 induces a metrics $ds^2 : T_PM \times T_PM \longrightarrow R$, where T_PM is the tangent space at $P \in M$. That generates the complex structure of the Riemann surface M.

We can choose the coordinates (u, v) on M so that ds^2 is a *conformal metrics.* This means that the vectors f_u and f_v are ortogonal and of equal positive length in R^3 at every point $f(P)$. Under such parametrization, which we call *conformal*, the first fundamental form is given by the matrix

$$g = (g_{ij}) = 4e^{2\hat{u}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $\hat{u} : M \longrightarrow R$.

The eigenvalues of the matrix $g^{-1}b$, where b is a matrix of the second fundamental form of f, are *the principal curvatures* k_1 and k_2. This gives the following expressions for the *mean* and *Gaussian curvatures*

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2}tr(g^{-1}b), \tag{1.2}$$

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$$K = k_1 k_2 = det(g^{-1}b). \tag{1.3}$$

If the mean curvature of a surface is identically zero, one speaks about minimal surfaces and the study of these surfaces is a field for itself. If the mean curvature is constant, but not zero, the surfaces are called *CMC surfaces*, which we are now considering.

The spheres and round cylinders are the first few examples of CMC surfaces. Much latter Delaunay classified all revolutional CMC surfaces and called them Delaunay surfaces. For some time, no new CMC surfaces were found. The question was opened whether there are any compact CMC surfaces other then the spheres. 1986. Wente was proved that must exist the CMC tori and futhermore there exist infinitely many constant mean curvature tori. Moreover, for each integer $g \geq 2$, there is a compact constant mean curvature surface of genus g.

The CMC surfaces can be regarded as interface surfaces in nature. Particulary, the interface shape is described by a constant mean curvature surface that satisfies some particular conditions. The interface shape separates a gas layer within a superhydrophic surface consisting of a square lattice of posts from a pressurized liquid above the surface.

2 Visualization of some CMC surfaces using Mathematica computer program

Based on the results from [3], [5], [8] and [10], we get interesting examples of the CMC surfaces. We use Mathematica computer program to present the structure of these surfaces.

2.1 Sphere

The sphere is the simplest example of the surfaces of nonzero mean curvature. It is easely shown to be the next parametrization with constant mean curvature $H = \frac{1}{2}$.

```
sphere[u_,v_]:={2Cos[u]Cos[v],2Cos[u]Sin[v],2Sin[u]}
ParametricPlot3D[sphere[u,v], {u,-Pi/2,Pi/2},{v,0,2Pi},
               PlotPoints->27]
```

2.2 Cylinder

The second simple example of CMC surfaces is the round cylinder. The next program consists two parametrizations: the first which is conformal and the second which is uncorfomal. Both of them gives the cylinder of radius 1 and of main curvature $H = \frac{1}{2}$.

```
<<Graphics`ParametricPlot3D`
cylinder1[u_, v_] :={-4u,-Sin[4v],-Cos[4v]}
p=ParametricPlot3D[cylinder1[u,v],{v,0,2Pi,Pi/72},{u,0,1,1},
```

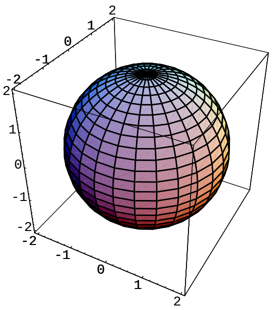


Figure 1: *A sphere*

```
                        ViewPoint->{1,1,1}, Boxed->False,Axes->False,
                        DisplayFunction->Identity]
cylinder2[u_,v_]:={Cos[v],Sin[v],u};
q=ParametricPlot3D[cylinder2[u,v],{v,0,2Pi,Pi/24},{u,0,2,2},
                        ViewPoint->{1,2,-2}, Boxed->False,Axes->False,
                        DisplayFunction->Identity]
Show[GraphicsArray[{p,q}],DispalayFunction->$DisplayFunction]
```

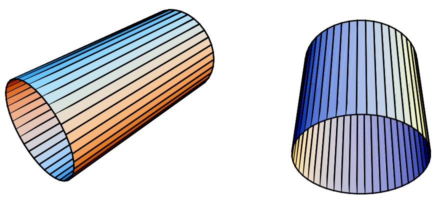


Figure 2: *The cylinders*

2.3 Delaunay surfaces

The locus of an ellipse as the point of contact rolls along a straight line in a plane is called the undulary. The locus of a focus of a hyperbola as the point of contact rolls along a straight line in a plane forms the curve which we call the nodary. Rotating each of the roulettes about its axis of rolling produces five types of surfaces with constant mean curvature in Euclidean space R^3, called *Delaunay surfaces*: the catenoids (by rolling a parabola which are the minimal surfaces), unduloids, nodoids, right circular cylinders (which are unduloids made by rolling a circle), and spheres (by rolling a degenerate ellipse of eccentricity 0).

2.3.1 Unduloid

Let the ellipse be given by the next equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$. The parametric equation of the curve in the plane-undulary is

$$undulary(u)=(x(u),y(u)), \tag{2.3.1.1}$$

where

$$x(u)=\int_0^u\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}d\phi+\frac{(a+\sqrt{a^2-b^2}\cos u)\sqrt{a^2-b^2}\sin u}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}, \tag{2.3.1.2}$$

$$y(u)=\frac{b(a+\sqrt{a^2-b^2}\cos u)}{\sqrt{a^2\sin^2u+b^2\cos^2u}}. \tag{2.3.1.3}$$

The unduloid comes by rotation of the undulary about the axe of rotation and has a equation:

$$unduloid(u,v)=(x(u),y(u)\cos v,y(u)\sin v). \tag{2.3.1.4}$$

The next program gives an undulary and unduloid for $a=1,5$ and $b=1$ and the functions x and y like (2.3.1.2) and (2.3.1.3).

```
<<Graphics‘ParametricPlot3D‘
undulary[u_]:={x,y}
unduloid[u_,v_]:={x, y Cos[v], y Sin[v]}
ParametricPlot[undulary[u],{u,0,4Pi},PlotRegion->{{0.1,0.8},
                {0.2,0.8}},PlotStyle->RGBColor[1,0,0]]
p=ParametricPlot3D[unduloid[u,v],{u, Pi,4Pi,Pi/12},
                {v, - Pi / 2, Pi /2,Pi/12}, PlotPoints->{40,20},
                Boxed->False,Axes->False,DisplayFunction->Identity];
q=ParametricPlot3D[unduloid[u,v],{u, Pi,4Pi,Pi/12},
                {v, - Pi / 2,3Pi / 2,Pi/12}, PlotPoints->{40,20},
                Boxed->False,Axes->False,DisplayFunction->Identity];
Show[GraphicsArray[{p,q}],DisplayFunction->$DisplayFunction]
```

2.3.2 Nodoid

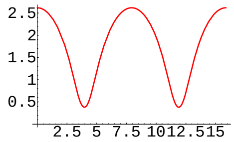
Let the hyperbola be given by the next equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>b>0$. The parametric equation of the curve in the plane-nodary is

$$nodary(u)=(x(u),y(u)), \tag{2.3.2.1}$$

where

$$x(u)=a(1-\cos u+\int_0^u\frac{\sin^2\phi}{\sqrt{sin^2\phi+b^2/a^2}}d\phi), \tag{2.3.2.2}$$

$$y(u)=a(\sin u+\sqrt{sin^2u+b^2/a^2}). \tag{2.3.2.3}$$



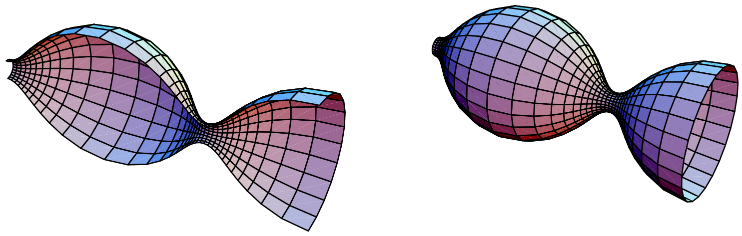


Figure 3: *An undulary, half and all of an unduloid*

The nodoid comes by rotation of the nodary about the axe of rotation and has the parametric equation

(2.3.2.4) $$nodoid(u,v) = (x(u), y(u)\cos v, y(u)\sin v).$$

For $a = 1,5$, $b = 1$ and (2.3.2.2), (2.3.2.3) we have the program:

```
<<Graphics‘ParametricPlot3D‘
nodary[u_]:={x,y}
nodoid[u_,v_]:={x,y*Cos[v],y*Sin[v]}
p=ParametricPlot[nodary[u],{u,0,4Pi},AspectRatio->Automatic,
                PlotStyle->RGBColor[1,0,0]]
q=ParametricPlot3D[nodoid[u,v],{u, - 5Pi / 2, 2Pi,Pi/12},
                    {v, - Pi/2, Pi/2,Pi/12 },PlotPoints->{40,20},
                    Boxed->False,Axes->False,DisplayFunction->Identity]
r=ParametricPlot3D[nodoid[u,v],{u, - 5 Pi / 2, 2Pi,Pi/12},
                    {v, - Pi,Pi,Pi/12 },PlotPoints->{40,20},Boxed->False,
                    Axes->False,DisplayFunction->Identity]
Show[GraphicsArray[{q,r}],DisplayFunction->$DisplayFunction]
```

2.4 Wente tori

The conformal parametrization of a torus is given by an immersion

$$f : C/\Gamma \to R^3,$$

where Γ is a 2-dimensional lattice. The simplest CMC tori were found by Wente and analytically stadied by Abresch and Walter. Walter proved that the set of all symetric tori which found by Wente are in one-to-one correspodence with the set

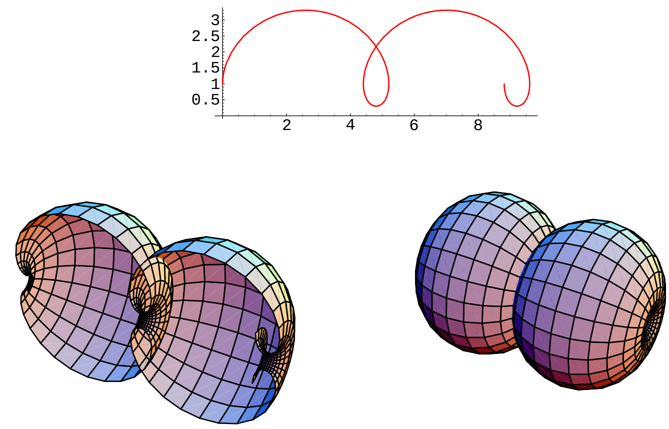


Figure 4: *A nodary, half and all of a nodoid*

of reduced fractions $l/n \in (1,2)$. For each l/n we call the corresponding symetric Wente torus $W_{l/n}$.

The parametric equation of the torus is:

$$torus(u,v) = (Z\cos(w-j) + \frac{\cos w}{2H}, Z\sin(w-j) + \frac{\sin w}{2H}, x_3), \tag{2.4.1}$$

where

$$Z = \sqrt{\frac{2}{H}}\frac{1}{\bar{\alpha}^2} \cdot \frac{((\bar{\alpha}^2 - b)\gamma^2\cos^2 u + p)\bar{\gamma}\cos v - (p\gamma^2\cos^2 u + (\bar{\alpha}^2 + b))\gamma\cos u}{\sqrt{p - 2b\gamma^2\cos^2 u - p\gamma^4\cos^4 u} \cdot (1 - T\cos u\cos v)},$$

$$w = \sqrt{H}\frac{2}{\alpha}\int_0^u \frac{1 + T^2\cos^2 t}{1 - T^2\cos^2 t}\frac{dt}{\sqrt{1 - k^2\sin^2 t}},$$

$$j = \tan^{-1}(\frac{\alpha}{2\sqrt{H}}\tan u\sqrt{1 - k^2\sin^2 u}) + (m-1)\pi, \quad \frac{(2m-3)\pi}{2} \leq u < \frac{(2m-1)\pi}{2},$$

$m \in N$,

$$x_3 = \frac{1}{\bar{\alpha}\sqrt{H}} \cdot (2T\frac{\cos u\sin v\sqrt{1 - \bar{k}^2\sin^2 v}}{1 - T\cos u\cos v} + \frac{1}{\bar{\gamma}}\int_0^v \frac{1 - 2\bar{k}^2\sin^2 t}{\sqrt{1 - \bar{k}^2\sin^2 t}}dt),$$

$$T = \gamma\bar{\gamma},$$

$$\gamma = \sqrt{\tan\theta}, \ \bar{\gamma} = \sqrt{\tan\bar{\theta}},$$

$$\alpha = \sqrt{4H\frac{\sin 2\bar{\theta}}{\sin 2(\theta + \bar{\theta})}}, \ \bar{\alpha} = \sqrt{4H\frac{\sin 2\theta}{\sin 2(\theta + \bar{\theta})}},$$

$$\bar{\theta} = 65.354955^\circ.$$

Some values of θ are given in the next table:

$W_{l/n}$	θ
$W4/3$	12.7898°
$W3/2$	17.7324°
$W6/5$	8.0983°
$W5/3$	21.4807°

Table 1: *The values of* θ

More values of θ one can find in [8].

For different values of θ we compute number g which present the number of the fundamental pieces. When l is odd (resp. l is even), $W_{l/n}$ consists of the union of $2n$ (resp. n) fundamental pieces. The number g determine the range of u.

In the sequel we give some Wente tori with their cut-aways using the next program:

```
<<Graphics`ParametricPlot3D`
ParametricPlot3D[{Z*Cos[w-j]+Cos[w]/(2H), Z*Sin[w-j]+Sin[w]/(2H),x3},
                {u,U1,U2,Pi/24},{v,-Pi,Pi,Pi/24},
                Boxed->False,Axes->False,ViewPoint->{Vx,Vy,Vz}]
```

We use the value $H = 1/2$. By variation of the range of u and v and the change of ViewPoint, (table 2.), we make the pictures which enable to perceive the structure of tori.

Figure	**range of u**	**range of v**	**ViewPoint**
5.a	$(-\pi/2, 5\pi/2)$	$(-\pi, \pi)$	$(-1, 1, 1)$
5.b	$(-\pi, 5\pi)$	$(-\pi/6, \pi/6)$	$(-1, 3, 5)$
5.c	$(-\pi/3, 5\pi/3)$	$(-\pi/3, \pi/3)$	$(-1, 1, 1)$
5.d	$(-2\pi/3, 4\pi)$	$(-\pi, \pi/24)$	$(-1, 1, 1)$
5.e	$(-\pi/3, 5\pi/3)$	$(-\pi, \pi/24)$	$(-1, 1, -1)$
5.f	$(-\pi/3, 5\pi/3)$	$(-\pi, \pi/24)$	$(0, -1, -1)$
6.a	$(-\pi/2, 7\pi/2)$	$(-\pi, \pi)$	$(-1, -1, 1)$
6.b	$(0, 7\pi/2)$	$(-\pi/4, \pi)$	$(-1, -1, 1)$
6.c	$(-\pi/2, 7\pi/2)$	$(-\pi/2, \pi/32)$	$(-1, -1, 1)$
6.d	$(-\pi/3, 7\pi/3)$	$(-\pi/3, \pi/3)$	$(-1, -1, 1)$
6.e	$(-\pi, 7\pi)$	$(-\pi/2, \pi/24)$	$(-1, -1, 1)$
7.a	$(-\pi/2, 9\pi/2)$	$(-\pi, \pi)$	$(-1, 1, 1)$
7.b	$(-\pi, 9\pi)$	$(-\pi/8, \pi/8)$	$(-1, -1, 1)$
7.c	$(-\pi/2, 9\pi/2)$	$(-\pi/2, \pi/16)$	$(-1, -1, 1)$
7.d	$(-\pi/3, 3\pi)$	$(-\pi/2, \pi/16)$	$(-1, -1, 1)$
8.a	$(-\pi/2, 15\pi/2)$	$(-\pi, \pi)$	$(-1, 1, -3)$
8.b	$(-\pi/2, 15\pi/2)$	$(-\pi, \pi)$	$(-1, 1, 1)$

Table 2: *The range of* u *and* v *and ViewPoint*

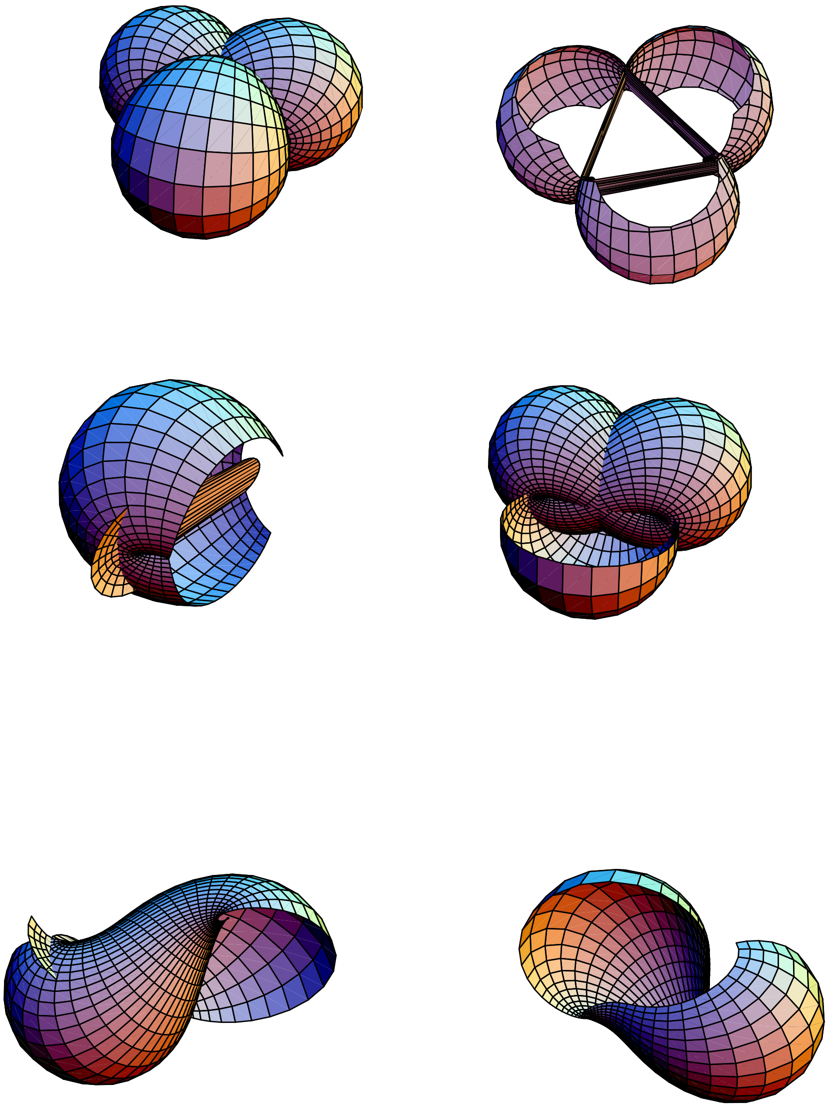


Figure 5(a-f): *A Wente torus* $W_{4/3}$ *with cut-aways*

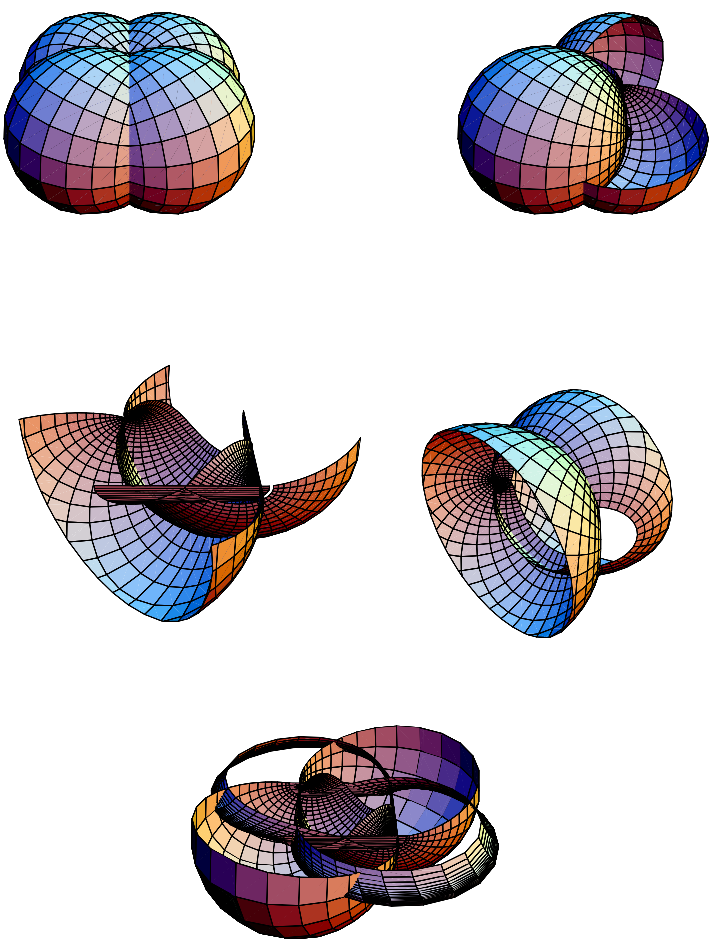


Figure 6(a-e): *A Wente torus* $W_{3/2}$ *with cut-aways*

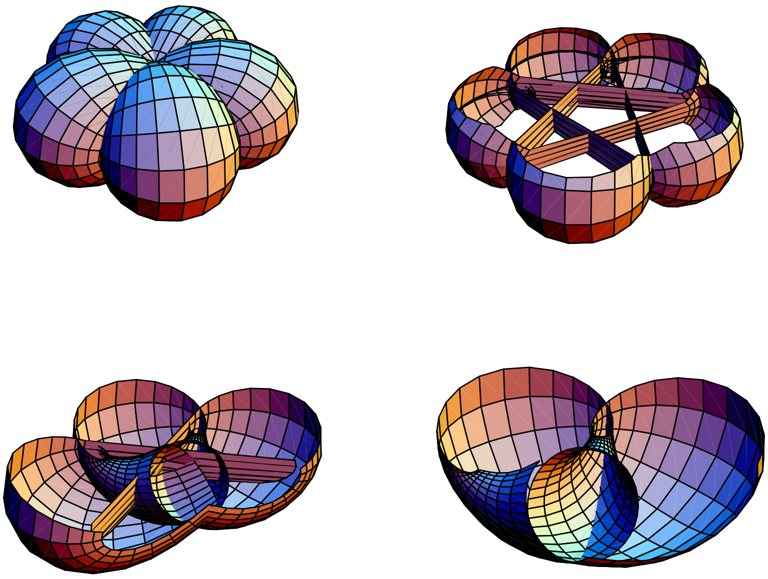


Figure 7(a-d): *A Wente torus* $W_{6/5}$ *with cut-aways*

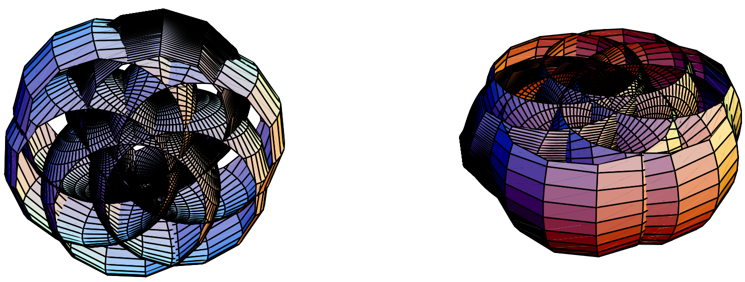


Figure 8(a-b): *Cut-aways of Wente torus* $W_{5/4}$

References

[1] A. I. Bobenko, Surfaces in terms of of 2 by 2 matrices. Old and new integrable systems, *Braunschweig Wiesbaden: Vieweg*, (1994), 483–127.

[2] A. I. Bobenko, Exploring Surfaces through Methods from the Theory of Integrable Systems. Lectyres on the Bonnet problem, *Fachberereich Mathematik, Technische Universitat Berlin, Strasse des 17. Juni 136.*

[3] M. S. Ćirić, Graphical presentation of some constant mean curvature surfaces, *moNGeometrija*, (2008), 38–47.

[4] S. Fujimori, S. Kobayashi and W. Rossman, Loop Methods for Constant Mean Curvature Surfaces, *arXiv: math/0602570v1 [math.DG]*, (2006).

[5] H. Gollek, On projective duals of vector fields, *Institute of Mathematics, Humboldt University, Berlin* , (2007).

[6] W. Hongyou and De Kalb, The PDE describing constant mean curvature surfaces, *Mathematica Bohemica*, (2001), 531–540.

[7] G. Kamberov, Prescribing mean curvature: Existence and uniqueness problems, *Electronic research announcement of the American math. society*, **9** (1998), 4–11.

[8] W. Rossman, The Morse Index of Wente Tori, *Faculty of Science, Kobe Univ., Japan*, (2008).

[9] W. Rossman, Computers and mathematics: Applications of computers to integrable systems methods for constructing constant mean curvature surfaces, (2006).

[10] K. Toyama, Self-Parallel Constant Mean Curvature Surfaces, *EG-models, http://www.math.tu-berlin.de/eg-models*, (2002), 4–11.

[11] R. Walter, Explicit examples to the H-problem of Heinz Hopf' Geom, *Dedicata 23*, (1987), 187–213.

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