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P*th* MEAN ASYMPTOTIC STABILITY AND INTEGRABILITY OF ITÔ–VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

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Abstract

Sufficient conditions for the *pth* mean stability and integrability of the solutions to non-linear Itô–Volterra integrodifferential equations with non-convolution drift and diffusion terms are investigated in this paper. Asymptotic convergence rates in *pth* moment sense are also discussed for the convolution case with infinite delay.

1 Introduction

In many fields of physical, engineering and social sciences there is a large number of nonlinear dynamical systems dependent of random excitations of a Gaussian white noise type. For example, the behavior of a non-linear dynamical system can be represented by the following differential equation

$$\ddot{y} + f(t, \dot{y}, y) = g(t, \dot{y}, y) \cdot \xi(t, \omega), \quad t \geq 0,$$

where $\xi(t, \omega)$ is a Gaussian white noise perturbation and $\omega \in \Omega$ are random events. Since $\xi(t, \omega) = \dot{w}(t, \omega)$, where $w(t, \omega)$ is a Brownian motion, this equation can be transformed into the following stochastic system,

$$\begin{aligned} dy(t) &= x(t)\,dt \\ dx(t) &= -f\Big(t, x(t), c + \int_0^t x(s)\,ds\Big)dt + g\Big(t, x(t), c + \int_0^t x(s)\,ds\Big)dw(t), \end{aligned} \tag{1}$$

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where $y(0) = c$ and ω is usually omitted, as we will do throughout the paper. The second equation in this system is the stochastic Volterra integrodifferential equation, which is a special case of the equation considered in the present paper,

$$dx(t) = \left[f(t,x(t)) + g\left(t, x(t), \int_0^t G(t,s)\, g_1(s,x(s))\, ds\right)\right] dt \tag{2}$$
$$+h\left(t, x(t), \int_0^t H(t,s)\, h_1(s,x(s))\, ds\right) dw(t), \quad t \geq 0,\ x(0) = x_0.$$

In a different direction, there are several open problems concerning the solutions of these equations. The researchers' interest is focused, among other things, on exploring the behavior of the solutions by comparing them in some sense with the solutions of the corresponding deterministic Volterra integrodifferential equations

$$\dot{x}(t) = f(t,x(t)) + g\left(t, x(t), \int_0^t G(t,s) g_1(s,x(s))\, ds\right), \quad t \geq 0,\ x(0) = x_0, \tag{3}$$

or, in a simpler form, with the solutions of ordinary differential equations $\dot{x}(t) = f(t,x(t))$, $t \geq 0$, $x(0) = x_0$. One of the priority is to emphasize conditions under which the asymptotically stable solution to Eq. (3) remains to be asymptotically stable in the presence of stochastic excitations described by the diffusion coefficient of Eq. (38).

It is well-known that there is significant and very rich literature discussing special techniques to study almost sure and *pth* mean exponential stability of solutions to various classes of stochastic differential equations (see monographs [12] and [15] by X. Mao, and [6, 9, 11, 19], for instance). However, less attention has been devoted to asymptotic stability of Itô–Volterra integrodifferential equations (see [1, 2, 3, 13, 14] and literature cited therein, among other things). For example, the exponential mean square asymptotic stability and integrability for the convolution Itô–Volterra equation $dx(t) = f(t,x(t))\, dt + h\left(t, \int_0^t H(t-s)\, x(s)\, ds\right) dw(t)$ is considered in [13] under the assumptions that the kernel H decays exponentially fast. On the other hand, even in the deterministic linear case, that is, for the equation $\dot{x}(t) = -a\, x(t) + \int_0^t H(t-s)\, x(s)\, ds$, exponential stability cannot always be guaranteed if the kernel does not vanish exponentially fast (see Murakami [16], for instance). Because of that, it is important to provide conditions guaranteeing *pth* mean asymptotic stability and integrability of the solutions to Eq. (2). We highlight paper [1] by J. Appleby where the relations between integrability in *pth* mean and almost sure, as well as *pth* mean asymptotic stability of the non-linear time-homogeneous Itô–Volterra equation with convolution kernels, $dx(t) = \left[f_1(x(t)) + \int_0^t K_1(t-s) g_1(x(s))\, ds\right] dt + \left[f_2(x(t)) + \int_0^t K_2(t-s)\, g_2(x(s))\, ds\right] dw(t)$ is studied under various assumptions for the coefficients of the equation. Note that paper [1] contains detailed literature about similar problems for various deterministic and stochastic differential equations.

In the present paper, our investigation is essentially based on the recent paper [14] by X. Mao and M. Riedle, where they studied asymptotic mean square stability and integrability of one restriction of Eq. (2), that is, of the following Itô–Volterra integrodifferential equation

$$dx(t) = \left[f(t, x(t)) + g\left(t, \int_0^t G(t,s)\, x(s)\, ds\right)\right] dt \tag{4}$$
$$+h\left(t, \int_0^t H(t,s)\, x(s)\, ds\right) dw(t), \quad t \geq 0,\ x(0) = x_0.$$

In fact, we generalize the results of paper [14] in two directions: First, we discuss Eq. (2) which is more complex than Eq. (4). Second, we study asymptotic *pth* mean stability, $p \geq 2$, and *pth* mean integrability of the solutions to Eq. (2). Note that the stochastic Volterra integrodifferential equation in the system (1) is not of the type (4).

The organization of the paper is as follows: Section 2 describes the problems to be studied and contains the main results of the paper. Precisely, we provide sufficient conditions under which the solutions to Eq. (2) are *pth* mean asymptotically stable and integrable. In Section 3, the convolution case of Eq. (2) with infinite delay is considered. Likewise, some *pth* mean asymptotic convergence rates are discussed, based on comparison with the appropriate results [20] by D.F. Shea and S. Wainger for deterministic Volterra integrodifferential equations. The concluding Section 4 contains some comments on extension of the previous results to cover the *pth* asymptotic stability and integrability of more complex Itô–Volterra equations. This section also contains some examples that illustrate the usefulness of the theoretical considerations.

2 Main results

As usual, we first estabilish some standard notations and notions. Our initial assumption is that all random variables and processes considered here are defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a standard n-dimensional Brownian motion $w = \{w(t), t \geq 0\}$ (i.e. $\mathcal{F}_t = \sigma\{w(s), 0 \leq s \leq t\}$). Let $|x|$ stand for the Euclidean norm of $x \in R^k$ and $||A||$ for the operator norm of a matrix $A \in R^{k\times l}$, $k, l \in N$, while A^T is the transpose of a matrix or vector. Let also $L^{\frac{p}{2}}(R_+)$ be the space of integrable functions $B : R_+ \to R^{k\times l}$ with $||B||_{L^{\frac{p}{2}}} := \int_0^\infty ||B(u)||^{\frac{p}{2}}\, du$, where $p \geq 2$ and $R_+ = [0, \infty)$.

The topic of our analysis is Eq. (2), that is, its equivalent integral form,

$$x(t) = x_0 + \int_0^t \left[f(s, x(s)) + g\left(s, x(s), \int_0^s G(s,u)\, g_1(u, x(u))\, du\right)\right] ds \tag{5}$$
$$+ \int_0^t h\left(s, x(s), \int_0^s H(s,u)\, h_1(u, x(u))\, du\right) dw(s),\ t \geq 0.$$

We suppose that x_0 is a random vector in R^d with $E|x_0|^p < \infty$ and independent of w. The functions $f : R_+ \times R^d \to R^d$, $g : R_+ \times R^d \times R^{d_1} \to R^d$, $g_1 : R_+ \times R^d \to R^{d_2}$,

$h : R_+ \times R^d \times R^{d_3} \to R^{d\times n}$, $h_1 : R_+ \times R^d \to R^{d_4}$ are assumed to be Borel measurable in their domains and satisfy the conditions $f(t,0) \equiv 0$, $g(t,0,0) \equiv 0$, $g_1(t,0) \equiv 0$, $h(t,0,0) \equiv 0$, $h_1(t,0) \equiv 0$ for all $t \geq 0$. The kernels $G : J \to R^{d_1\times d_2}$ and $H : J \times R^{d_3\times d_4}$ are continuous on J, where $J = \{(t,s) : 0 \leq s \leq t\}$, $G \equiv 0$, $H \equiv 0$ on J^c, and they are integrable in the sense

$$||G||_B := \sup_{t\geq 0} \int_0^t ||G(t,s)||^{\frac{p}{2}}\, ds < \infty, \tag{6}$$
$$||H||_B := \sup_{t\geq 0} \int_0^t ||H(t,s)||^{\frac{p}{2}}\, ds < \infty.$$

The equation (5) was earlier studied in several papers [17, 18] by M.G. Murge, B.G. Pachpatte and [8, 10] by S. Janković and M. Jovanović, for instance, and also in the basic paper [4] by M.A. Berger, V.J. Mizel in somewhat simpler form, that is, when the equation is linear with respect to all the Lebesgue and Itô integrals. The basic existence-and-uniqueness theorem based on Picard method of iterations was proved in [17]. Under the assumptions that all the functions in Eq. (5) are globally Lipschitzian and satisfy the linear growth conditions uniformly in t, s, and if $E|x_0|^2 < \infty$, then there exists a unique a.s. continuous and adapted strong solution $x(t;x_0)$ to Eq. (5) satisfying $\sup_{t\in[0,T]} E|x(t;x_0)|^2 < \infty$ for all $T > 0$. Moreover, if $E|x_0|^p < \infty$, $p > 0$, then $\sup_{t\in[0,T]} E|x(t;x_0)|^p < \infty$.

Throughout the paper, we assume, with no emphasis on conditions, that there exists a unique strong solution $x(t;x_0), t \geq 0$ to Eq. (5) satisfying $\sup_{t\in[0,T]} E|x(t;x_0)|^p < \infty$, $p \geq 2$, and that all the Lebesgue and Itô integrals employed further are well defined. Obviously, the above assumptions ensure that if $x_0 = 0$ a.s., then Eq. (5) has the zero solution, that is, $x(t,0) = 0$ a.s. for all $t \geq 0$.

As we saw above, the main purpose of the paper is to estabilish conditions under which the solution $x(t;x_0)$ to Eq. (5) is the *pth* mean asymptotically stable in the sense that

$$\lim_{t\to\infty} E|x(t;x_0)|^p = 0 \tag{7}$$

whatever the value of the initial condition $x(0) = x_0$. For simplicity, we will further use the notation $x(t)$ instead of $x(t;x_0)$ to denote the solution to Eq. (5) for a given initial condition x_0. Likewise, we will sometimes write x_t instead of $x(t)$ because of the clarity and shorter notation.

Theorem 1 *Let there exist a symmetric $(d \times d)-$matrix Q, continuous functions $\xi, \eta, \rho, \theta : R_+ \to R_+$ and constants $k_1, k_2 > 0$ such that for all $t \geq 0, x \in R^d, y \in R^{d_1}, z \in R^{d_3}$ the following conditions hold:*

(H_1) $|x|^2 \leq x^T Q x$,

(H_2) $p\, x^T Q f(t,x) \leq -\xi(t) \cdot x^T Q x$,

(H_3) $|Qg(t,x,y)| \leq \eta(t) \cdot (|x| + |y|)$,

(H_4) $|Qh(t,x,z)| \leq \rho(t) \cdot (|x| + |z|)$,

(H_5) $trace[h^T(t,x,z) Q h(t,x,z)] \leq \theta(t) \cdot (|x|^2 + |z|^2)$,

(H_6) $\quad |g_1(t,x)| \le k_1\,|x|,\ \ |h_1(t,x)| \le k_2\,|x|.$

If the functions

$$a(t) := \xi(t) - \Big[(p-1)\big(2\eta(t) + 2(p-2)\rho^2(t) + \theta(t)\big) \\ + k_1^{\frac{p}{2}}\, \|G\|_B\, \eta(t)\, t^{\frac{p}{2}-1}\Big], \tag{8}$$

$$b(t,s) := k_1^{\frac{p}{2}} \eta(t)\, t^{\frac{p}{2}-1} \|G(t,s)\|^{\frac{p}{2}} \\ + k_2^{p}\, ||H||_B\, t^{p-2}\big(\theta(t) + 2(p-2)\rho^2(t)\big) \cdot \|H(t,s)\|^{\frac{p}{2}} \tag{9}$$

for all $(t,s) \in J$ *satisfy the conditions*

$$a(t) > 0 \textit{ for all large } t \textit{ and } \int_0^\infty a(s)\, ds = \infty, \tag{10}$$

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_0^t b(t,s)\, ds < 1, \tag{11}$$

$$\lim_{t\to\infty} \frac{1}{a(t)} \int_0^T b(t,s)\, ds = 0 \textit{ for each } T \ge 0, \tag{12}$$

then the solution x *to Eq. (5) is pth moment asymptotically stable, that is,*

$$\lim_{t\to\infty} E|x(t)|^p = 0.$$

Proof. Let us denote that

$$y(s) = \int_0^s G(s,u)x(u)\, du, \quad z(s) = \int_0^s H(s,u)x(u)\, du, \quad s \ge 0.$$

Then Eq. (5) can be rewritten in its differential form as

$$dx(t) = [f(t,x(t)) + g(t,y(t))]\, dt + h(t,z(t))\, dw(t).$$

If we take $U(t) = (x_t^T Q x_t)^{p/2}$ and apply the Itô formula, we obtain for $t \ge 0$ that

$$dU(t) = p\,(x_s^T Q x_s)^{\frac{p}{2}-1}\Big[x_t^T Q f(t,x_t) + x_t^T Q g(t,x_t,y_t) \\ + \frac{1}{2}\, trace[h^T(t,x_t,z_t) Q h(t,x_t,z_t)] \\ + p\left(\frac{p}{2}-1\right)(x_t^T Q x_t)^{\frac{p}{2}-2} |x_t^T Q h(t,x_t,z_t)|^2\Big]\, dt \\ + p\,(x_s^T Q x_s)^{\frac{p}{2}-1} x_s t^T Q h(t,x_t,z_t)\, dw(t).$$

Let us define the function $V(t) := EU(t)$. Then, (H_2) yields for $0 \le t_1 \le t$,

$$V(t) \le V(t_1) - \int_{t_1}^t \xi(s) V(s)\, ds + I_1(t) + I_2(t) + I_3(t), \tag{13}$$

where

$$I_1(t) = p\,E\int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-1} x_s^T Q g(s,x_s,y_s)\,ds,$$
$$I_2(t) = \frac{p}{2}\,E\int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-1} trace[h^T(s,x_s,z_s) Q h(s,x_s,z_s)]\,ds, \tag{14}$$
$$I_3(t) = p\left(\frac{p}{2}-1\right) E\int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-2} |x_s^T Q h(s,x_s,z_s)|^2\,ds.$$

It remains to estimate these integrals.

In order to estimate $I_1(t)$, we will apply (H_1) and (H_3) and the following version of the Young inequality: $a^{\nu-1}b \le \frac{\nu-1}{\nu}\,a^\nu + \frac{1}{\nu}\,b^\nu$, $a,b\ge 0, \nu \ge 1$. Thus, we have

$$\begin{aligned}
I_1(t) &\le p\,E\int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-1} |x_s|\,|Qg(s,x_s,y_s)|\,ds \\
&\le p\,E\int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-1} \eta(s)\big(|x_s|^2 + |x_s|\,|y_s|\big)\,ds \\
&\le p\int_{t_1}^{t} \eta(s)\,V(s)\,ds + p\,E\int_{t_1}^{t} \eta(s)\left[\frac{\frac{p}{2}-1}{\frac{p}{2}}\,(x_s^T Q x_s)^{\frac{p}{2}} + \frac{2}{p}\,|x_s|^{\frac{p}{2}} |y_s|^{\frac{p}{2}}\right] ds \\
&= 2(p-1)\int_{t_1}^{t} \eta(s)\,V(s)\,ds + 2\,E\int_{t_1}^{t} \eta(s)|x_s|^{\frac{p}{2}} |y_s|^{\frac{p}{2}}\,ds.
\end{aligned}$$

The last integral can be estimated by applying the Hölder inequality to $|y_s|^{\frac{p}{2}}$, by using (H_6) and the property (6) of the kernel G. Hence,

$$\begin{aligned}
&2\,E\int_{t_1}^{t} \eta(s)|x_s|^{\frac{p}{2}} |y_s|^{\frac{p}{2}}\,ds \\
&\quad = 2\,E\int_{t_1}^{t} \eta(s)|x_s|^{\frac{p}{2}} \left|\int_0^s G(s,u)\,g_1(u,x_u)\,du\right|^{\frac{p}{2}} ds \\
&\quad \le 2k_1^{\frac{p}{2}} E\int_{t_1}^{t} \eta(s)\,s^{\frac{p}{2}-1}\left(\int_0^s \|G(s,u)x_u\|^{\frac{p}{2}} |x_s|^{\frac{p}{2}} |x_u|^{\frac{p}{2}}\,du\right) ds \\
&\quad \le k_1^{\frac{p}{2}} E\int_{t_1}^{t} \eta(s)\,s^{\frac{p}{2}-1}\left(\int_0^s \|G(s,u)\|^{\frac{p}{2}}\big(|x_s|^p + |x_u|^p\big)\,du\right) ds \\
&\quad \le k_1^{\frac{p}{2}}\|G\|_B \int_{t_1}^{t} \eta(s)\,s^{\frac{p}{2}-1} V(s)\,ds \\
&\qquad + k_1^{\frac{p}{2}} \int_{t_1}^{t} \eta(s)\,s^{\frac{p}{2}-1}\left(\int_0^s \|G(s,u)\|^{\frac{p}{2}} V(u)\,du\right) ds.
\end{aligned}$$

Therefore,

$$I_1(t) \le \int_{t_1}^{t} \eta(s)\left[2(p-1) + k_1^{\frac{p}{2}}\,||G||_B\,s^{\frac{p}{2}-1}\right] V(s)\,ds \tag{15}$$

$$+k_1^{\frac{p}{2}} \int_{t_1}^{t} \eta(s)\, s^{\frac{p}{2}-1} \left(\int_0^s \|G(s,u)\|^{\frac{p}{2}} V(u)\, du \right) ds.$$

Similarly, by using (H_5) we get

$$\begin{aligned} I_2(t) &\le \frac{p}{2}\, E \int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-1} \theta(s) \big(|x_s|^2 + |z_s|^2\big)\, ds \\ &\le \frac{p}{2}\, E \int_{t_1}^{t} \theta(s)\, V(s)\, ds + \frac{p}{2} \int_{t_1}^{t} \theta(s) \left[\frac{\frac{p}{2}-1}{\frac{p}{2}} (x_s^T Q x_s)^{\frac{p}{2}} + \frac{2}{p} |z_s|^p \right] ds \\ &= (p-1) \int_{t_1}^{t} \theta(s)\, V(s)\, ds + E \int_{t_1}^{t} \theta(s) |z_s|^p\, ds. \end{aligned}$$

To estimate the second integral, we apply the Hölder inequality for $\mu = p, \nu = \frac{p}{p-1}, \frac{1}{\mu} + \frac{1}{\nu} = 1$, as well as (H_6) and (6). So,

$$\begin{aligned} & E \int_{t_1}^{t} \theta(s) |z_s|^p\, ds \qquad (16) \\ &= E \int_{t_1}^{t} \theta(s) \left| \int_0^s H(s,u)\, h_1(u, x_u)\, du \right|^p ds \\ &\le k_2^p\, E \int_{t_1}^{t} \theta(s) \left(\int_0^s \|H(s,u)\|^{\frac{p}{2(p-1)}}\, du \right)^{p-1} \\ &\qquad \times \left(\int_0^s \|H(s,u)\|^{\frac{p}{2}} |h_1(u,x_u)|^p du \right) ds \\ &\le k_2^p\, ||H||_B \int_{t_1}^{t} \theta(s)\, s^{p-2} \left(\int_0^s \|H(s,u)\|^{\frac{p}{2}}\, V(u)\, du \right) ds. \end{aligned}$$

Hence,

$$\begin{aligned} I_2(t) &\le (p-1) \int_{t_1}^{t} \theta(s)\, V(s)\, ds \qquad (17) \\ &\quad + k_2^p\, ||H||_B \int_{t_1}^{t} \theta(s)\, s^{p-2} \left(\int_0^s \|H(s,u)\|^{\frac{p}{2}}\, V(u)\, du \right) ds. \end{aligned}$$

By repeating the previous procedures and (H_4), we see that

$$\begin{aligned} I_3(t) &\le p \left(\frac{p}{2} - 1 \right) E \int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-2} |x_s|^2 |Qh(s, x_s, z_s)|^2\, ds \qquad (18) \\ &\le p(p-2)\, E \int_{t_1}^{t} (x_s^T Q x_s)^{\frac{p}{2}-2} \rho^2(s)\, |x_s|^2 \left(|x_s|^2 + |z_s|^2 \right) ds \\ &\le p(p-2) \int_{t_1}^{t} \rho^2(s)\, V(s)\, ds \\ &\quad + p(p-2)\, E \int_{t_1}^{t} \rho^2(s) (x_s^T Q x_s)^{\frac{p}{2}-1} |z_s|^2\, ds \end{aligned}$$

$$\leq 2(p-1)(p-2)\int_{t_1}^{t}\rho^2(s)\,V(s)\,ds$$
$$+\,2(p-2)k_2^p||H||_B\int_{t_1}^{t}\rho^2(s)\,s^{p-2}\left(\int_0^s\|H(s,u)\|^{\frac{p}{2}}V(u)\,du\right)ds.$$

Finally, (13) together with (15), (17) and (18) implies that

$$V(t)\leq V(t_1)+\int_{t_1}^{t}\left(-a(s)V(s)+\int_0^s b(s,u)V(u)\,du\right)ds, \tag{19}$$

where the functions a and b are determined with (8) and (9), respectively.

Since the remainder of the proof is the same as the proof of Theorem 1 in [14], we will omit details and briefly describe the basic idea: The inequality (19) causes for $t \geq 0$ that

$$D_+V(t)\leq -a(t)V(t)+\int_0^t b(t,s)V(s)\,ds,$$

where $D_+V(t)$ is the right Dini derivative of $V(t)$. The essence of the proof is to compare $V(t)$ with the solution of the initial value problem

$$\dot{y}(t)=-a(t)y(t)+\int_0^t b(t,s)y(s)\,ds,\ t\geq 0,\ y(0)=V(0). \tag{20}$$

According to Theorem 2.2.2 in [5] (see also [7]), there exists a unique solution y to Eq. (20). Moreover, the conditions (10)–(12) imply that $y(t)\to 0$ as $t\to\infty$. Since $V(t)\leq y(t)$ for $t\geq 0$, then $E|x(t)|^p\leq V(t)\to 0$ as $t\to\infty$. $\diamondsuit$

The following theorem contains sufficient conditions under which the solution x to Eq. (5) is *pth* mean integrable.

Theorem 2 *Let the conditions (H_1)–(H_6) in Theorem 1 hold and let the functions a and b be defined with (8) and (9), respectively. If*

$$\sup_{s\in[0,T]}\int_s^\infty b(t,s)\,dt<\infty \text{ for every } T\geq 0, \tag{21}$$

$$\limsup_{s\to\infty}\left(-a(s)+\int_s^\infty b(t,s)\,dt\right)<0, \tag{22}$$

then the solution x to Eq. (5) is the pth moment integrable, that is,

$$\int_0^\infty E|x(t)|^p\,dt<\infty.$$

Proof. The proof is analogous to the one of Theorem 2 in [14]. We will briefly present it for completeness.

From (22), it follows that there exist $T \geq 0$ and a constant $k > 0$ such that for every $s \geq T$,

$$-a(s) + \int_s^\infty b(t,s)\,dt < -k,$$

A simple calculation yields from (19) that, for $t \geq T$,

$$V(t) \leq V(0) + \int_0^T \left(-a(s) + \int_s^\infty b(u,s)\,du\right) V(s)\,ds - k\int_T^t V(s)\,ds. \qquad (23)$$

Since (21) implies that $\int_T^t V(s)\,ds \leq c = \text{const}$, independently of t, from (23) we see that $\int_0^\infty E|x(ts)|^p\,ds \leq \int_0^\infty V(s)\,ds < \infty$. $\diamondsuit$

3 Itô–Volterra equations with convolution kernels

In this section, we will discuss *pth* mean asymptotic stability and integrability of the solutions to Eq. (5) with continuous convolution kernels $G, H \in L^{\frac{p}{2}}(R_+)$, that is, to the equation

$$dx(t) = \left[f(t,x(t)) + g\left(t, x(t), \int_0^t G(t-s)\,g_1(s,x(s))\,ds\right)\right] dt \qquad (24)$$
$$+h\left(t, x(t), \int_0^t H(t-s)\,h_1(s,x(s))\,ds\right) dw(t),\ t \geq 0,\ x(0) = x_0,$$

where the functions f, g, g_1, h, h_1 are defined as above. Recall that $||G||_B = \int_0^\infty ||G(u)||^{\frac{p}{2}}\,du = ||G||_{L^{\frac{p}{2}}}$ and $||H||_B = ||GH||_{L^{\frac{p}{2}}}$ in this case.

Having in mind the results from Section 2, we can formulate the following assertion.

Corollary 1 *Let there exist a symmetric $(d \times d)-$matrix Q, a continuous function $\xi : R_+ \to R_+$ such that $t^{p-2} = o\big(\xi(t)\big)$ as $t \to \infty$ and positive constants $\eta, \rho, \theta, k_1, k_2$ such that the conditions (H_1)–(H_6) hold with $\eta(t) \equiv \eta, \rho(t) \equiv \rho, \theta(t) \equiv \theta$. Then the solution x to Eq. (24) obeys* $\lim_{t\to\infty} E|x(t)|^p = 0$.

Proof. From (8) and (9), we have

$$a(t) = \xi(t) - \left[(p-1)\big(2\eta + 2(p-2)\rho^2 + \theta\big) + k_1^{\frac{p}{2}}\,\|G\|_{L^{\frac{p}{2}}}\,\eta\,t^{\frac{p}{2}-1}\right],$$
$$\sup_{t\geq 0}\int_0^t b(t,s)\,ds \leq k_1^{\frac{p}{2}}\eta\,||G||_{L^{\frac{p}{2}}}\,t^{\frac{p}{2}-1} + k_2^p\,||H||^2_{L^{\frac{p}{2}}}\big(\theta + 2(p-2)\rho^2\big)\,t^{p-2}.$$

Since the conditions (10)–(12) hold, the proof follows straightforwardly by applying Theorem 1. $\diamondsuit$

Especially, if $\xi(t) = \xi \cdot t^\alpha$, $\xi = \text{const} > 0$, the above assertion holds for $\alpha > p-2$.

Recall that the main square asymptotic stability and integrability for Eq. (4) with convolution kernels in paper [14] were considered under the assumptions (H_1)–(H_3) and (H_5) with positive constants ξ, η, ρ, θ. However, these assumptions do not allow us to apply of Theorem 1 and Theorem 2 in order to estabilish *pth* mean asymptotic stability and integrability of the solutions to Eq. (24) since the conditions (10) and (21) do not hold, for instance. Because of that, we will appropriately modify (H_2)–(H_5) to verify the *pth* mean asymptotic stability and integrability of the solutions to Eq. (24) and to significantly simplify the conditions (10)–(12) and (21)–(22).

Theorem 3 *Let there exist a symmetric $(d \times d)-$matrix Q and positive constants $\xi, \eta, \rho, \theta, k_1, k_2$ such that for all $t > 0, x \in R^d, y \in R^{d_1}, z \in R^{d_3}$ the conditions (H_1) and (H_6) are valid and the following assumptions hold:*

(H_7) $\quad p\, x^T Q f(t,x) \leq -\xi \cdot x^T Q x,$

(H_8) $\quad |Qg(t,x,y)| \leq \eta \cdot t^{\frac{2}{p}-1} \cdot |y|,$

(H_9) $\quad |Qh(t,x,z)| \leq \rho \cdot t^{\frac{2}{p}-1} \cdot |z|,$

(H_{10}) $\quad trace[h^T(t,x,z) Q h(t,x,z)] \leq \theta \cdot t^{2\left(\frac{2}{p}-1\right)} \cdot |z|^2.$

If

$$2k_1^{\frac{p}{2}}\, \eta\, ||G||_{L^{\frac{p}{2}}} + k_2^p\big((p-2)\rho^2 + \theta\big)\, ||H||^2_{L^{\frac{p}{2}}} < \xi - \frac{p-2}{2}\big(2\eta + (p-2)\rho^2 + \theta\big), \quad (25)$$

then the solution x to Eq. (24) satisfies

$$\lim_{t\to\infty} E|x(t)|^p = 0, \qquad \int_0^\infty E|x(t)|^p\, dt < \infty. \quad (26)$$

Proof. We can start from the relation (13) which, because of (H_7), becomes

$$V(t) \leq V(t_1) - \xi \int_{t_1}^t V(s)\, ds + I_1(t) + I_2(t) + I_3(t), \quad (27)$$

where $I_i(t), i = 1,2,3$ are determined with (14). By using (H_8), we find for $0 \leq t_1 < t$ that

$$\begin{aligned} I_1(t) &\leq p\eta\, E\int_{t_1}^t (x_s^T Q x_s)^{\frac{p}{2}-1} s^{\frac{2}{p}-1} |x_s|\, |y_s|\, ds \\ &\leq p\eta\, E\int_{t_1}^t \left[\frac{\frac{p}{2}-1}{\frac{p}{2}}\, (x_s^T Q x_s)^{\frac{p}{2}} + \frac{2}{p}\, s^{1-\frac{p}{2}} |x_s|^{\frac{p}{2}} |y_s|^{\frac{p}{2}}\right] ds \\ &= (p-2)\eta \int_{t_1}^t V(s)\, ds + 2\eta\, E\int_{t_1}^t s^{1-\frac{p}{2}} |x_s|^{\frac{p}{2}} |y_s|^{\frac{p}{2}}\, ds. \end{aligned}$$

Since

$$\begin{aligned} 2\, E\int_{t_1}^t s^{1-\frac{p}{2}} |x_s|^{\frac{p}{2}} |y_s|^{\frac{p}{2}}\, ds \leq k_1^{\frac{p}{2}} \|G\|_{L^{\frac{p}{2}}} &\int_{t_1}^t V(s)\, ds \\ &+ k_1^{\frac{p}{2}} \int_{t_1}^t \left(\int_0^s \|G(s-u)\|^{\frac{p}{2}} V(u)\, du\right) ds, \end{aligned}$$

then

$$I_1(t) \leq \left[(p-2)\eta + k_1^{\frac{p}{2}}\eta||G||_{L^{\frac{p}{2}}}\right]\int_{t_1}^{t} V(s)\,ds \qquad (28)$$
$$+k_1^{\frac{p}{2}}\eta\int_{t_1}^{t}\left(\int_0^s \|G(s-u)\|^{\frac{p}{2}}V(u)\,du\right)ds.$$

By applying (H_9), (H_{10}) and the procedure analogous to (16), we find that

$$I_2(t) \leq \frac{p-2}{2}\,\theta\int_{t_1}^{t} V(s)\,ds \qquad (29)$$
$$+k_2^p\,\theta\,||H||_{L^{\frac{p}{2}}}\int_{t_1}^{t}\left(\int_0^s \|H(s-u)\|^{\frac{p}{2}}\,V(u)\,du\right)ds.$$

$$I_3(t) \leq \frac{(p-2)^2}{2}\,\rho^2\int_{t_1}^{t} V(s)\,ds \qquad (30)$$
$$+(p-2)\,k_2^p\,\rho^2\,||H||_{L^{\frac{p}{2}}}\int_{t_1}^{t}\left(\int_0^s \|H(s-u)\|^{\frac{p}{2}}V(u)\,du\right)ds.$$

Substituting (28)–(30) into (3) yields the relation (19), where

$$a(s) \equiv a = \xi - \left[\frac{p-2}{2}\big(2\eta + (p-2)\rho^2 + \theta\big) + k_1^{\frac{p}{2}}\,\eta\,||G||_{L^{\frac{p}{2}}}\right], \qquad (31)$$
$$b(s,u) := \tilde{b}(s-u) \qquad (32)$$
$$= k_1^{\frac{p}{2}}\eta\,||G(s-u)||^{\frac{p}{2}} + k_2^p\,\big((p-2)\rho^2+\theta\big)\,||H||_{L^{\frac{p}{2}}}\cdot||H(s-u)||^{\frac{p}{2}}.$$

It is easy to verify that the conditions (10)–(12) in Theorem 1 as well as (21)–(22) in Theorem 2 can be replaced by the common condition (25). For instance, (25) implies that

$$\limsup_{t\to\infty}\frac{1}{a}\int_0^t b(t,s)\,ds = \frac{1}{a}\left[k_1^{\frac{p}{2}}\eta\,||G||_{L^{\frac{p}{2}}} + k_2^p\,\big((p-2)\rho^2+\theta\big)\,||H||^2_{L^{\frac{p}{2}}}\right] < 1.$$

Therefore, (26) holds by applying Theorem 1 and Theorem 2, which completes the proof. $\diamondsuit$

Corollary 2 *Let there exist a symmetric* $(d\times d)-$*matrix* Q*, a continuous function* $\xi: R_+ \to R_+$ *such that* $\xi(t)\to\infty$ *as* $t\to\infty$ *and positive constants* η,ρ,θ,k_1,k_2 *such that the conditions* (H_1)*,* (H_2)*,* (H_6)*,* (H_8)*–*(H_{10}) *hold. Then the solution* x *to Eq. (24) obeys* $\lim_{t\to\infty}E|x(t)|^p = 0$.

Proof. The proof follows from Theorem 1 since (10)–(12) are valid. $\diamondsuit$

Note that for $p = 2$, $g_1(t,x) \equiv x, h_1(t,x) \equiv x$, $K = 1$, the condition (25) is reduced to the one from paper [14] discussing Eq. (4).

In the remainder of this section, we will briefly discuss asymptotic convergence rates for the *pth* mean of the solutions to Eq. (24). More precisely, we explain conditions that are easy to verify guaranteeing, in dependence on some rate functions, the pth mean asymptotic stability and pth mean integrability of the solutions to Eq. (24), to provide a full insight into the study on this type of stochastic Volterra convolution equations. Since our discussion is motivated by paper [14] and since the proofs of the assertions presented below contain insignificant modifications with respect to the ones from [14], we will just formulate them and omit their proofs. For details, we refer the reader to Theorem 4, Corollary 5 and Corollary 6) in [14].

Our discussion is based on the above results [20] by D. Shea and S. Wainger (see also [5, 7]) on convergence rates for the deterministic Volterra integrodifferential equation with unbounded delay,

$$\dot{y}(t) = \int_{[0,t]} y(t-s)\,\mu(ds), \quad t \geq 0, \tag{33}$$

where μ is a finite Borel function satisfying $\int_0^\infty \varphi(s)|\mu|(ds) < \infty$ for a weight function φ and the total variation $|\mu|$ of μ. Hence, we first briefly introduce some notions and present Shea and Wainger's results needed in our discussion.

A function $\varphi : R \to R_+$ is said to be a weight function if $\varphi(0) = 1$, φ is measurable, locally bounded and locally bounded away from zero and $\varphi(t+s) \leq \varphi(t)\cdot\varphi(s)$ for $t, s \in R$. The last property causes (see [7], Lemma 4.4.1, p.120, for example) the existence of the limit

$$\pi_\varphi = -\lim_{t\to\infty} \frac{\ln\varphi(t)}{t}.$$

There are some of the weight functions:

$$\begin{aligned}
&\varphi(t) = e^{\alpha t},\ \alpha \in R, \text{ with } \pi_\varphi = -\alpha,\\
&\varphi(t) = (1+|t|)^\alpha,\ \alpha \geq 0,\ \text{with } \pi_\varphi = 0,\\
&\varphi(t) = (1+\ln(1+|t|))^\alpha,\ \alpha \geq 0, \text{ with } \pi_\varphi = 0.
\end{aligned}$$

If the so-called characteristic function of the measure μ,

$$h_\mu(z) := z - \int_0^\infty e^{-zu}\mu(du)$$

satisfies $h_\mu(z) \neq 0$ for every $z \in \mathcal{C}$ with $Re z \geq \pi_\varphi$, it is shown in [20] (see also [7], Theorems 4.4.13 and 4.4.16) that the solution y to Eq. (33) obeys

$$\lim_{t\to\infty} \varphi(t)\,y(t) = 0, \quad \int_0^\infty \varphi(t)\,y(t)\,ds < \infty. \tag{34}$$

We can now formulate the following assertion:

Theorem 4 *Let there exist a symmetric $(d \times d)-$matrix Q and positive constants $\xi, \eta, \rho, \theta, k_1, k_2$ such that for all $t > 0, x \in R^d, y \in R^{d_1}, z \in R^{d_3}$ the conditions (H_1), (H_6), (H_7)–(H_{10}) hold. If there exists a weight function φ such that*

$$\int_0^\infty \varphi(u) \left(||G(u)||^{\frac{p}{2}} + ||H(u)||^{\frac{p}{2}}\right) du < \infty, \tag{35}$$

and if

$$h(z) := z - a - \int_0^\infty e^{-zu}\, \tilde{b}(u)\, du \neq 0 \quad \text{for } z \in \mathcal{C},\ Re\, z \geq \pi_\varphi,$$

where a and $\tilde{b}$ are defined by (31) and (32), respectively, then the solution x to Eq. (24) obeys

$$\lim_{t\to\infty} \varphi(t)\, E|x(t)|^p = 0, \qquad \int_0^\infty \varphi(t)\, E|x(t)|^p\, dt < \infty. \tag{36}$$

The conditions guaranteeing the properties (36) for the solutions to Eq. (24) could be simplified significantly for weight functions φ with $\pi_\varphi = 0$ and $\varphi(t) = e^{\alpha t}$, which will be seen from the following assertions.

Corollary 3 *Let the conditions (H_1), (H_6), (H_7)–(H_{10}) hold. If there exists a weight function φ with $\pi_\varphi = 0$ such that (35) and (25) hold, then the solution x to Eq. (24) obeys (36).*

Corollary 4 *Let the conditions (H_1), (H_6), (H_7)–(H_{10}) and (25) hold. If there exist constants $\alpha_1, \alpha_2 > 0$ such that*

$$\int_0^\infty e^{\alpha_1 s} \|G(s)\|^{\frac{p}{2}}\, ds < \infty, \qquad \int_0^\infty e^{\alpha_2 s} \|H(s)\|^{\frac{p}{2}}\, ds < \infty, \tag{37}$$

then there exists a constant $\beta > 0$ such that the solution x to Eq. (24) obeys

$$\lim_{t\to\infty} e^{\beta t}\, E|x(t)|^p = 0, \qquad \int_0^\infty e^{\beta t}\, E|x(t)|^p\, dt < \infty.$$

4 Some comments and examples

First, let us give some conclusions and comments.

• As the general conclusion of the previous considerations, it is appropriate to say that under the noted sufficient conditions presence of noise does not destabilize in *pth* moment sense the asymptotically stable zero solution to deterministic equations (3) or $\dot{x}(t) = f(t, x(t))$. Precisely, the above assertions show how much dynamical system can tolerate stochastic perturbations without losing the property of *pth* asymptotic stability.

• In the paper, we have focused on the Itô–Volterra integrodifferential equation (2) and the appropriate convolution equation (24). However, the obtained results could be extended to more complex equations, for instance, to the equation

$$
\begin{aligned}
dx(t) = & \left[f(t, x(t)) + g\left(t, x(t), \int_0^t \tilde{g}(t, s, x(s))\, ds\right)\right] dt \\
& + h\left(t, x(t), \int_0^t \tilde{h}(t, s, x(s))\, ds\right) dw(t), \quad t \ge 0,
\end{aligned} \tag{38}
$$

with initial data $x(0) = x_0$, where $f : R_+ \times R^d \to R^d$, $g : R_+ \times R^d \times R^{d_1} \to R^d$, $h : R_+ \times R^d \times R^{d_2} \to R^{d\times n}$, $\tilde{g} : J \times R^d \to R^{d_1}$, $\tilde{h} : J \times R^d \to R^{d_2}$.

As usual, with no emphasis on conditions, we require that there exists a unique a.s. continuous and adapted solution to Eq. (38) (see [17, 18] for details). We suppose, for instance, that there exists a function $\xi : R_+ \to R_+$ with $t^{p-1} = o(\xi(t))$ as $t \to \infty$ such that $-p\, x^T f(t, x) \ge \xi(t)\, |x|^2$ for all $t \ge 0$ and $x \in R^d$. We also suppose that $|g(t, x, y)| \le K_1(|x| + |y|)$ and $||h(t, x, z)|| \le K_2(|x| + |z|)$ for some constants $K_1, K_2 > 0$ and all $t \ge 0, x \in R^d, y \in R^{d_1}, z \in R^{d_2}$, as well as that $|\tilde{g}(t, s, x)| \le G(t, s)\, |x|$ and $|\tilde{h}(t, s, x)| \le H(t, s)\, |x|$, where G and H are continuous and $L^{\frac{p}{2}}$-integrable scalar functions. If $f(t, 0) \equiv 0$, $g(t, 0, 0) \equiv 0$, $h(t, 0, 0) \equiv 0$, $\tilde{g}(t, s, 0) \equiv 0$, $\tilde{h}(t, s, 0) \equiv 0$, all the conditions of Corollary 1 are valid with $Q = I$, where I is a unit d-matrix, and, therefore, $\lim_{t\to\infty} E|x(t)|^p = 0$. Moreover, if (21) and (22) hold, Theorem 2 yields that $\int_0^\infty E|X(t)|^p < \infty$.

• Having in mind the results in [1], our future investigation has the benefit of connecting the almost sure asymptotic stability and *pth* mean asymptotic stability and integrability of the solutions to Eqs. (3) and (24) by using, among other things, the procedures applied in the present paper.

In the remainder, we will examine the validity of the preceding considerations by applying them to the following examples.

Example 1. Let us consider, for instance, the $4th$ mean asymptotic stability and integrability of the scalar Itô–Volterra equation with non-convolution kernels,

$$
\begin{aligned}
dx(t) = & \left[-a\, t^\alpha x(t) + \frac{1}{(1+t)^2} \ln\left(1 + |x(t)| + \left|\int_0^t \frac{8 \sin x(t)}{(2+t-s)^{\frac{5}{2}}}\, ds\right|\right)\right] dt \\
& + \frac{1}{1+t^2} \sin\left(x(t) + \int_0^t \frac{e^s\, x(s)}{(1+e^t)(1+s+|x(s)|)}\, ds\right) dw(t), \; x(0) = x_0.
\end{aligned} \tag{39}
$$

Here, a and $\alpha \ge 0$ are constants and

$$
\begin{aligned}
& f(t, x) = -a\, t^\alpha x, \quad G(t, s) = \frac{1}{(2+t-s)^{\frac{5}{2}}}, \quad H(t, s) = \frac{e^s}{1+e^t} \\
& g(t, x, y) = \frac{1}{(1+t)^2} \ln(1 + |x| + |y|), \quad g_1(t, x) = 8 \sin x, \\
& h(t, x, z) = \frac{1}{1+t^2} \sin(x + z), \quad h_1(t, x) = \frac{x}{1+t+|x|}.
\end{aligned}
$$

All the functions are locally Lipschitzian and satisfy the linear growth conditions, which guarantees the existence and uniqueness of the solution to Eq. (39) for an arbitrary Cauchy problem. Likewise, there exists a zero solution.

It is easy to check that for $p = 4$ the conditions (H_1)–(H_6) are valid with

$$\xi(t) = a\,t^{\alpha},\ \eta(t) = \frac{1}{(1+t)^2},\ \rho(t) = \frac{1}{1+t^2},\ \theta(t) = \frac{2}{(1+t^2)^2},$$
$$||G||_{L^2} = \frac{1}{64},\ ||H||_{L^2} = \frac{1}{2},\ k_1 = 8,\ k_2 = 1.$$

From (8) and (9) we find that

$$a(t) = a\,t^{\alpha} - \frac{6+t}{(1+t)^2} - \frac{18}{(1+t^2)^2},$$
$$b(t,s) = \frac{64t}{(2+t-s)^5} + \frac{3t^2}{(1+t^2)^2} \cdot \frac{e^{2s}}{(1+e^t)^2}.$$

Clearly, the condition (10) holds for $a > 1$, $\alpha = 1$ or $a > 0$, $\alpha > 1$. A simple calculation shows that $\int_0^t b(t,s)\,ds = t + o(t)$ as $t \to \infty$ and $\int_s^{\infty} b(t,s)\,dt = s + o(s)$ as $s \to \infty$, and, therefore, (11), (12), (21) and (22) hold for $a > 0$, $\alpha > 1$. What remains is to apply Theorem 1 and Theorem 2 to conclude that the solutions to Eq. (39) satisfy $\lim_{t\to\infty} E|x(t)|^4 = 0$ and $\int_0^{\infty} E|x(t)|^4\,dt < \infty$.

Note that $\eta(t) \le 1$, $\rho(t) \le 1$, $\theta(t) \le 2$. If we take an arbitrary convolution kernel $H \in L^{\frac{p}{2}}(R_+)$ instead of the kernel H in Eq. (39), the application of Corollary 1 causes that $\lim_{t\to\infty} E|x(t)|^4 = 0$ for $a > 0$, $\alpha > 2$.

Example 2. Let us now consider, for instance, the $6th$ mean asymptotic stability and integrability of the scalar Itô–Volterra equation with convolution kernels,

$$dx(t) = \left[f(t,x(t)) + 2\left(1+t^2+|x(t)|\right)^{-\frac{1}{3}} \int_0^t e^{-3(t-s)} x(s)\,ds\right] dt \tag{40}$$
$$+ \left(1+t+x^2(t)\right)^{-\frac{2}{3}} \int_0^t e^{-(t-s)} \sin x(s)\,ds\,dw(t), \quad t \ge 0, \quad x(0) = x_0,$$

where we suppose that $f(t,0) \equiv 0$ and that the condition (H_2) holds, that is, $6xf(t,x) \le -\xi \cdot x^2$ for a constant $\xi > 0$. Here,

$$g(t,x,y) = 2y\left(1+t^2+|x|\right)^{-\frac{1}{3}}, \quad g_1(t,x) = x, \quad G(t) = e^{-3t},$$
$$h(t,x,z) = z\left(1+t+x^2\right)^{-\frac{2}{3}}, \quad h_1(t,x) = \sin x, \quad H(t) = e^{-t}.$$

In addition to the zero solution and the fact that g, g_1, h, h_1 are Lipschitzian and satisfy the linear growth conditions, we suppose the existence and uniqueness of the solution of an arbitrary initial problem. It is easy to verify that the conditions (H_1), (H_6), (H_7)–(H_{10}) hold with $\eta = 2$, $\rho = \theta = 1$, $k_1 = k_2 = 1$, $||G||_{L^3} = 1/9$, $||H||_{K^3} = 1/3$. By applying Theorem 3, we find from (25) that $\xi > 19$ is the sufficient condition for the $6th$ mean asymptotic stability and integrability of the solutions to Eq. (40).

Moreover, the conditions (37) hold for $0 < \alpha_1 < 9$, $0 < \alpha_2 < 3$. Hence, the application of Corollary 4 implies that the solutions are exponentially asymptotically stable and L^6-integrable in the sense that there exists a constant $\beta > 0$ such that $\lim_{t\to\infty} e^{\beta t} E|x(t)|^6 = 0$ and $\int_0^\infty e^{\beta t} E|x(t)|^6\, dt < \infty$.

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